# Skew propagation time

by

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# DEDICATION

I would like to dedicate this thesis to my mother Monica, father Jerry, and sisters Corinne, Kaleigh, and Marissa. I wish also to include in this dedication Andrew Ylvisaker and his amazing family. Without their infinite support and encouragement I would not have been able to complete my research. I would also like to express my sincere gratitude those who have become my friends and family in the Iowa State mathematics department, especially Melanie Erickson, Dr. Leslie Hogben, and Dr. Wolfgang Kliemann for their support and guidance during the process of completing this work. I would not be where I am without all of you. Thank you for never giving up on me.

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# ABSTRACT

The zero forcing number has long been used as a tool for determining the maximum nullity of a graph, and has since been extended to skew zero forcing number, which is the zero forcing number used when the matrices corresponding to the graphs in question are required to have zeros as their diagonal entries. Skew zero forcing is based on the following specific color change rule: for a graph $G$ (that does not contain any loops), where some vertices are colored blue and the rest of the vertices are colored white, if a vertex has only one white neighbor, that vertex forces its white neighbor to be blue. The minimum number of blue vertices that it takes to force the graph using this color change rule is called the skew zero forcing number. A set of blue vertices of order equal to the skew zero forcing number of the graph, such that when the skew zero forcing process is carried out to completion the entire graph is colored blue, is called a minimum skew zero forcing set. More recently, the concept of propagation time of a graph was introduced. Propagation time of a graph is how fast it is possible to force the entire graph blue over all possible minimum zero forcing sets. In this thesis, the concept of propagation time is extended to skew propagation time. We discuss the tools used to study extreme skew propagation time, and examine the skew propagation time of several common families of graphs. Finally, we include a brief discussion on loop graph propagation time, where the graphs in question are allowed to contain loops (in other words, these graphs are loop graphs, not simple graphs, as in the skew zero forcing case).

# CHAPTER 1. INTRODUCTION

The zero forcing number of a graph, as it applies to combinatorial matrix theory, is an upper bound for maximum nullity and has been studied extensively since its introduction in [1]. The zero forcing number was extended to the skew zero forcing number in [23]. Propagation time $\mathrm{pt}(G)$ of a graph $G$ for standard zero forcing on simple graphs was introduced in [10] and [21]. Zero forcing was studied in control of quantum systems as well, as in [7], [8], and [26]; the idea of propagation in particular is evident in [26]. The propagation time problem for standard zero forcing has since been extended to the propagation time problem for positive semidefinite zero forcing and here is extended to loop graph zero forcing, and in particular, skew zero forcing.

## 1.1 Graph theory basics

A *simple, undirected graph* $G$ is denoted $G = (V, E)$ where $V(G)$ (or just $V$) is the set of vertices and $E(G)$ (or just $E$) is a set of unordered pairs representing edges between the vertices of the graph. The pair $\{u, v\} \in E$ is an edge between vertices $u$ and $v$ in the graph $G$ and can be denoted $uv$. For a simple graph, $u \neq v$ if $uv \in E$. Sometimes, however, it is desirable to talk about an edge between a vertex $v$ and itself. This type of edge is called a *loop*. In contrast to a simple graph $G$, a graph $\mathfrak{G} = (\mathfrak{V}, \mathfrak{E})$ that allows loops is called a *loop graph*. Note that although a loop graph is allowed to have loops, this does not mean it is required to have any loops. Figures 1.1–1.3 show examples of loop graphs where some of the vertices have loops, all of the vertices have loops, and none of the vertices have loops, respectively.

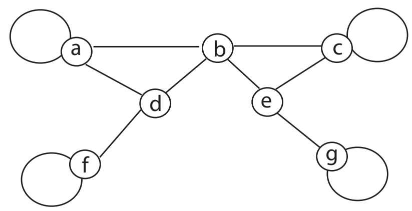



Figure 1.1 An example of a loop graph with loops on some vertices.

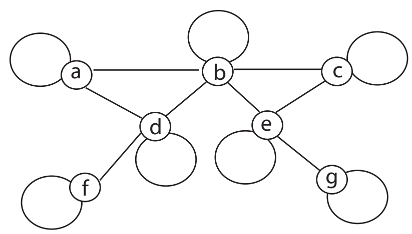



Figure 1.2 An example of a loop graph with loops on all vertices.

For the remainder of this work, with the exception of Chapter 3, a graph $G = (V, E)$ is a simple graph or can be viewed as a loop graph without loops, by considering parameters that account for this distinction. For any graph, we say that two distinct vertices $u$ and $v$ are adjacent if $uv$ is an edge. If $G = (V, E)$ such that $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}$, then $G$ is *path* on $n$ vertices, denoted $P_n$. If $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n,\ v_nv_1\}$, then $G$ is a *cycle* on $n$ vertices, denoted $C_n$. A graph $G$ is *connected* if for any two vertices $x, y \in V$ there is a path between $x$ and $y$. A *tree* is a connected graph that is *acyclic*, meaning it does not contain any cycles.

If $G$ is a graph, and $U \subset V$, then the *induced subgraph* obtained by deleting the vertices in $U$ along with their incident edges is denoted $G[V \backslash U]$ or just $G - U$. The induced subgraph obtained by deleting all the vertices in $V$ that are not in $U$ along with

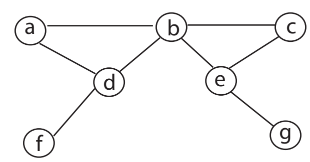



Figure 1.3   An example of a loop graph with no loops.

their incident edges is simply denoted $G[U]$. We say that a graph $H$ is a *subgraph* of $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Let $G_1$ and $G_2$ be any two graphs. The *cartesian product* of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G = (V, E)$ where $V = V_1 \times V_2$ and there is an edge between the vertices $(u, v)$ and $(u', v')$ if and only if $u = u'$ and $vv' \in E_2$ or $v = v'$ and $uu' \in E_1$. The cartesian product of $G_1$ and $G_2$ is denoted $G_1 \Box G_2$. The $n$th hypercube $Q_n$ is defined by induction on $n$ where $Q_1 = P_2$ and $Q_{n+1} = Q_n \Box P_2$. The *corona* of $G_1$ with $G_2$ is formed by taking a single copy of $G_1$ and $|G_1|$ copies of $G_2$ and making every vertex in the $i$th copy of $G_2$ adjacent to the $i$th vertex of $G_1$. The corona of $G_1$ with $G_2$ is denoted $G_1 \circ G_2$ and $|G_1 \circ G_2| = |G_1|(|G_2| + 1)$.

If $G_1$ and $G_2$ are any two graphs, then the *join* of $G_1$ and $G_2$, denoted $G_1 \vee G_2$, has the disjoint union of $V(G_1)$ and $V(G_2)$ as the vertex set, and the edge set includes the edges of $G_1$, the edges of $G_2$, and for every vertex $u \in V(G_1)$ and $v \in V(G_2)$, $u$ and $v$ are adjacent.

The distance between any two vertices $u$ and $v$ in a graph $G$, denoted $d_G(u, v)$, is the length of the shortest possible path between $u$ and $v$ in $G$. The diameter of the graph $G$ is the greatest distance between any two vertices in $G$ and is denoted $\text{diam}(G)$.

## 1.2 Literature review

### 1.2.1 Standard zero forcing

Standard zero forcing (sometimes referred to as graph propagation or graph infection) is based on the standard *color change rule*: for a simple graph $G$ in which some of the vertices are colored blue, if each vertex of $G$ is colored either white or blue, and vertex $v$ is a blue vertex with only one white neighbor $w$, then change the color of $w$ to blue. Applying the color change rule to a vertex $v$ with single white neighbor $w$ is called a *force*, and we write $v \to w$ to say that $v$ forces $w$. A *zero forcing set* for a graph $G$ is an initial set $B$ of blue vertices such that the set of blue vertices that results from applying the color change rule until no more changes are possible is the entire set of vertices of $G$. A *minimum zero forcing set* of a graph $G$ is a zero forcing set of the smallest possible cardinality, and the *zero forcing number* $\mathrm{Z}(G)$ is $|B|$ where $B$ is a minimum zero forcing set.

**Example 1.2.1.** For the simple graph in Figure 1.4, there is no single vertex and no vertex set of size 2 that when initially colored blue will allow the entire graph to be forced. There is a zero forcing set of cardinality 3, namely the set $\{a, f, g\}$. We know from [4] that if a graph has the complete graph $K_r$ as a subgraph, then the zero forcing number is at least $r - 1$. Hence the zero forcing number of the graph in Figure 1.4, which contains $K_4$ as a subgraph, is 3. The zero forcing process for this set is shown in Figure 1.4, and proceeds as follows: the only white neighbor of $f$ is $d$ and the only white neighbor of $g$ is $e$. So the first two forces are $f \to d$ and $g \to e$. Once these vertices are forced, the only white neighbor of $a$ is $b$, and so $a \to b$. Finally then, the only white neighbor of $b$ is $c$, and the force $b \to c$ results in the entire graph being colored blue.

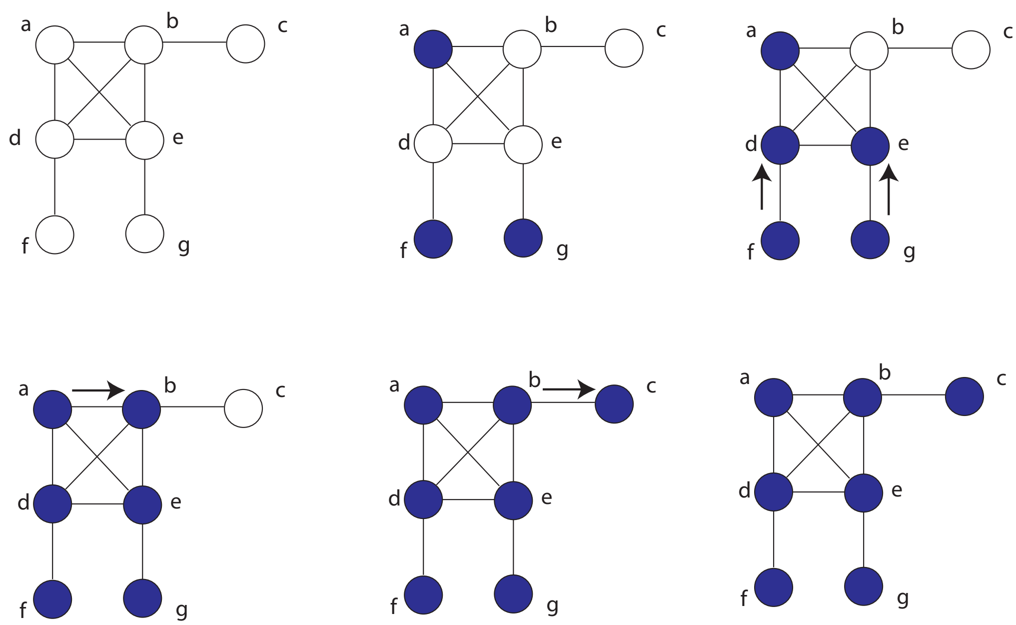



Figure 1.4 An example of a simple graph with zero forcing number 3.

As stated, zero forcing is an important tool in combinatorial matrix theory because the zero forcing number of a graph is an upper bound for its maximum nullity. Consider the set of real symmetric $n$-square matrices, $S_n(\mathbb{R})$. For a matrix $A \in S_n(\mathbb{R})$, the graph of $A$, denoted $\mathcal{G}(A)$, is the graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{v_i v_j | a_{ij} \neq 0 \text{ and } i \neq j\}$. Conversely, the set of symmetric matrices that correspond to a graph $G$ contains all matrices in $S_n(\mathbb{R})$ such that $\mathcal{G}(A) = G$, and is denoted $\mathcal{S}(G)$. (Note that the diagonal entries of these matrices can be zero or nonzero.) The *minimum rank* of a graph $G$ is $\operatorname{mr}(G) = \min\{\operatorname{rank} A | A \in \mathcal{S}(G)\}$, and the *maximum nullity* of $G$ is $\operatorname{M}(G) = \max\{\operatorname{null} A | A \in \mathcal{S}(G)\}$.

An obvious relationship between $\operatorname{mr}(G)$ and $\operatorname{M}(G)$ is that $\operatorname{mr}(G) + \operatorname{M}(G) = |G|$. For a more in depth discussion of the relationship between zero forcing and the minimum rank problem for graphs, see [1], [17] and [18].

### 1.2.2 Standard propagation time

The study of propagation time of a graph aims to discover how quickly, or in how many discrete time steps, an entire graph $G$ can be forced blue over all possible minimum zero forcing sets using standard zero forcing. Propagation time for standard zero forcing was defined in [21] and [10].

**Definition 1.2.2.** Let $G = (V, E)$ be a simple graph, and $B$ a zero forcing set for $G$. Define $B_0 = B$, and for $t \geq 0$, let $B_{t+1}$ be the set of vertices $w$ for which there exists any blue vertex $v$ in $G$ such that $w$ is the only white neighbor of $v$ not in $\cup_{s=0}^{t} B_s$. The *propagation time* of $B$ in $G$, denoted $\text{pt}(G, B)$, is the smallest integer $t_0$ such that $V_G = \cup_{t=0}^{t_0} B_t$.

Let $G = (V, E)$ be a graph and $B$ a zero forcing set for $G$, and let $t = 0$ so that $B_0 = B$. While $t = 0$, we require that every vertex in $B_0$ that can perform a force does perform a force, and when this process is complete, we set $t = 1$ and let $B_1$ be the new set of blue vertices. So in general, if $\cup_{i=0}^{t} B_i$ is the current set of blue vertices, and once any vertex in $\cup_{i=0}^{t} B_i$ that can perform a force does perform a force, we set $t = t + 1$; we can continue this process while $B_t \subset V$. The first value of $t$ for which $B_t = V$ is called the *propagation time* of $B$ in $G$, and is denoted $\text{pt}(G, B)$.

The *minimum propagation time* of $G$ is

$$\text{pt}(G) = \min\{\text{pt}(G, B) | B \text{ is a minimum zero forcing set of } G\}.$$

A subset $B$ of vertices of a graph $G$ is an *efficient zero forcing set* for $G$ if $B$ is a minimum zero forcing set of $G$ and $\text{pt}(G, B) = \text{pt}(G)$.

The *maximum propagation time* of $G$ is

$$\text{PT}(G) = \max\{\text{pt}(G, B) | B \text{ is a minimum zero forcing set of } G\}.$$

Two minimum zero forcing sets $B$ and $B'$ of a graph $G$ are *isomorphic* if there is a graph automorphism $\phi$ of $G$ such that $\phi(B) = B'$. With the notion of isomorphic zero forcing sets we can make the following observations.

**Observation 1.2.3.** *Any two isomorphic zero forcing sets achieve the same propagation time, so if $B$ is an efficient minimum zero forcing set, then any minimum zero forcing set isomorphic to $B$ is also efficient.*

**Observation 1.2.4.** *Simple graphs can have nonisomorphic minimum zero forcing sets that achieve minimum propagation time (See Example 1.5 in* [21]*), but it is not the case that every minimum zero forcing set of a graph necessarily achieves the same propagation time.*

**Example 1.2.5.** Let $G$ be the graph in Figure 1.4. The zero forcing number of $G$ is 3, as described in Example 1.2.1. Propagation time for the zero forcing set $B = \{a, f, g\}$ is $\text{pt}(G, B) = 3$. Propagation time for the zero forcing set $B' = \{a, b, d\}$ is $\text{pt}(G, B') = 2$, since at time step 1 $a \to e$ and at time step 2 $d \to f, e \to g$, and $b \to c$. From Remark 1.2.6 below, we note that if $\text{pt}(G) = 1$, then $\text{Z}(G) \geq \frac{|G|}{2}$. Since $\text{Z}(G) = 3 < \frac{7}{2}$, $\text{pt}(G) \neq 1$. Therefore we can conclude definitively that $\text{pt}(G) = 2$, since it cannot equal 1 and we have shown a zero forcing set $B'$ for which $\text{pt}(G, B') = 2$.

The next two remarks were observed in [21].

**Remark 1.2.6.** Since $\text{Z}(G) \geq 1$, the following must be a lower bound on $\text{pt}(G)$ for a simple graph:

$$\frac{|G| - \text{Z}(G)}{\text{Z}(G)} \leq \text{pt}(G).$$

This is because the number of vertices that must be forced blue initially is $|G| - \text{Z}(G)$, and at each time step at most $\text{Z}(G)$ vertices can be forced, giving $|G| - \text{Z}(G) \leq \text{pt}(G) \cdot \text{Z}(G)$. Dividing both sides by the $\text{Z}(G)$ gives the inequality.

**Remark 1.2.7.** For any graph $G$, $0 \leq \text{pt}(G) \leq |G| - 1$. It is clear that the zero forcing process must take some nonnegative number of time steps, and since the zero forcing number of a graph is strictly greater than zero, $\text{pt}(G)$ can never be equal to the order of the graph for standard zero forcing.

The following are known results regarding extreme propagation time for standard zero forcing.

**Observation 1.2.8.** [21] *Let $G$ be a graph. The following are equivalent for standard propagation time:*

1. $\text{pt}(G) = 0$
2. $\text{PT}(G) = 0$
3. *$G$ has no edges.*

Characterizing graphs with $\text{pt}(G) = 1$ proved to be a more difficult problem, and will be discussed later in this thesis when comparing graphs with standard propagation time 1 and skew propagation time 1.

For high standard propagation time, we have the following results due to [21].

**Proposition 1.2.9.** [21] *Let $G$ be a graph. The following are equivalent:*

1. $\text{pt}(G) = |G| - 1$
2. $\text{PT}(G) = |G| - 1$
3. $\text{Z}(G) = 1$
4. *$G$ is a path.*

**Observation 1.2.10.** [21] *For a simple graph $G$:*

1. $\text{pt}(G) = |G| - 2$ *implies* $\text{PT}(G) = |G| - 2$*, but not conversely.*
2. $\text{pt}(G) = |G| - 2$ *if and only if* $\text{Z}(G) = 2$ *and exactly one force is performed at each time step for every minimum zero forcing set.*

3. $\mathrm{PT}(G) = |G| - 2$ *if and only if* $\mathrm{Z}(G) = 2$ *and there exists a minimum zero forcing set such that exactly one force is performed at each time step.*

In [21], the *standard propagation time interval* of a simple graph $G$ was defined as

$$[\mathrm{pt}(G), \mathrm{PT}(G)] \;=\; [\mathrm{pt}(G), \mathrm{pt}(G)+1, \ldots, \mathrm{PT}(G)-1, \mathrm{PT}(G)]$$

and the *propagation time discrepancy* was defined as

$$\mathrm{pd}(G) = \mathrm{PT}(G) - \mathrm{pt}(G).$$

It was asked in that paper whether or not the propagation time interval is full for standard zero forcing. In other words, given a graph $G$ and its propagation time interval $[\mathrm{pt}(G), \mathrm{PT}(G)]$, does every integer in the propagation time interval represent the propagation time of an efficient zero forcing set for $G$? It was shown in [21] that this is not the case by considering the generalized star $S(e_1, e_2, e_3)$ having three arms, where $e_1 \le e_2 \le e_3$. Specifically, where $1 < e_1 < e_2 < e_3$ and the vertices are given labels as in the diagram shown in Example 1.2.11. The leaves are denoted $u_1, u_2$ and $u_3$, the vertex of degree 3 is denoted $v$, and the neighbors of $v$ are denoted $w_1, w_2$ and $w_3$, respectively.

**Example 1.2.11.** The graph $S(2, 5, 7)$ has minimum propagation time of 8 and maximum propagation time of 12, yielding the propagation time interval $[8, 12]$. It is shown in the table below that $8, 9, 11$, and 12 are all possible propagation times of efficient zero forcing sets for $S(2, 5, 7)$, but 10 is not a possible propagation time. Hence the standard propagation time interval is not full for the generalized star $S(2, 5, 7)$.

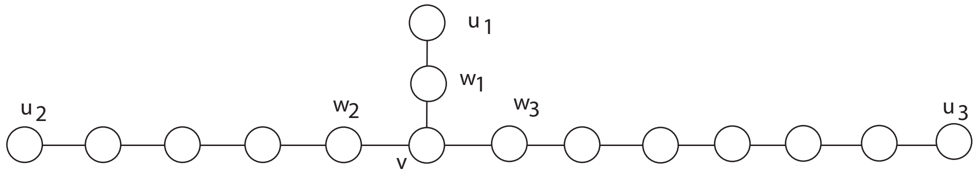



Figure 1.5 A graph $S(2, 5, 7)$ for which the propagation time interval is not full. [21]

Table 1.1 Minimum zero forcing sets and propagation times of the graph $S(2,5,7)$. [21]

| $B$ | $\mathrm{pt}(S(e_1,e_2,e_3),B)$ | $\mathrm{pt}(S(2,5,7),B)$ |
|---|---|---|
| $\{u_2,u_3\}$ | $e_1+e_3-1$ | 8 |
| $\{u_1,w_3\}$ | $e_1+e_3-1$ | 8 |
| $\{u_1,w_2\}$ | $e_1+e_3$ | 9 |
| $\{u_1,u_2\}$ | $e_2+e_3-1$ | 11 |
| $\{u_3,w_2\}$ | $e_2+e_3-1$ | 11 |
| $\{u_1,u_3\}$ | $e_2+e_3-1$ | 11 |
| $\{u_2,w_3\}$ | $e_2+e_3-1$ | 11 |
| $\{u_3,w_1\}$ | $e_2+e_3$ | 12 |
| $\{u_2,w_1\}$ | $e_2+e_3$ | 12 |

### 1.2.3 Positive semidefinite zero forcing

For positive semidefinite zero forcing, the color change rule differs from the color change rule for standard zero forcing. For a set $S$ of blue vertices, let $W_1, W_2, \ldots, W_k$ be the sets of vertices of the connected components of $G-S$. The *positive semidefinite color change rule* states: If $w \in W_i$ is the only white neighbor of vertex $b \in S$ in the induced subgraph $G[W_i \cup S]$, then change the color of $w$ to blue. If when the positive semidefinite color change rule is carried out to completion with an initial set $B$ of blue vertices the entire graph is colored blue, then $B$ is a *positive semidefinite zero forcing set*. The *positive semidefinite zero forcing number* of a graph $G$ is denoted $\mathrm{Z}_+(G)$ and is the minimum possible cardinality $|B|$ over all possible positive semidefinite minimum zero forcing sets $B$.

**Example 1.2.12.** In this example we illustrate a positive semidefinite zero forcing process on the graph $G=(V,E)$ in Figure 1.6. The positive semidefinite zero forcing number of $G$ is 1. Begin with the initial set of blue vertices $B=\{e\}$. The following steps describe the positive semidefinite zero forcing process.

1. The graph $G$ with $B=\{e\}$ colored blue.

2. $G[V \backslash B]$ has 4 connected components.

3. Connecting $B$ back to the first component gives $e \to f$.

4. Connecting $B$ back to a second component gives $e \to d$.

5. Connecting $B$ back to a third component gives $e \to b$.

6. Connecting $B$ back to the last component gives $e \to c$.

7. The set of blue vertices $B_0 \cup B_1$ is $\{b, c, d, e, f\}$.

8. $G[V \backslash (B_0 \cup B_1)]$ has 4 connected components.

9. Connecting $B_0 \cup B_1$ back to the first component gives $d \to h$.

10. Connecting $B_0 \cup B_1$ back to a second component gives $d \to i$.

11. Connecting $B_0 \cup B_1$ back to a third component gives $b \to a$.

12. Connecting $B_0 \cup B_1$ back to the last component gives $f \to g$.

13. The set of blue vertices $B_0 \cup B_1 \cup B_2$ is $\{a, b, c, d, e, f, g, h, i\}$.

14. $G[V \backslash (B_0 \cup B_1 \cup B_2)]$ has 2 connected components.

15. Connecting $B_0 \cup B_1 \cup B_2$ back to the first component gives $g \to j$.

16. Connecting $B_0 \cup B_1 \cup B_2$ back to the second component gives $g \to k$.

17. The set of blue vertices $B_0 \cup B_1 \cup B_2 \cup B_3$ is $\{a, b, c, d, e, f, g, h, i, j, k\}$.

18. $G[V \backslash (B_0 \cup B_1 \cup B_2 \cup B_3)]$ is a single isolated vertex.

19. Connecting $B_0 \cup B_1 \cup B_2 \cup B_3$ back to this vertex gives $k \to m$.

20. Finally, the entire graph is forced blue.

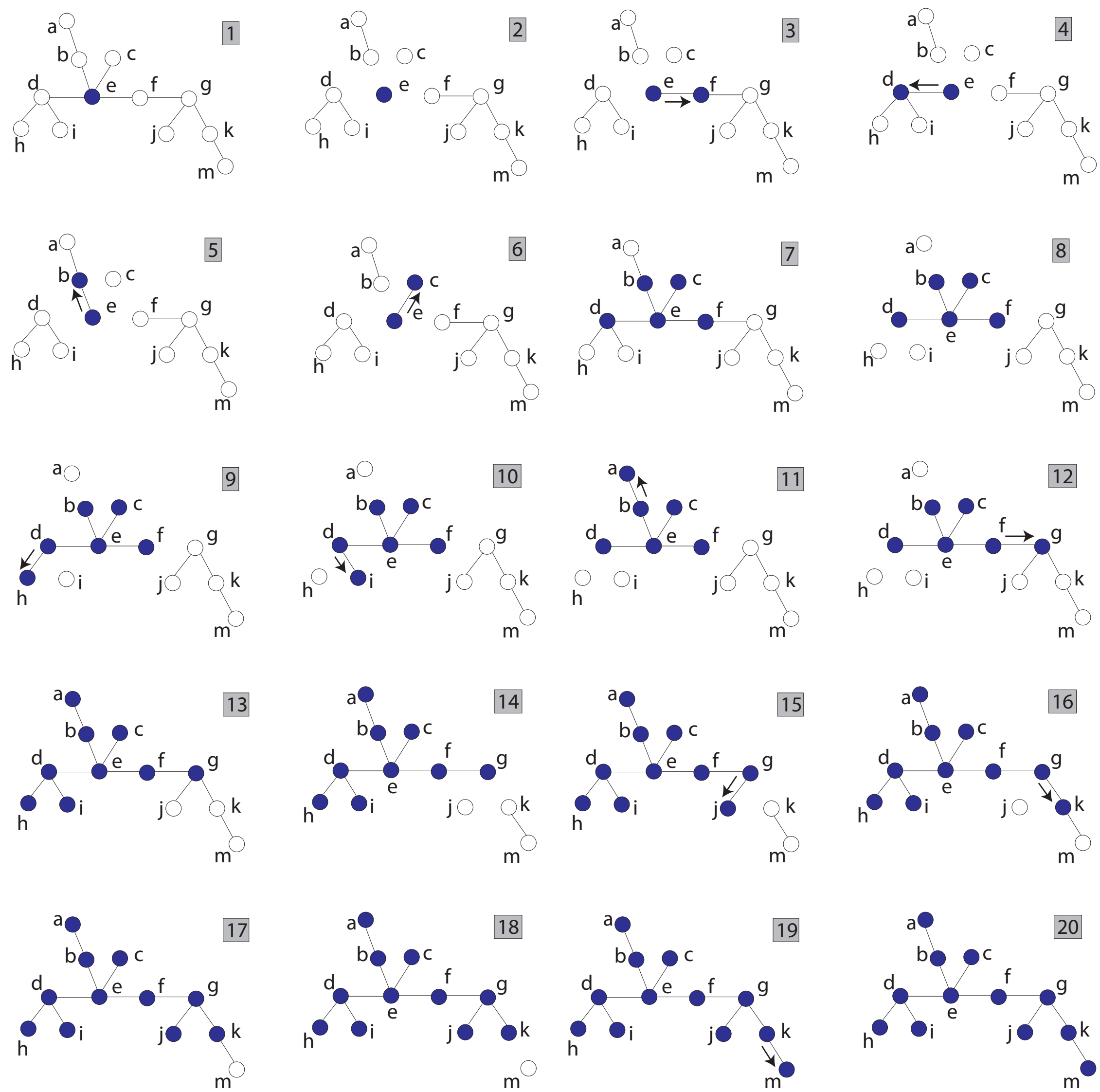



Figure 1.6 An example of a simple graph $G$ with positive semidefinite zero forcing number 1.

The positive semidefinite zero forcing number is an upper bound for positive semidefinite maximum nullity, where the *positive semidefinite minimum rank* of a graph $G$ is

$$\mathrm{mr}_+(G) = \min\{\mathrm{rank}\, A | A \in \mathcal{S}(G) \text{ and } A \text{ is positive semidefinite}\}$$

and the *positive semidefinite maximum nullity* of $G$ is

$$\mathrm{M}_+(G) = \max\{\mathrm{null}\, A | A \in \mathcal{S}(G) \text{ and } A \text{ is positive semidefinite}\}.$$

A relationship between $\text{mr}(G)$ and $\text{M}(G)$ is that $\text{mr}(G) + \text{M}(G) = |G|$, and for $\text{mr}_+(G)$ and $\text{M}_+(G)$ it is also true that $\text{mr}_+(G) + \text{M}_+(G) = |G|$. Positive semidefinite zero forcing and the positive semidefinite minimum rank are discussed in [3], [18], and [24].

### 1.2.4 Positive semidefinite propagation time

Propagation time for positive semidefinite zero forcing is defined similarly to propagation time for standard zero forcing, but following the positive semidefinite zero forcing definition, in [28].

Let $G = (V, E)$ be a graph and $B^+$ a positive semidefinite zero forcing set, and set $t = 0$ so that $B_0^+ = B^+$. We first consider the set $\{W_1, W_2, \ldots, W_j\}$ of vertex sets of the connected components of $G[V \backslash B_0^+]$. Reattaching $B_0^+$ to each $W_i$ separately, we require that every vertex in $B_0^+$ that can perform a force in $G[W_i \cup B_0^+]$ does perform a force in $G[W_i \cup B_0^+]$. When this process is complete, we set $t = 1$ and let $B_1^+$ be the set of vertices forced blue while $t = 0$. Then in general, if $B_t^+$ is the set of blue vertices obtained by connecting $B_{t-1}^+$ to each of the connected components of $G[V \backslash B_{t-1}^+]$ and carrying out the aforementioned forcing process, we can repeat the process until $\cup_{s=0}^{t} B_s^+ = V$. The first such $t$ for which $\cup_{s=0}^{t} B_s^+ = V$ is the positive semidefinite propagation time of $B^+$ in $G$, and is denoted $\text{pt}_+(G, B^+)$.

Other analogous definitions include the *minimum positive semidefinite propagation time* of $G$, which is the smallest possible $\text{pt}_+(G, B)$ taken over all minimum positive semidefinite zero forcing sets $B$ of $G$ and denoted $\text{pt}_+(G)$, and the *maximum positive semidefinite propagation time* of $G$, which is the largest possible $\text{PT}_+(G, B)$ taken over all minimum positive semidefinite zero forcing sets $B$ of $G$ and denoted $\text{PT}_+(G)$. Finally, if $G$ is a graph then a minimum positive semidefinite zero forcing set $B$ is *efficient* if $\text{pt}_+(G, B) = \text{pt}_+(G)$.

**Example 1.2.13.** For the graph $G$ in Figure 1.6, $\text{pt}_+(G, \{e\}) = 4$. This is because in the positive semidefinite zero forcing process, forces in different connected components, although they are independent of each other, occur simultaneously. In other words, a single vertex in the positive semidefinite zero forcing set can perform more than one force at any given time step. At $t = 1$, $e \to f$, $e \to d$, $e \to b$ and $e \to c$. At $t = 2$, $d \to h$, $d \to i$, $b \to a$ and $f \to g$. At $t = 3$, $g \to j$ and $g \to k$. Finally, at $t = 4$, $k \to m$, and the entire graph has been forced.

### 1.2.5 Skew zero forcing

Skew zero forcing on a simple graph $G$ is based on this distinct *color change rule* [23]: if each vertex of $G$ is colored either white or blue, and $v$ is a vertex with only one white neighbor $w$, then change the color of $w$ to blue. Skew zero forcing differs from standard zero forcing in that for skew zero forcing, a vertex does not need to be blue in order to perform a force. A *skew zero forcing set* for a graph $G$ is an initial set $B^-$ of blue vertices such that applying the skew color change rule until no more changes are possible results in all vertices of $G$ being colored blue. A *minimum skew zero forcing set* for $G$ is a zero forcing set of $G$ of minimum cardinality, and the *skew zero forcing number*, denoted $\mathrm{Z}^-(G)$, is the cardinality of a minimum zero forcing set.

Recall that for standard zero forcing on a simple graph $G$, $\mathrm{Z}(G)$ is an upper bound for the maximum nullity of the associated matrices in $\mathcal{S}(G)$. The skew zero forcing number $\mathrm{Z}^-(G)$ is an upper bound on the maximum nullity of another set of matrices associated with $G$.

In linear algebra, we say that a square matrix is *skew-symmetric* if $A^T = -A$ (as opposed to symmetric, $A^T = A$). The set of *skew-symmetric matrices described by a graph* $G$ is

$$\mathcal{S}^-(G) = \{A \mid A^T = -A, \mathcal{G}(A) = G\}.$$

The *minimum skew rank* of a graph $G$ is defined as

$$\mathrm{mr}^-(G) = \min\{\mathrm{rank}\, A \mid A \in \mathcal{S}^-(G)\},$$

and the *maximum skew nullity* of $G$ is defined as

$$\mathrm{M}^{-}(G) = \max\{\operatorname{null} A \mid A \in \mathcal{S}^{-}(G)\}.$$

The skew zero forcing number is an upper bound for skew maximum nullity, and it is true that $\mathrm{mr}^{-}(G) + \mathrm{M}^{-}(G) = |G|$ for skew maximum nullity and minimum rank. Many more results regarding skew minimum rank and maximum nullity can be found in [23].

As we will observe in Section 3 of this thesis, the color change rule for loop graphs is the same as the color change rule for skew zero forcing, but now a vertex $v$ is a neighbor of itself if and only if it has a loop. Thus, the loop graph zero forcing number of a simple graph $G$, regarded as a loop graph that itself does not contain any loops, is the same as the skew zero forcing number of $G$.

The zero forcing numbers (including standard, skew, positive semidefinite, and loop) are graph parameters that are independent of the matrices used to describe whatever applicable graphs correspond to the matrices in question. The connection between zero forcing number and matrices is used to obtain the upper bound on maximum nullity that has been established and stated for standard, positive semidefinite, skew, and loop graph zero forcing. This is because of a process in which zeros are literally forced in the null vectors of a matrix that describes a graph. This process utilizes only the zero-nonzero pattern of the matrix and not the numerical values of the entries of the matrix. Therefore it is not surprising that the skew zero forcing number is the same as the loop graph zero forcing number of a loop graph without loops. That is, $\mathrm{Z}^{-}(G)$ is an upper bound on both the maximum nullity of skew-symmetric matrices associated with $G$ and the maximum nullity of symmetric matrices (with zero diagonal entries) associated with $G$ viewed as a loop graph containing no loops.

**Example 1.2.14.** For the graph $G$ in Figure 1.7, the set $\{a\}$ is a skew zero forcing set. Since $d$ and $e$ are the only white neighbors of $f$ and $g$, respectively, $f \to d$ and $g \to e$. Then $b$ and $c$ are each other's only white neighbors, so $b \to c$ and $c \to b$. Finally $d \to f$

and $e \to g$, and the whole graph is forced. Note that although the vertex $a$ is the only vertex in the skew zero forcing set, $a$ does not actually perform a force in this skew zero forcing process.

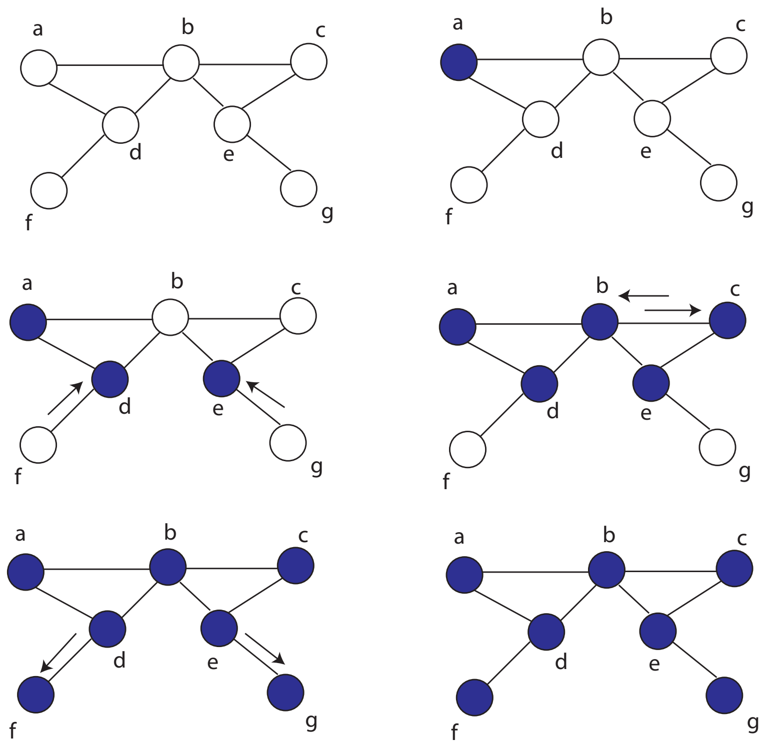



Figure 1.7 An example of a simple graph with skew zero forcing number 1 and the skew zero forcing process illustrated.

## 1.3 Organization of this thesis

In Chapter 2 we introduce and study skew propagation time. Section 2.1 contains definitions and observations; in particular the idea of white vertex forcing and skew zero forcing sets of size 0. Section 2.2 presents tools for analyzing skew propagation time, including uniqueness of zero forcing sets, reversals, matchings, and extreme skew zero forcing number. In Section 2.3 we give a counterexample to the fullness of the skew propagation time interval. In Section 2.4 both high and low skew propagation times are analyzed, it is shown that $\mathrm{pt}^-(G) = |G|$ is impossible, and graphs having high skew propagation time $\mathrm{pt}^-(G) = |G| - 1$ and $\mathrm{Z}^-(G) = 0$ are characterized. In Section 2.5 the skew zero forcing numbers and skew propagation time of select graph families are given, including those of paths, cycles, wheels, certain products of graphs, and trees.

Chapter 3 briefly introduces the concept of the propagation time of graphs that allow loops. Some definitions and observations about loop graph propagation time are given, including some similarities and differences to the properties of skew propagation time.

# CHAPTER 2. SKEW PROPAGATION TIME

## 2.1 Definitions and Observations

Skew propagation time is defined as follows based on the skew zero forcing process.

**Definition 2.1.1.** Let $G = (V, E)$ be a simple graph, and $B^-$ a skew zero forcing set of $G$. Define $B^-_{(0)} = B^-$, and for $t \geq 0$, let $B^-_{(t+1)}$ be the set of vertices $\{w\}$ for which there exists any vertex $v$ in the graph $G$, blue or white, such that $w$ is the only white neighbor of $v$ not in $\cup_{s=0}^{t} B^-_{(s)}$. The *skew propagation time* of $B^-$ in $G$, denoted $\mathrm{pt}^-(G, B^-)$, is the smallest integer $t_0$ such that $V = \cup_{t=0}^{t_0} B^-_{(t)}$.

**Definition 2.1.2.** The *minimum skew propagation time of* $G$ is

$$\mathrm{pt}^-(G) = \min\{\mathrm{pt}^-(G, B^-) | B^- \text{ is a minimum skew zero forcing set of } G\}.$$

**Definition 2.1.3.** The *maximum skew propagation time of* $G$ is

$$\mathrm{PT}^-(G) = \max\{\mathrm{PT}^-(G, B^-) | B^- \text{ is a minimum skew zero forcing set of } G\}.$$

Note that since we are mainly interested in the minimum skew propagation time, from now on minimum skew propagation time may simply be referred to as skew propagation time of a graph $G$.

**Definition 2.1.4.** A subset $B^-$ of vertices of $G$ is an *efficient skew zero forcing set* for $G$ if $B^-$ is a minimum skew zero forcing set for $G$ and $\mathrm{pt}^-(G, B^-) = \mathrm{pt}^-(G)$.

**Definition 2.1.5.** A *white vertex force* occurs when a vertex that has not yet been colored blue forces some other vertex in the graph blue.

**Example 2.1.6.** Notice that the Petersen graph $P$ has $\mathrm{Z}(P) = 5$ and $\mathrm{pt}(P) = 1$. For skew zero forcing, $P$ has $\mathrm{Z}^-(P) = 4$, and $\mathrm{pt}^-(P) = 1$ because of white vertex forcing.

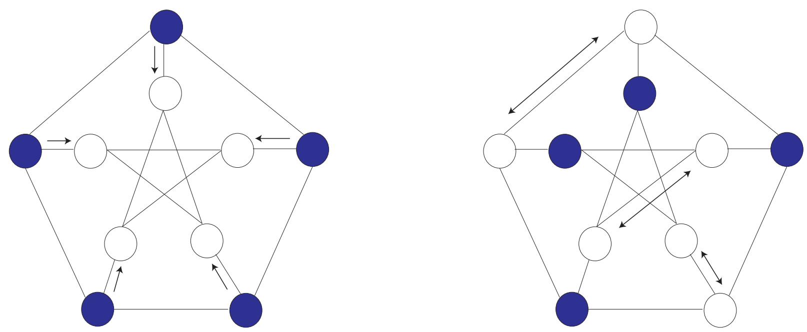


Figure 2.1 An efficient standard and efficient skew zero forcing set (where white vertex forcing occurs) for the Petersen graph.

We note the following two observations that distinguish standard zero forcing from skew zero forcing.

**Observation 2.1.7.** *Simple graphs can have minimum skew zero forcing sets of size 0. Furthermore, white vertex forcing can occur in the skew propagation process on simple graphs.*

Observation 2.1.7 implies that $B^- = \emptyset$ is a possible minimum skew zero forcing set for a graph, and forcing chains in the skew zero forcing process may be 2-cycles instead of paths.

**Remark 2.1.8.** The possibility that $\mathrm{Z}^-(G) = 0$ for a graph $G$ implies that the bound

$$\frac{|G| - \mathrm{Z}(G)}{\mathrm{Z}(G)} \leq \mathrm{pt}(G)$$

as stated cannot be extended to skew propagation time.

**Example 2.1.9.** Consider the path $P_2$ (Figure 2.2) on two vertices, $v$ and $w$. The graph has order 2, skew zero forcing number 0, and skew propagation time 1, since $v \to w$ and

$w \to v$ simultaneously and there are no other vertices to force. $\mathrm{Z}^-(P_2) = 0$ implies that $\emptyset$ is an efficient skew zero forcing set for $P_2$.

Figure 2.2 The path $P_2$ has $\mathrm{Z}^-(P_2) = 0$ and $\mathrm{pt}^-(P_2) = 1$.

**Observation 2.1.10.** *As with standard zero forcing, any minimum skew zero forcing set that is isomorphic to an efficient zero forcing set is also efficient, but 2 different skew zero forcing sets can have different skew propagation times.*

**Example 2.1.11.** Consider $P_7$, the path on 7 vertices, $\{a, b, c, d, e, f, g\}$, where $a$ and $g$ are the endpoints of the path and $d$ is the middle-most vertex in the path. Then $B = \{c\}$ and $B' = \{e\}$ are two isomorphic skew zero forcing sets that have skew propagation time 2. $B'' = \{a\}$ is a zero forcing set for the graph that is minimum, but not efficient because $\mathrm{pt}^-(P_7, B'') = 3$.

**Proposition 2.1.12.** *The propagation time of a simple graph $G$ for standard zero forcing is greater than or equal to the skew propagation time for the graph $G$, as long as $\mathrm{Z}(G) = \mathrm{Z}^-(G)$.*

*Proof.* Let $G$ be a simple graph such that $\mathrm{Z}(G) = \mathrm{Z}^-(G)$. Consider the standard propagation process of some efficient zero forcing set $B$ of $G$ with standard zero forcing. If a force happens at time step $t$ in the standard zero forcing process, then the corresponding force happens at time $t$ or earlier in the skew zero forcing process for the graph $G$ with the same (now skew) zero forcing set $B$. This is because the same list of forces can be followed if the vertex in question has not already been forced blue by being the unique white neighbor of another vertex at a previous time step (i.e. there was a white vertex force). □

**Example 2.1.13.** There exists a simple graph $G$ such that $\mathrm{Z}^{-}(G) < \mathrm{Z}(G)$ and the propagation time for skew zero forcing on $G$ is strictly less than the propagation time for standard zero forcing on $G$. The path on 4 vertices, $P_4$, has $\mathrm{Z}(P_4) = 1$ and $\mathrm{pt}(P_4) = 3$, but $\mathrm{Z}^{-}(P_4) = 0$ and $\mathrm{pt}^{-}(P_4) = 2$. Furthermore, there exists a simple graph $G$ such that $\mathrm{Z}^{-}(G) < \mathrm{Z}(G)$ and the propagation time for skew zero forcing on $G$ is strictly greater than the propagation time for standard zero forcing on $G$. Consider the paw $H$ on 4 vertices, which consists of a cycle on 3 vertices and a leaf on exactly one vertex in the cycle. The paw has $\mathrm{Z}(H) = 2$ and $\mathrm{pt}(H) = 2$, but $\mathrm{Z}^{-}(H) = 0$ and $\mathrm{pt}^{-}(H) = 3$.

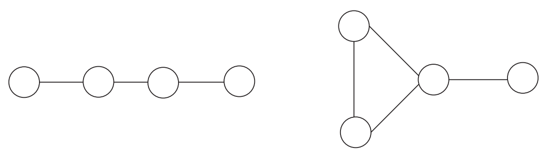


Figure 2.3 The path $P_4$ and the paw graph described in Example 2.1.13.

As is the case for standard propagation time in [21] the skew propagation time of a graph and the diameter of the same graph are not comparable.

**Example 2.1.14.** Consider the Full House graph on 5 vertices, shown in Figure 2.4, and call this graph $H$. This graph has $\mathrm{Z}^{-}(H) = 1$, $\mathrm{pt}^{-}(H) = 3$, and $\mathrm{diam}(H) = 2$. The cycle $C_4$ has $\mathrm{diam}(C_4) = 2$, and has $\mathrm{Z}^{-}(C_4) = 2$, $\mathrm{pt}^{-}(C_4) = 1$.

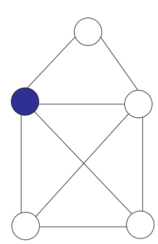


Figure 2.4 The full house graph $H$ described in Example 2.1.14.

**Example 2.1.15.** Let $G$ be a graph such that $\mathrm{Z}^-(G) < \mathrm{Z}(G)$, and let $B$ be an efficient zero forcing set for $G$. We can ask the following questions, all of which can be answered in the negative with a single counterexample.

1. Is every $\mathrm{Z}^-(G)$ element subset of $B$ an efficient skew zero forcing set for $G$?

2. Is every efficient skew zero forcing set of $G$ contained in an efficient standard zero forcing set for $G$?

3. Does every efficient standard zero forcing set of $G$ have a subset of $\mathrm{Z}^-(G)$ vertices that is an efficient skew zero forcing set for $G$?

4. Does there exist an efficient skew zero forcing set that can be augmented with $\mathrm{Z}(G) - \mathrm{Z}^-(G)$ vertices to become an efficient standard zero forcing set $B'$ for $G$?

The generalized star $S$ on 7 vertices illustrated in Figure 2.5 provides a counterexample. Up to isomorphism, the two efficient standard zero forcing sets are $\{a, e\}$ and $\{b, g\}$. The only efficient skew zero forcing set is the middle-most vertex $\{c\}$. Therefore not every, and in fact no, $\mathrm{Z}^-(S)$ element subset of an efficient standard zero forcing set is a skew zero forcing set for $S$, since none of $\{a\}$,$\{b\}$,$\{e\}$ or $\{g\}$ is an efficient skew zero forcing sets for $S$. Not every, and in fact no, efficient skew zero forcing set for $S$ is contained in an efficient standard zero forcing set for $S$ since $\{c\} \not\subseteq \{a, e\}$ and $\{c\} \not\subseteq \{b, g\}$.

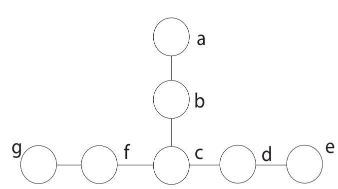



Figure 2.5 The generalized star as described in Example 2.1.15.

**Observation 2.1.16.** *If $G$ is disconnected simple graph with connected components $W_1, W_2, \dots W_k$, then* $\mathrm{pt}(G) = \max\{\mathrm{pt}(W_i)\}$, $\mathrm{PT}(G) = \max\{\mathrm{PT}(W_i)\}$, $\mathrm{pt}^-(G) = \max\{\mathrm{pt}^-(W_i)\}$, *and* $\mathrm{PT}^-(G) = \max\{\mathrm{PT}^-(W_i)\}$.

## 2.2 Tools for analyzing skew propagation time

### 2.2.1 Uniqueness of skew zero forcing sets

Given a graph $G$ and a set of blue vertices $B$, the *derived set* of $B$ in $G$ is determined by applying the color change rule to the set $B$ and continuing on each new set of blue vertices containing $B$ until no more color changes are possible. When $B$ is a zero forcing set, we define the *chronological list of forces* to be an ordered list of the forces in the order they occur using $B$ to force $G$. Note that when $B$ is a zero forcing set, the derived set of $B$ in $G$ is the entire set of vertices of $G$. The chronological list of forces is usually not unique. We prove the uniqueness of a derived set of a skew zero forcing set for a simple graph $G$.

**Theorem 2.2.1.** *If $D_1$ and $D_2$ are derived sets of a graph, obtained by repeatedly applying the skew color change rule until no more forces can occur on some initial set of blue vertices $B$, then $D_1 = D_2$.*

*Proof.* Let $B$ be the initial set of blue vertices in the graph $G$, and let $D_1$ and $D_2$ be the derived sets obtained by carrying out the skew zero forcing process to completion on $G$ (i.e., applying the skew color change rule repeatedly until no more forces are possible, but by making different choices as to the order in which the vertices are forced). Let $\mathcal{F}_1$ be a chronological list of forces corresponding to $D_1$ and $\mathcal{F}_2$ be a chronological list of forces corresponding to $D_2$.

Let $x_k$ denote the vertex forced blue at step $k$ in the chronological list of forces $\mathcal{F}_1$, and let $X_k = \{x_1, x_2, \dots, x_k\}$ and $X_0 = \emptyset$. We show that if $X_k \subseteq D_2$ and $k < n$, then $X_{k+1} \subseteq D_2$. Assume $X_k \subseteq D_2$ (and clearly $B \subset D_2$). Then there exists a vertex $v$ such that $v \to x_{k+1}$ at time step $k+1$ in $\mathcal{F}_1$. Therefore every neighbor of $v$ except $x_{k+1}$ is contained in $B \cup X_k \subseteq D_2$. Therefore $x_{k+1} \in D_2$, since no more forces are possible after the chronological list of forces $\mathcal{F}_2$ is complete. Therefore $X_{k+1} \subseteq D_2$ and hence $D_1 \subseteq D_2$. A similar argument can be made to show that $D_2 \subseteq D_1$, and thus that the two derived sets are equal, as claimed. □

Since in propagating skew zero forcing, several forces may occur simultaneously, it makes more sense to talk about these forces as a set of forces rather than a chronological list of forces, which was useful in talking about the standard and skew zero forcing numbers themselves. The definition of derived set remains the same for skew zero forcing.

**Definition 2.2.2.** Let $G = (V, E)$ be a graph and $B$ be a skew zero forcing set for $G$. We refer to the unordered list of forces in a chronological list of forces of $B$ simply as a *set of forces* for $B$ in $G$.

In [3] it was shown that for a connected graph of order at least 2, there must be more than one minimum zero forcing set, but this is clearly not the case for skew zero forcing. The path on 2 vertices, and in fact any even length path whatsoever, has the empty set as a minimum zero forcing set, and hence a unique efficient zero forcing set.

**Definition 2.2.3.** [21] If a subset $B$ of vertices of $G$ is an efficient zero forcing set for $G$ and thus $\operatorname{pt}(G, B) = \operatorname{pt}(G)$, we define the following set of all efficient zero forcing sets of $G$ by

$$\operatorname{Eff}(G) = \{B \mid B \text{ is an efficient zero forcing set of } G\}.$$

This definition can be extended directly to skew zero forcing.

**Definition 2.2.4.** If a subset $B$ of vertices of $G$ is an efficient skew zero forcing set for the graph $G$ and thus $\operatorname{pt}^-(G, B) = \operatorname{pt}^-(G)$, we define the following set of all efficient skew zero forcing sets of $G$ by

$$\operatorname{Eff}^-(G) = \{B \mid B \text{ is an efficient skew zero forcing set of } G\}.$$

**Remark 2.2.5.** In [[21], Example 2.15] it was shown that $\bigcap_{B\in \text{Eff}(G)} B = \emptyset$ is not necessarily true. For example, $\bigcap_{B\in \text{Eff}(W_5)} B = \{a\}$ where $W_5$ is the wheel on 5 vertices and $a$ is the center vertex of the wheel. The same is true for skew propagation time. Consider the generalized star $S$ in Example 2.1.14. The only minimum skew zero forcing set for this graph is $\{c\}$. Therefore $\bigcap_{B\in \text{Eff}(S)} B = \{c\} \neq \emptyset$.

### 2.2.2 Reversals

We define several more terms in the context of skew zero forcing:

**Definition 2.2.6.** Let $G$ be a simple graph, $B^-$ be a skew zero forcing set for $G$, and $\mathcal{F}$ a set of forces for $B^-$ in $G$. We use the notation $\text{Term}^-(\mathcal{F})$ to denote the *skew terminus* of $\mathcal{F}$, which is the set of vertices in $B^-$ that do not perform any forces in the forcing process $\mathcal{F}$. We say that the *reverse set of skew forces* of $\mathcal{F}$, with the notation $\text{Rev}^-(\mathcal{F})$, is the result taking every force in $\mathcal{F}$ and reversing it. A *skew forcing chain* of a set of forces $\mathcal{F}$ is a sequence of vertices $v_1, \ldots, v_k$ such that $v_j \to v_{j+1}$ in $\mathcal{F}$. A *maximal skew forcing chain* is a skew forcing chain such that there is no other skew forcing chain of which it is a proper subset.

**Remark 2.2.7.** Recall that each vertex in a simple graph can skew force at most one other vertex, and can be forced by at most one other vertex, so the maximal skew zero forcing chains of a graph are disjoint for skew zero forcing and every maximum skew zero forcing chain is a path or a 2-cycle. Furthermore, the elements of the skew zero forcing set $B^-$ are the initial vertices of the maximal skew forcing paths. The cardinality of the skew terminus of a skew zero forcing set $B^-$ is $|B^-|$, since when reversing a 2-cycle in a set of forces, the result of reversing the 2-cycle is itself and the result of reversing a path in a set of forces is the path in reverse order.

**Observation 2.2.8.** *A skew forcing chain may have only one vertex if it is a member of the skew zero forcing set of the graph and performs no force in $\mathcal{F}$. In this case the* *vertex is in both the original skew zero forcing set and* $\mathrm{Term}^-(\mathcal{F})$*.*

The 'skew terminus' is given its name because of the fact that any vertex does not perform a force in a skew zero forcing process $\mathcal{F}$ if and only if it is the last vertex in some maximal skew forcing chain.

In [3] it is shown that, for standard zero forcing, if $B$ is a zero forcing set of the simple graph $G$ and $\mathcal{F}$ is a chronological list of forces for $B$ in $G$, then the terminus of $\mathcal{F}$ is also a zero forcing set for $G$ with the set of forces in reverse order, and it is observed in [21] that this extends to sets of forces. Here we establish the analogous result for skew zero forcing.

**Theorem 2.2.9.** *If* $B^-$ *is a skew zero forcing set for a graph* $G$*, then so is any skew terminus of* $B^-$*.*

*Proof.* We proceed by induction on the order of $G$. For a graph $G$ on 1 or 2 vertices there is nothing to show. So suppose that for all graphs $H$ of order $1, \ldots, k$, any skew terminus of a skew zero forcing set for $H$ is also a skew zero forcing set for $H$.

Now consider a graph $G$ of order $k+1$, $B^-$ a skew zero forcing set for $G$, and $\mathcal{F}$ a chronological list of forces for $B^-$. Let $B' = \mathrm{Term}^-(\mathcal{F})$ and $\mathcal{F}' = \mathrm{Rev}^-(\mathcal{F})$. Now suppose that $u \to v$ is the first force in $\mathcal{F}'$. Then all neighbors of $u$ except $v$ must be in $B'$, since when the last force $v \to u$ was done in $\mathcal{F}$, each of them had $u$ as a white neighbor and thus did not force any other vertices previously in the original set of vertices in its original order. Thus $u \to v$ is a valid force for $\mathrm{Term}^-(\mathcal{F})$. Then what remains is a graph $\hat{H} = G - u$ on $k$ vertices, with $\hat{B} = B^-$ as a skew zero forcing set for $\hat{H}$, and $\hat{\mathcal{F}} = \mathcal{F} \backslash \{v \to u\}$ a chronological list of forces for $\hat{B}$. Let $\hat{B}' = \mathrm{Term}^-(\hat{\mathcal{F}}) = (B' \backslash \{u\}) \cup \{v\}$ and $\hat{\mathcal{F}}' = \mathrm{Rev}^-(\hat{F})$. Then since $\mathcal{F}'$ consists of $u \to v$ as its first force, the chronological list of forces $\mathcal{F}'$ for $B'$ is $\hat{\mathcal{F}}' \cup \{u \to v\}$, which is a valid chronological list of forces for $B'$ since $\hat{\mathcal{F}}'$ is a valid list of forces for $\hat{B}'$. Then $B'$ is a valid skew zero forcing set for $G$ by the induction hypothesis, which concludes the proof. $\square$

**Corollary 2.2.10.** *Let $G$ be a graph, $B^-$ a minimum skew zero forcing set of $G$, and $\mathcal{F}$ a set of forces of $B$. Then* $\mathrm{Rev}^-(\mathcal{F})$ *is a set of forces of* $\mathrm{Term}^-(\mathcal{F})$ *and* $B^- = \mathrm{Term}^-(\mathrm{Rev}^-(\mathcal{F}))$.

**Observation 2.2.11.** *Let $G$ be a graph, $B^-$ a minimum skew zero forcing set for $G$, and $\mathcal{F}$ a set of forces of $B^-$. Then the* $\mathrm{Rev}^-(\mathcal{F}) = \mathcal{F}$ *if and only if all of the forcing chains for $\mathcal{F}$ of $B^-$ are 2-cycles or paths of order 1 (isolated vertices), if and only if* $\mathrm{Term}^-(\mathcal{F}) = B^-$*; equivalently, no blue vertex performs a force.*

**Definition 2.2.12.** Let $G$ be a graph and suppose that $B^-$ is a skew zero forcing set for the graph $G$. Let $\mathcal{F}$ be a set of forces for $B^-$. Start with $\mathcal{F}_{(0)} = B^-$ and for $t \geq 0$, let $\mathcal{F}_{(t+1)}$ be the set of vertices $w$ such that the force $v \to w$ is a force in the set of forces $\mathcal{F}$, $w$ is not in $\cup_{s=0}^{t}\mathcal{F}_{(s)}$, and $w$ is the only neighbor of $v$ not in $\cup_{s=0}^{t}\mathcal{F}_{(s)}$. We define the *skew propagation time* of $\mathcal{F}$ in $G$, which we denote by $\mathrm{pt}^-(G,\mathcal{F})$, to be the smallest integer $t_0$ such that $V = \cup_{t=0}^{t_0}\mathcal{F}_{(t)}$.

Let $G$ be a graph and $B^-$ be a skew zero forcing set of $G$. Furthermore, let $\mathcal{F}$ be a set of forces of $B^-$. Then $\cup_{s=0}^{t}\mathcal{F}_{(s)} \subseteq \cup_{s=0}^{t}B^-_{(i)}$ for all $t = 0, \ldots, \mathrm{pt}^-(G, B^-)$.

**Definition 2.2.13.** Let $G$ be a graph, $B^-$ a skew zero forcing set of $G$, and $\mathcal{F}$ a set of forces of $B^-$. Define $D_0(\mathcal{F}) = \mathrm{Term}^-(\mathcal{F})$ and for $t = 1, \ldots, \mathrm{pt}^-(G,\mathcal{F})$, define $D_t$ to the the set of vertices $v$ such that there exists a vertex $w$ in $\mathcal{F}_{(\mathrm{pt}^-(G,\mathcal{F})-t+1)}$ such that $v \to w$.

**Observation 2.2.14.** *Let $V$ be the vertex set of the graph $G$. Then* $V = \cup_{s=0}^{\mathrm{pt}^-(G,\mathcal{F})}D_s(\mathcal{F})$.

**Lemma 2.2.15.** *Let $G$ be a graph and let $B^-$ be a skew zero forcing set for $G$. Suppose $\mathcal{F}$ is a set of forces for $B^-$. Then* $D_t(\mathcal{F}) \subseteq \cup_{s=0}^{t}\mathrm{Rev}^-(\mathcal{F})_{(s)}$.

*Proof.* We know from previous results that $\mathrm{Rev}^-(\mathcal{F})$ is a set of forces for $\mathrm{Term}^-(\mathcal{F})$. We proceed by induction on $t$. To start the process $D_0(\mathcal{F}) = \mathrm{Term}^-(\mathcal{F}) = \mathrm{Rev}^-(\mathcal{F})_{(0)}$. Assume that for $0 \leq r \leq t$, $D_r(\mathcal{F}) \subseteq \cup_{s=0}^{r}\mathrm{Rev}^-(\mathcal{F})_{(s)}$. This is the induction hypothesis. Now let $v \in D_{t+1}(\mathcal{F})$. Then in $\mathcal{F}$, for some $u$, $v \to u$ at time $\mathrm{pt}^-(G,\mathcal{F}) - t$. If $w \in$

$N(u) - \{v\}$, then in $\mathcal{F}$, it must be the case that $w$ cannot perform a force before time step $\mathrm{pt}^-(G, \mathcal{F}) - t + 1$. Therefore $w$ is not in the union $\cup_{s=0}^{t} D_s(\mathcal{F}) \subseteq \cup_{s=0}^{t} \mathrm{Rev}^-(\mathcal{F})_{(s)}$. So if $v \notin \cup_{s=0}^{t} \mathrm{Rev}^-(\mathcal{F})_{(s)}$, then $v \in \mathrm{Rev}^-(\mathcal{F})_{(t+1)}$. Therefore we can conclude that $D_{t+1}(\mathcal{F}) \subseteq \cup_{s=0}^{t+1} \mathrm{Rev}^-(\mathcal{F})_{(s)}$. □

**Corollary 2.2.16.** *If $G$ is a graph, $B^-$ is a minimum skew zero forcing set of $G$, and $\mathcal{F}$ is a propagating set of forces for $B^-$ in $G$, then* $\mathrm{pt}^-(G, \mathrm{Term}^-(\mathcal{F})) \leq \mathrm{pt}^-(G, B^-)$*.*

### 2.2.3 Graphs and perfect matchings

Let $G = (V, E)$ be a graph. The degree of a vertex $v \in V$ is the number of vertices to which $v$ is adjacent, and is denoted $\deg_G(v)$. When the graph $G$ is clear from the context, we simply write $\deg(v)$. The *minimum degree* of $G$, is the minimum degree over all possible vertices $v \in V$, and the *maximum degree* is the maximum degree over all such vertices. The notation for minimum and maximum degree of $G$ are $\delta(G)$ and $\Delta(G)$, respectively.

For a graph $G = (V, E)$, a *matching* is a set of edges,

$$M = \{\{i_1, j_1\}, \{i_2, j_2\}, \ldots, \{i_k, j_k\}\} \subseteq E,$$

where no vertex in $V$ can be an endpoint of more than one edge in $M$. If a graph has a matching for which every vertex is an endpoint of some edge in $M$, we say that the graph has a *perfect matching*. Even if there is no perfect matching in a graph $G$, we may still be interested in the maximum number of edges over all possible matching in the graph $G$. This number is called the *matching number* of the graph, and is denoted $\mathrm{match}(G)$. A matching that realizes whose order is $\mathrm{match}(G)$ is called a *maximum matching*.

**Observation 2.2.17.** *Any graph that has a perfect matching must be on an even number of vertices.*

We know that for any graph, $0 \leq \mathrm{Z}^-(G) \leq |G|$ and $\mathrm{Z}^-(G) = |G|$ if and only if the graph is made up of isolated vertices [23].

**Proposition 2.2.18.** *Let $n \geq 4$ be even. Then there exists a graph $G$ on $n$ vertices such that $\mathrm{Z}^-(G) = 0$, $G$ contains a unique perfect matching, and $\mathrm{pt}^-(G) = j$ for $j = 2, \ldots, n-1$.*

*Proof.* Let $G$ be a cycle on $\frac{n}{2}$ vertices such that each vertex has exactly one leaf. Then $G$ has $\mathrm{Z}^-(G) = 0$, has a unique perfect matching of each leaf with its neighbor, and $\mathrm{pt}^-(G) = 2$. For $\mathrm{pt}^-(G) = 3$ let $G$ be a cycle on $\frac{n+2}{2}$ vertices such that $\frac{n-2}{2}$ of the vertices have leaves, and the vertices that do not have leaves are neighbors. Then each leaf forces its neighbor at time step 1, at time step 2 the two vertices without leaves force each other and each cycle vertex of degree 3 that does not have a degree 2 neighbor forces its leaf neighbor, and at time step 3 the remaining white leaves are forced by their neighbors. For each of these two cases, $G$ has $n$ vertices and a unique perfect matching. We can continue in this way by forming a cycle on $\frac{n+2i}{2}$ vertices (so there are $\frac{n-2i}{2}$ leaves), for $i = 2, 3, \ldots, \frac{n-4}{2}$ to get skew propagation times $4, 5, \ldots, \frac{n}{2}$.

We use a different technique to obtain skew propagation times $n-1, \ldots, \frac{n}{2}+2, \frac{n}{2}+1$, Taking a path on $n-2$ vertices and forming a 3-cycle on the end of the path using the last vertex in the path and two additional vertices has a unique perfect matching, has skew zero forcing number 0, and has minimum skew propagation time $n-1$. Taking a path on $n-(3+2i)$ for $i = 1, \ldots, \frac{n-2}{2}$ and forming a cycle of order $3+2i$ using the last vertex in the path and $2+2i$ additional vertices has a unique perfect matching and $\mathrm{pt}^-(G) = n-2, n-3, \ldots, \frac{n+2}{2}$ for $i = 1, \ldots, \frac{n-2}{2}$ respectively. □

**Corollary 2.2.19.** *For every even $n$, there exist graphs with skew propagation time $1, \ldots, n-1$.*

*Proof.* The complete graph on $n$ vertices has skew propagation time 1, and by Theorem 2.2.18, there exists a graph $G$ on $n$ vertices such that $\mathrm{Z}^-(G) = 0$, $G$ contains a unique perfect matching, and $\mathrm{pt}^-(G) = j$ for $j = 2, \ldots, n-1$. □

**Observation 2.2.20.** *In order for a graph $G$ to have $Z^-(G) = 0$, the graph must be on an even number of vertices, but a perfect matching in the graph does not guarantee $Z^-(G) = 0$ (consider the cycle on 4 vertices). In order to have a forcing chain on anything other than 2 vertices, the forcing chain must begin with some blue vertex $b$ in the skew zero forcing set $B$ for $G$.*

**Remark 2.2.21.** Any tree with a perfect matching has skew zero forcing number 0, by the algorithm for finding the matching number of a tree found in [23].

**Theorem 2.2.22.** [23] *If a tree $T$ does not have a perfect matching then the skew zero forcing number of $T$ is $Z^-(T) = |T| - 2 \cdot \text{match}(G)$.*

### 2.2.4 Extreme skew zero forcing number

Skew zero forcing number is crucial to the study of skew propagation time. We state some results regarding graphs with extreme skew zero forcing number, which will be used in the study of skew propagation time.

First we consider graphs with high skew zero forcing number $Z^-(G) = |G|$ and $Z^-(G) = |G| - 2$. It is not possible to have zero forcing number $Z^-(G) = |G| - 1$ for any graph.

**Observation 2.2.23.** *The only graph $G$ with $Z^-(G) = |G|$ is a set of isolated vertices.*

**Theorem 2.2.24.** [23] *Let $G$ be a connected graph with $|G| \geq 2$. The following are equivalent:*

1. $\text{mr}^-(G) = 2$,
2. $G = K_{n_1,n_2,\ldots,n_t}$ *for some* $t \geq 2$, $n_i \geq 1, i = 1\ldots, t$,
3. *$G$ does not contain $P_4$ or the paw as an induced subgraph.*

The next corollary was proved independently in the preprint [11].

**Corollary 2.2.25.** *Let $G$ be a graph of order greater than one. Then* $\mathrm{Z}^-(G) = |G| - 2$ *if and only if $G$ is a disjoint union of a complete multipartite graph of the form* $K_{n_1,n_2,\ldots,n_s}$ *with a (possibly empty) set of isolated vertices. If* $G \neq K_{n_1,n_2,\ldots,n_s} \dot{\cup} rK_1$, *then* $\mathrm{Z}^-(G) \leq |G| - 4$. *Observe that* $\mathrm{Z}^-(K_{n_1,n_2,\ldots,n_s} \dot{\cup} rK_1) = \mathrm{Z}^-(K_{n_1,n_2,\ldots,n_s}) + r$.

*Proof.* First let $G = K_{n_1,n_2,\ldots,n_s}$ with $s \geq 2$ and $n_i \geq 1$ for each $i$. By Theorem 2.2.24, $\mathrm{Z}^-(G) \geq \mathrm{M}^-(G) = |G| - 2$. If $u$ and $w$ are in different partite sets then $V - \{u, w\}$ is skew zero forcing set, so $\mathrm{Z}^-(K_{n_1,n_2,\ldots,n_s}) = |K_{n_1,n_2,\ldots,n_s}| - 2$. If $G$ has at least 2 components $G_1$ and $G_2$ that have edges, then $\mathrm{Z}^-(G) = \sum(G_i) \leq |G_1| - 2 + |G_2| - 2 \leq |G| - 4$.

Now suppose $G$ is connected and is not a complete multipartite graph. So $G$ does contain a $P_4$ or the paw as an induced subgraph by Theorem 2.2.24. Then $B = V \backslash \{v_1, v_2, v_3, v_4\}$, where $v_1, v_2, v_3$, and $v_4$ are the 4 vertices which induce the $P_4$ or paw subgraph, is a skew zero forcing set for $G$, and hence $\mathrm{Z}^-(G) \leq |G| - 4$. □

An algorithm for determining low skew zero forcing number $\mathrm{Z}^-(G) = 0$ for a graph $G$ was given in [22].

### 2.2.5 Dominating vertices

A vertex $v$ is called a *dominating vertex* if it is a neighbor of every other vertex in the graph. In other words, we say that $v$ *dominates* the graph.

**Observation 2.2.26.** *If* $K_1 = (\{v\}, \emptyset)$ *then the vertex $v$ dominates* $G \vee K_1$ *for any graph* $G$.

**Proposition 2.2.27.** *Let $G$ be graph and* $v \in V(G)$. *Then* $\mathrm{Z}^-(G) - 1 \leq \mathrm{Z}^-(G - v) \leq \mathrm{Z}^-(G) + 1$.

*Proof.* Suppose $B$ is a minimum skew zero forcing set of $G - v$. Then $B \cup \{v\}$ is a skew zero forcing set for $G$ and hence $\mathrm{Z}^-(G) \leq \mathrm{Z}^-(G - v) + 1$. Now let $B$ be a skew minimum zero forcing set with a set of forces $\mathcal{F}$. If there is a vertex $u$ such that there is a force

$v \to u$ in $\mathcal{F}$, then $B \cup \{u\}$ is a minimum skew forcing set with set of forces $\mathcal{F}\backslash\{v \to u\}$. Otherwise $B$ (or $B\backslash\{v\}$ if $v \in B$) is a skew zero forcing set with the same skew set of forces $\mathcal{F}$ as $G$. Then $\mathrm{Z}^-(G - v) \leq \mathrm{Z}^-(G) + 1$ implies $\mathrm{Z}^-(G) - 1 \leq \mathrm{Z}^-(G - v)$. □

**Proposition 2.2.28.** *Let $G$ be a graph. Then $\mathrm{Z}^-(G) - 1$ if $\delta(G) = 0$, and $\mathrm{Z}^-(G \vee K_1) = \mathrm{Z}^-(G) + 1$ if $\delta(G) \geq 1$.*

*Proof.* Let $G$ be a graph and consider the join $G \vee K_1$, where $u$ is the vertex of $K_1$. By Proposition 2.2.27, $\mathrm{Z}^-(G \vee K_1) \geq \mathrm{Z}^-(G) - 1$. Let $B$ be a minimum skew zero forcing set for $G$.

Suppose $\delta(G) = 0$ and let $v$ be an isolated vertex of $G$. Necessarily, $v \in B$ where $B$ is a minimum skew zero forcing set for $G$. We show $B\backslash\{v\}$ is a skew zero forcing set for $G \vee K_1$, implying $\mathrm{Z}^-(G \vee K_1) \leq \mathrm{Z}^-(G) - 1$. The formerly isolated vertex $v$ will force the dominating vertex $u$. Then the skew zero forcing process on $G - v$ continues as the skew zero forcing process of $B$ on $G$. Finally, $u$ can force $v$.

Now suppose $\delta(G) \geq 1$. Let $\hat{B}$ be an minimum skew zero forcing set for $G \vee K_1$. We show that there is a skew zero forcing set $B$ with the same cardinality that includes the dominating vertex $u$. If $u \notin \hat{B}$, then there exists a vertex $y$ such that $y \to u$ is the first force in the skew zero forcing process. Then $y$ has all blue neighbors in $G \vee K_1$ except $u$. Since $\delta(G) \geq 1$, there exists a vertex $z \in \hat{B}$ such that $z$ is a neighbor of $y$ and $(\hat{B}\backslash\{z\}) \cup \{u\}$ is a minimum skew zero forcing set of the same cardinality. Therefore we can assume the dominating vertex $u$ is in $B$. Since $u$ cannot perform a force until all but one vertex of $G$ is blue, $B\backslash\{u\}$ is a skew zero forcing set for $G$. Hence $\mathrm{Z}^-(G \vee K_1) \geq \mathrm{Z}^-(G) + 1$. Then by Proposition 2.2.27, $\mathrm{Z}^-(G \vee K_1) = \mathrm{Z}^-(G) + 1$. □

**Proposition 2.2.29.** *Let $G$ be a graph. Then $\mathrm{pt}^-(G \vee K_1) = \mathrm{pt}^-(G)$ if $\delta(G) \geq 1$.*

*Proof.* Let $\hat{B}$ be a skew zero forcing set for $G \vee K_1$ and let $u$ be the dominating vertex. If $u \in \hat{B}$ we claim that $u$ need not perform a force in the skew zero forcing process for $\hat{B}$. This is because if $u$ performs a force, then $u$ performs the last force in the skew zero forcing process since $\deg u = |G|$, and another vertex $v$ can perform the same force because $\delta(G) \geq 1$ and all other vertices are blue at the last time step. Then at any time step, if $x$ can perform a force in $G \vee K_1$, then $x$ can perform the same force in $G$. Since $u \in \hat{B}$ anyway, $u$ does not need to be forced. Therefore $\mathrm{pt}^-(G \vee K_1) = \mathrm{pt}^-(G)$.

Now suppose that $u \notin \hat{B}$. Let $v$ be the vertex that performs the first force in the skew zero forcing process. Then $v$ has all blue neighbors except some vertex $w$ when $v \to w$. Hence $w$ is the vertex $u$ since $u$ is a neighbor of every vertex. Let $x$ be a neighbor of $v$ such that $x \in B$ (such a vertex exists because $\delta(G) \geq 1$), and consider the set $(B \cup \{u\}) \backslash \{x\}$. Then this set $B'$ has the same cardinality as $B$, and the zero forcing process for $B'$ is at least as fast as for $B$, since $v$ need not force the vertex $u$ at time step 1, and instead $v$ can force $x$ at time step 1 since now $x$ is the only white vertex neighbor of $v$; $B$ may also force other vertices at time step 1. For all $t \geq 1$, after time step $t$ the set of blue vertices using $B$ is contained in the blue vertices using $B'$. Therefore there exists an efficient skew zero forcing set that contains the vertex $u$ for any graph $G \vee K_1$, and thus $\mathrm{pt}^-(G \vee K_1) = \mathrm{pt}^-(G)$ by the above argument. □

## 2.3 Skew propagation time interval

The *skew propagation time discrepancy* is the maximum propagation time for a graph minus the minimum propagation time for the same graph:

$$\mathrm{pd}^-(G) = \mathrm{PT}^-(G) - \mathrm{pt}^-(G)$$

**Definition 2.3.1.** The *skew propagation time interval* of a graph $G$ is defined as

$$[\mathrm{pt}^-(G), \mathrm{PT}^-(G)] = [\mathrm{pt}^-(G), \mathrm{pt}^-(G)+1, \ldots, \mathrm{PT}^-(G)-1, \mathrm{PT}^-(G)].$$

The skew propagation time interval is *full* if every integer in the propagation time interval can be realized as the skew propagation time of a graph.

The following example shows that the skew propagation time is not necessarily full for an arbitrary graph $G$.

**Example 2.3.2.** Consider the graph on 7 vertices shown in Figure 2.6. The skew zero forcing number of this graph is 1. Possible skew propagation times are 2 and 4, but 3 is not a possible skew propagation time. Coloring vertex $u$ yields a skew propagation time of 2. Coloring vertex $v$ gives skew propagation time 4. Because of symmetry, it is easy to see that coloring any other single vertex either gives skew propagation time 2, 4, or does not make up a skew zero forcing set.

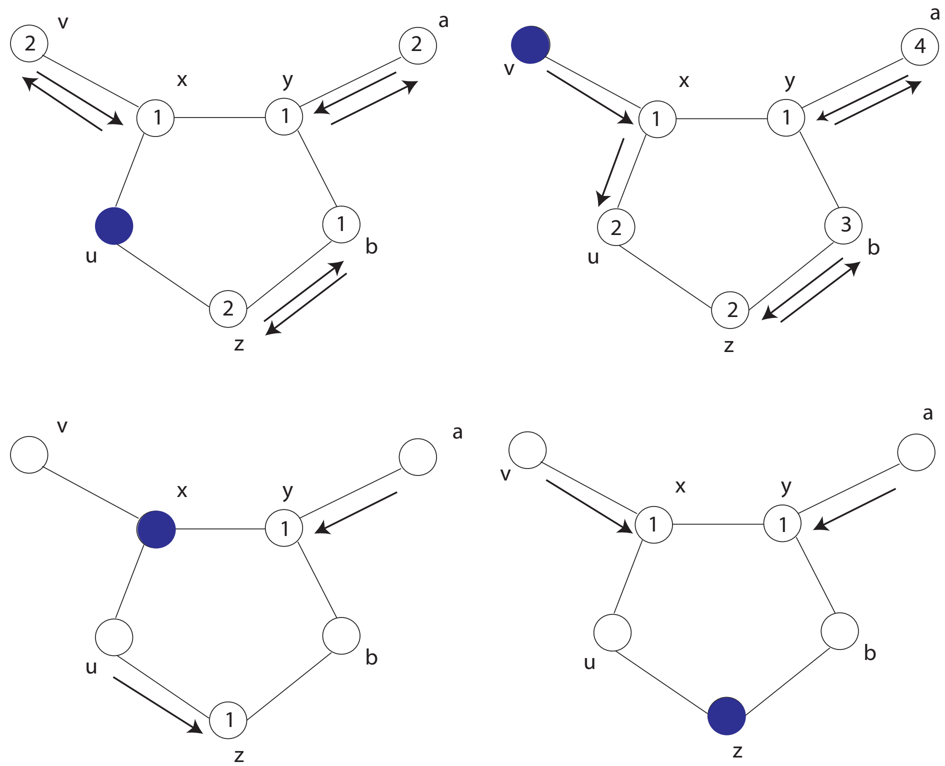



Figure 2.6 A graph where the skew propagation time interval is not full.

## 2.4 Extreme skew propagation time

In this section we consider graphs with extreme high skew propagation time $|G|-1$, and $|G|-2$, and low skew propagation time 0, 1, and 2. For standard zero forcing, it is known that that $0 \leq \mathrm{pt}(G) \leq |G|-1$ since $\mathrm{Z}(G) \geq 1$ for any simple graph and a set of isolated vertices has $\mathrm{Z}(G)$ equal to the order of the graph and $\mathrm{pt}(G) = 0$.

### 2.4.1 Low skew propagation time

We consider graphs with low skew propagation time 0, 1, and 2. First we observe that as with standard propagation time, the only way to have skew propagation time 0 is if every vertex in the graph is in the skew zero forcing set for the graph.

**Remark 2.4.1.** A graph $G$ has $\mathrm{pt}^-(G) = \mathrm{PT}^-(G) = 0$ if and only if $\mathrm{Z}^-(G) = |G|$ and if and only if $G$ is a set of isolated vertices. Suppose the graph has an edge. Then there are two vertices $u$ and $v$ that are neighbors. Without loss of generality, if $u$ and $v$ are white and there are no other white vertices, then $v \leftrightarrow u$ is a possible force. Therefore the skew zero forcing number of the graph can be at most $|G|-2$ and $\mathrm{pt}^-(G) > 0$ if the graph has an edge. In other words, the introduction of an edge to a graph requires that there be at least one force in the skew zero forcing process.

Next we discuss graphs with $\mathrm{pt}^-(G) = 1$.

**Observation 2.4.2.** *There is no connected graph other than $P_2$ with skew zero forcing number 0 and skew propagation time 1.*

**Proposition 2.4.3.** *Let $G$ be a connected graph with $|G| \geq 2$. If there exists a minimum skew zero forcing set $B$ such that $G[V \backslash B] = sK_2$ ($s$ disjoint copies of $K_2$), then* $\mathrm{pt}^-(G) = 1$ *and $B$ is efficient.*

*Proof.* Let $G = (V, E)$ be a graph on $n$ vertices and let $B$ be a skew zero forcing set such that $G[V \backslash B] = sK_2$. Then since the skew propagation time of each $K_2$ is 1, the blue vertices in $B$ are not needed to perform any forces in a skew zero forcing process of $B$ in $G$, $\text{pt}^-(G) = 1$, and $B$ is efficient. □

The converse to Proposition 2.4.3 is false in general.

**Example 2.4.4.** The graph $K_2 \Box K_5$ shown in Figure 2.7 does not have a minimum skew zero forcing set $B$ such that $G[V \backslash B] = sK_2$. The graph has $\text{Z}^-(G) = 5$: If any 4 vertices are colored, at most one vertex force can occur before the process halts. Clearly 5 vertices is sufficient to force the graph in one time step, so $\text{Z}^-(G) = 5$. The blue skew forcing set labeled in the figure has $\text{pt}^-(G, B) = 1$. Note that whenever the minimum skew zero forcing set has $\text{pt}^-(G) = 1$ and $|G \backslash B|$ is odd, there is no way for $G \backslash B$ to equal $sK_2$ for any minimum skew zero forcing set.

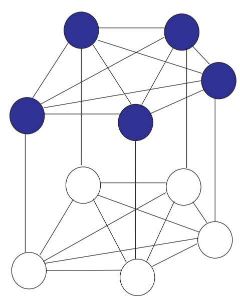


Figure 2.7 A graph with $\text{pt}^-(G) = 1$ where there does not exist a minimum skew zero forcing set $B$ such that $G[V \backslash B] = sK_2$

**Proposition 2.4.5.** *The converse of Theorem 2.4.3 is true when none of the vertices in $B$, a minimum skew zero forcing set for a graph $G$, perform any forces in the skew zero forcing process.*

*Proof.* Suppose $G$ is a graph with $|G| = n$ and $\text{pt}^-(G) = 1$. Suppose no vertex in $B$ performs a force in $G$. Then $V \backslash B$ must be self-forcing in a single time step, and hence it must be the case that $V \backslash B$ is a set of isolated 2-cycles. □

**Corollary 2.4.6.** *The converse for Theorem 2.4.3 is true when $G$ is a complete multipartite graph.*

*Proof.* The zero forcing number for such a graph $G$ is the order of the graph minus 2, and no vertex in a minimum skew zero forcing set $B$ needs to perform a force in the skew zero forcing process on $G$. □

**Proposition 2.4.7.** *If $B$ is a minimum skew zero forcing set for $G$, then $G \backslash B$ does not have an isolated vertex.*

*Proof.* Suppose that $G \backslash B$ has an isolated vertex $u$. Then any neighbor of $u$ must be in $B$. Then the vertex that forces $u$ in $G$ can be removed from the skew zero forcing set $B$ and hence $B$ is not a minimum skew zero forcing set. □

**Definition 2.4.8.** [21] Let $H = (V_1, E_1)$ and $H = (V_2, E_2)$ be graphs on an equal number of vertices, and suppose $\mu : V_1 \to V_2$ is a bijection. Then the matching graph $(H_1, H_2, \mu)$ is defined as the disjoint union of the graphs $H_1$ and $H_2$ and all edges $(v, \mu(v))$ with $v \in V_1$.

In [21] the following result was proved about $\operatorname{pt}(G) = 1$ with standard zero forcing.

**Proposition 2.4.9.** [21] *Let $G = (V, E)$ be a graph. Then any two of the following imply the third.*

1. $|G| = 2\,\mathrm{Z}(G)$
2. $\operatorname{pt}(G) = 1$
3. *$G$ is a matching graph.*

We do not expect the same result for standard zero forcing regarding matching graphs, $|G| = 2 \cdot \mathrm{Z}(G)$, and $\operatorname{pt}(G) = 1$, to translate to skew zero forcing due to the nature of the skew forcing process. In fact it is much harder to get $\mathrm{Z}^{-}(G) = \frac{G}{2}$ and $\operatorname{pt}^{-}(G) = 1$ in a matching graph than for the standard zero forcing case.

**Remark 2.4.10.** It is true that $|G| = 2\operatorname{Z}^{-}(G)$ and G being a matching graph imply $\operatorname{pt}^{-}(G) = 1$. For any matching graph $(H_1, H_2, \mu)$, $\{V_1\}$ is a skew zero forcing set of order $\frac{|G|}{2}$ and $\operatorname{pt}^{-}(G) = 1$ since at time step $t = 1$, $v \to \mu(v)$ and the entire graph is forced.

However, the other two pairs of conditions in Proposition 2.4.9 do not imply the third, as illustrated in the next two examples.

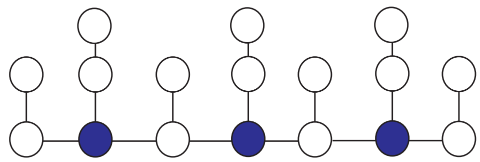
**Example 2.4.11.** Let $H_1$ be a path on the three vertices $v_1, v_2, v_3$ and $H_2$ be the path on three vertices $u_1, u_2, u_3$. Let $\mu$ be the bijection that maps $v_1$ to $u_1$, $v_2$ to $u_2$, and $v_3$ to $u_3$. Then clearly $G = (H_1, H_2, \mu) = P_3 \Box P_2$ is a matching graph. It is easy to see that $\operatorname{Z}^{-}(P_3 \Box P_2) = 2$, and by choosing $B = \{u_2, v_2\}$ that $\operatorname{pt}^{-}(G) = 1$. Therefore the facts that $\operatorname{pt}^{-}(G) = 1$ and $G$ is a matching graph do not imply $|G| = 2\operatorname{Z}^{-}(G)$.

**Example 2.4.12.** Let $S$ be the star on 4 vertices. Then $\operatorname{Z}^{-}(S) = 2 = \frac{|S|}{2}$ and $\operatorname{pt}^{-}(S) = 1$, but $S$ is not a matching graph. Therefore $|G| = 2\operatorname{Z}(G)$ and $\operatorname{pt}^{-}(G) = 1$ do not imply that $G$ is a matching graph.

In [21] it was shown that the diameter of a graph $G$ can get arbitrarily larger than its minimum propagation time. This leads us to ask, can the diameter of a graph $G$ be arbitrarily large and still have $\operatorname{pt}^{-}(G) = 1$? The next example shows that the answer to this question is yes.

**Example 2.4.13.** Consider a graph $G$ made up of a path $P_n$ on an odd number of vertices $n$, where each odd numbered vertex in the path has one leaf, and each even numbered vertex in the path has a $P_2$ appended to it. See the graph in Figure 2.8 for an example of a such a graph on 17 total vertices. (Observe that $|G| = \frac{5n-1}{2}$.) Then a the size of a minimum skew zero forcing set for such a graph is $\frac{n-1}{2}$. By [Theorem 3.24 in [22]], if $u$ is a leaf and its neighbor is $v$, then $\operatorname{Z}^{-}(G - \{u, v\}) = \operatorname{Z}^{-}(G)$. This leaves a set of isolated $\frac{n-1}{2}$ isolated vertices, and hence $\operatorname{Z}^{-}(G) = \frac{n-1}{2}$. Finally, let $B$ be the set of even numbered vertices in the path. By Proposition 2.4.3, $\operatorname{pt}^{-}(G, B) = 1$, so $\operatorname{pt}^{-}(G) = 1$.

Figure 2.8 The diameter of a graph can be arbitrarily large and still have skew propagation time 1.



The graph in the next example shows that graph $G$ can be of arbitrarily large maximum degree, have skew zero forcing number $\mathrm{Z}^{-}(G) = 1$, and $\mathrm{pt}^{-}(G) = 1$.

**Example 2.4.14.** Consider the generalized star $S(s_1, s_2, \ldots, s_k)$ such that $s_1, s_2, \ldots, s_k = 2$. It is easy to see that $\mathrm{Z}^{-}(S(2, 2, , 2)) = 1$. Then the middle vertex of the generalized star is an efficient skew zero forcing set, with skew propagation time 1 by Proposition 2.4.3. To get a larger skew zero forcing number, say $\mathrm{Z}^{-}(G) = m$, and arbitrary maximum degree, use several disjoint copies of such generalized stars.

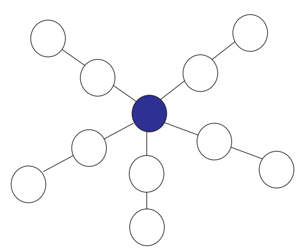


Figure 2.9 A generalized star $S$ with $\Delta(S) = 5$, skew zero forcing number 1 and skew propagation time 1.

Does there exist a connected graph with arbitrarily large zero forcing number and skew propagation time 1?

The answer to the above question is yes, and an example of a graph with arbitrarily high skew zero forcing number and skew propagation time 1 is the star on $n$ vertices, which has $Z^-(G) = n - 2$ and $\mathrm{pt}^-(G) = 1$. The complete graph $K_n$ and the complete multipartite graph $K_{n_1,n_2,\ldots,n_s}$, have these properties. Note that many of these graphs also provide examples for standard zero forcing and propagation time.

**Remark 2.4.15.** Let $G$ be a graph with $Z^-(G) \geq 1$ and $\mathrm{pt}^-(G) = 1$. Then any graph $G'$ obtained from $G$ by performing one or more vertex sums of $G$ with some number of 3-cycles consisting of one blue vertex of a skew zero forcing set $B$ of $G$ and 2 white vertices not in the vertex set of $G$ has $\mathrm{pt}^-(G') = 1$. Furthermore, for a graph $G$ and blue skew zero forcing set $B$, we can vertex sum any blue vertex in $B$ with $P_3$ to form $G'$ and still have $\mathrm{pt}^-(G') = 1$.

The construction in Remark 2.4.15 is illustrated in the next example.

**Example 2.4.16.** The gavel graph (see Figure 2.10) is constructed from a path on 3 vertices by summing two 3-cycles on an endpoint.

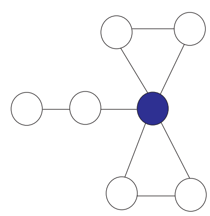


Figure 2.10 The gavel graph which has skew propagation time 1.

Finally, a remark regarding graphs with low skew propagation time $\mathrm{pt}^-(G) = 2$.

**Remark 2.4.17.** For any graph $G$, the corona $G \circ K_1$ has $Z^-(G \circ K_1) = 0$ and $\mathrm{pt}^-(G \circ K_1) \leq 2$, because the leaves force their neighbors, and then these neighbors force their corresponding leaves at the next time step. In the case of $G = K_n$ for $n \geq 2$, clearly $\mathrm{pt}^-(G \circ K_1) = 2$. Therefore a graph $G$ can be of arbitrarily large maximum degree and still have $Z^-(G) = 0$ and $\mathrm{pt}^-(G) = 2$. As we would expect based on the similar result for $\mathrm{pt}^-(G) = 1$, a graph can be of arbitrarily large diameter, have $Z^-(G) = 0$, and

$\mathrm{pt}^-(G) = 2$. Consider $P_n \circ K_1$ (also known as the comb graph). Because of the property of taking the corona of any graph $H$ with $K_1$ there are no such restrictions on the size of an induced cycle or length of an induced path in a graph $G$ with $\mathrm{pt}^-(G) = 2$.

There are numerous examples of graphs with $\mathrm{pt}^-(G) = 2$ and $\mathrm{Z}^-(G) \neq 0$, and in fact $\mathrm{Z}^-(G) = m$ for arbitrary positive integer $m$. These graphs are as difficult, if not even more difficult, to characterize than $\mathrm{pt}^-(G) = 1$.

### 2.4.2 High skew propagation time

Recall that for simple graphs, $\mathrm{pt}(G) = |G| - 1$ if and only if $G$ is a path, as shown in [21]. For skew propagation time, propagation on a path $P$ is much faster than $|P| - 1$ because of white vertex forcing, so the result for high standard propagation time on paths from [21] no longer holds and characterizing graphs with $\mathrm{pt}^-(G) = |G| - 1$ is a more challenging problem. To have high propagation time, a lower skew zero forcing number is necessary. For skew zero forcing $\mathrm{Z}^-(G) = 0$ is possible, and whether $\mathrm{Z}^-(G) = 0$ can be determined by Algorithm 3.24 in [22] (This is discussed further in Section **??**).

Since $\mathrm{Z}^-(G)$ can be 0, it seems that it might be possible for $0 \leq \mathrm{pt}^-(G) \leq |G|$. However we will show later that no graph can have skew propagation time equal to the order of the graph, and so $0 \leq \mathrm{pt}^-(G) \leq |G| - 1$ for skew propagation time as well.

**Remark 2.4.18.** Clearly there exist small graphs with $\mathrm{pt}^-(G) = |G| - 1$, such as $P_2$ and the paw graph.

**Lemma 2.4.19.** *For a graph $G = (V, E)$ with vertices $u$ and $v$, it is impossible to have $u \to v$ and $v \to u$ at consecutive time steps with skew zero forcing.*

*Proof.* Suppose $u \to v$ at time $t$. Then before time step $t$, the only white neighbor of $u$ is $v$. But this force does not affect the white neighbors of $v$ in any way, so having $u$ as the only white neighbor of $v$ after step $t$ and before time step $t + 1$ is impossible. Therefore

$u$ and $v$ are each others only white neighbors at time step $t$, and they force each other at that time step. □

The complement of a graph $G = (V, E)$, denoted $\overline{G}$, is the graph with the same vertex set as $G$, and edge set such that if $uv \in E$, then $uv$ is not in the edge set of $\overline{G}$, and vice versa. The half graph, denoted $H_s$ is the graph formed by one copy of the complete graph $K_s$ and one copy of $\overline{K_s}$, where $u_1, u_2, \ldots, u_s$ are the vertices in the vertex set of the $\overline{K_s}$ and $w_1, w_2, \ldots, w_s$ are the vertices of $K_s$, and the edge $u_i w_j$ is in the edge set of $H_s$ if and only if $i \geq j$. The half graph $H_3$ is shown in Figure 2.11.

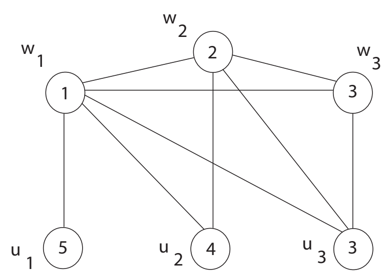
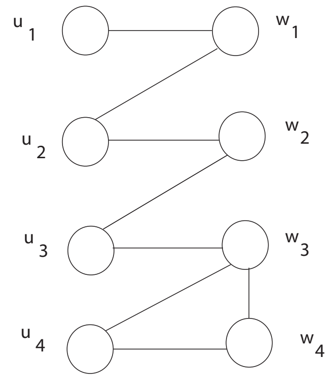



Figure 2.11 The half graph on 6 vertices has $\mathrm{pt}^{-}(G) = |G| - 1 = 5$ with the forces occurring at the time steps labeled on the vertices.

In fact, the half graph and certain subgraphs are the only graphs with $\mathrm{Z}^{-}(G) = 0$ and $\mathrm{pt}^{-}(G) = |G| - 1$, as the next result shows. An example of such a subgraph of $H_4$ is shown in Figure 2.12.

Figure 2.12 The smallest spanning subgraph of the half graph $H_4$ required for $\mathrm{pt}^-(G) = |G| - 1$.




**Theorem 2.4.20.** *There is no graph for which* $\mathrm{pt}^-(G) = |G|$. *If* $\mathrm{Z}^-(G) = 0$ *and* $\mathrm{pt}^-(G) = |G| - 1$, *then* $G$ *is a spanning subgraph of a half graph* $H_s$ *with the following 3 properties, where the vertices are labeled* $u_1 \dots u_s, w_1, \dots w_s$*;*

1. $N(u_1) = \{w_1\}$, $N(u_2) = \{w_1, w_2\}$.

2. *For* $i = 3, \dots, s : \{w_{i-1}, w_i\} \subseteq N(u_i), w_j \notin N(u_i)$ *for* $j > i$, *and* $u_j \notin N(u_j)$ *for all* $j$.

3. $w_{s-1} \in N(w_s)$.

*Proof.* Let $|G| = n$. By Observation 2.1.16, a disconnected graph cannot have $\mathrm{pt}^-(G) = |G|$, so we assume $G$ is connected. In order for $\mathrm{pt}^-(G) = |G|$ for a graph $G$, it must be the case that $\mathrm{Z}^-(G) = 0$. Since the graph has skew zero forcing number 0, it must be a graph on an even number of vertices, say $n = 2s$, and all the skew zero forcing chains of the graph must be 2-cycles. Let $v_i, v_2, \dots, v_n$ denote the entire set of vertices of the graph. In order for the graph to have $\mathrm{pt}^-(G) = |G|$, exactly one force must occur at each time step.

We introduce some additional notation for the vertices of the graph $G$. Since each skew forcing chain is a 2-cycle, in every forcing chain there is one white vertex force and one blue vertex force. Let $u_1, u_2, \ldots, u_s$ denote the vertices that perform white vertex forces in each forcing chain $u_i \leftrightarrow w_i$, and let $w_1, w_2, \ldots, w_s$ denote the vertices that are forced by $u_1, u_2, \ldots, u_s$ and perform blue vertex forces in each forcing chain $u_i \leftrightarrow w_i$.

Suppose at time step 1, $u_1 \to w_1$, at time step 2, $u_2 \to w_2$, and so on until some time step $k$ at which $u_k \to w_k$ and after which there is some blue vertex force. We claim that the force that occurs immediately before this blue vertex force is actually $u_s \to w_s$, and the time step at which this force occurs is time step $s$; i.e. all the white vertex forces that will occur in the skew zero forcing process have occurred before there are any blue forces. Suppose otherwise. Then the vertex forces that occur immediately after $u_k \to w_k$ are some set of blue vertex forces $w_{\ell_1} \to u_{\ell_1}$, $w_{\ell_2} \to u_{\ell_2}$, $\ldots$, $w_{\ell_i} \to u_{\ell_i}$ and then the white vertex force $u_{k+1} \to w_{k+1}$, in that order. Then in order for $u_{k+1}$ to force $w_{k+1}$ at this time step and no previous time step, the white vertex $u_{k+1}$ must be adjacent to $u_{\ell_i}$. But $u_{\ell_i}$ has already performed the force $u_{\ell_i} \to w_{\ell_i}$ because $w_{\ell_i} \to u_{\ell_i}$ is a blue vertex force, and hence $u_{\ell_i}$ can have no other white neighbors after that time step. Therefore $u_{k+1}$ is not adjacent to $u_{\ell_i}$, and our assumption that $w_{\ell_1} \to u_{\ell_1}$ is not $u_s \to w_s$ must be incorrect. Therefore all the white vertex forces that occur in the skew zero forcing process will have occurred before there are any blue forces, and time step $k$ is actually time step $s$.

Observe that $w_s$ is not adjacent to any vertex in $\{u_1, \ldots, u_{s-1}\}$ since each of these $u_i$ forced another vertex while $w_s$ is white. Then after $t = s - 1$, $w_s$ has only one white neighbor $u_s$ and so $u_s \leftrightarrow w_s$ at time step $s$, which is a contradiction to the claim that $\mathrm{pt}^-(G) = |G|$ because one force is no longer performed at every time step in the graph.

The remaining component of the proof is by construction using the only possible skew zero forcing process for a graph with $\mathrm{Z}^-(G) = 0$ and $\mathrm{pt}^-(G) = |G| - 1$. We use this method to show that such a graph must be a subgraph of a half graph $H_s$ having a specific form. We know from the first half of the proof that the graph must be on an even number of vertices $n = 2s$ and all of the skew forcing chains must be 2-cycles $u_i \leftrightarrow w_i$ with forcing process $u_1 \rightarrow w_1, \ldots, u_{s-1} \rightarrow w_{s-1}, w_s \leftrightarrow w_s, w_{s-1} \rightarrow u_{s-1}, \ldots, w_1 \rightarrow u_1$. Then before time step 1, there must be one possible vertex force $u_1 \rightarrow w_1$ and one possible vertex force only, hence exactly one degree 1 vertex, which is $u_1$. Then before time step 2, there must be one white vertex $u_2$ with neighbor $w_1$ and exactly one white neighbor $w_2$ and at time step 2, $u_2 \rightarrow w_2$. Hence $\deg(u_2) = 2$ and $N(u_2) = \{w_1, w_2\}$. Continuing this way, $u_i$ is adjacent to $w_i$, $u_i$ is adjacent to $w_{i-1}$, $u_i$ is not adjacent to $u_j$ for $i \neq j$ and $u_i$ is not adjacent to $w_j$ for $j > i$. When the forcing process reaches vertex $u_s$, $u_s$ must be adjacent to white vertex $w_s$ and the vertex $w_{s-1}$, and $w_s$ is adjacent to $w_{s-1}$. At this time step, $u_s$ and $w_s$ are each other's only white neighbors, and hence force each other at the same time step. Then the remaining vertex forces are $w_{s-1} \rightarrow u_{s-1}, \ldots, w_2 \rightarrow u_2, w_1 \rightarrow u_1$, each at a single time step. Since the only vertex forces that occurred simultaneously were $u_s \leftrightarrow w_s$, the entire subgraph of $H_s$ is forced in $|G| - 1$ time steps. Any subgraph of the half graph $H_s$ of the form just described may have any of the following additional edges as well and still have $\text{pt}^-(G) = |G| - 1$: for $3 \leq k \leq s$, vertex $u_k$ may have any of the neighbors $w_{k-2}, \ldots, w_1$. Additional edges between the $w_i$ are also permitted. There are no other graphs for which $\text{pt}^-(G) = |G| - 1$. □

**Observation 2.4.21.** *There is a graph $G$ of every possible even order that has $\text{pt}^-(G) = |G| - 1$. Note that the paw graph in Figure 2.3 is $H_2$.*

**Conjecture 2.4.22.** *Let $G$ be a graph, and let $B$ be a skew zero forcing set for $G$. Then $\text{pt}^-(G, B) \leq |G| - |B| - 1$.*

## 2.5 Skew propagation time of graph families

In this section are some common graph families with their skew zero forcing numbers observed and skew propagation times proved.

**Proposition 2.5.1.** *Any complete multipartite graph of the form $K_{n_1,n_2,\dots,n_s}$ where $s \geq 2$ has $\mathrm{Z}^-(K_{n_1,n_2,\dots,n_s}) = n_1 + n_2 + \cdots + n_s - 2$ and $\mathrm{pt}^-(K_{n_1,n_2,\dots,n_s}) = \mathrm{PT}^-(K_{n_1,n_2,\dots,n_s}) = 1$, and thus a skew propagation time discrepancy of 0.*

*Proof.* Note that $K_n = K_{1,1,\dots,1}$. By Corollary 2.2.25, $\mathrm{Z}^-(G) = |G| - 2$ for a complete multipartite graph $G$. By choosing two adjacent vertices white and the rest blue, then each white vertex forces the other in a single time step. Hence $\mathrm{pt}^-(K_{n_1,n_2,\dots,n_s}) = \mathrm{PT}^-(K_{n_1,n_2,\dots,n_s}) = 1$ since having two nonadjacent white vertices is not a skew zero forcing set. □

**Proposition 2.5.2.** *Consider the path $P_n$ on $n \geq 2$ vertices. Then for skew zero forcing, $\mathrm{Z}^-(P_n) = 0$ if $n$ is even, $\mathrm{Z}^-(P_n) = 1$ if $n$ is odd, and*

$$\mathrm{pt}^-(P_n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \\ \lfloor \frac{n+1}{4} \rfloor & \text{if } n \text{ is odd} \end{cases}$$

*Furthermore, $\mathrm{PT}^-(P_n) = \frac{n}{2}$ if $n$ is even, $\mathrm{PT}^-(P_n) = \frac{n-1}{2}$ if $n$ is odd, and the skew propagation time interval is full for any path. Hence, the skew propagation time discrepancy is 0 for $n$ even and $\frac{n-1}{2} - \lfloor \frac{n+1}{4} \rfloor$ for $n$ odd.*

*Proof.* Suppose $n \geq 2$ is even and $P_n = v_1 v_2 \dots v_{n-1} v_n$. Then $\mathrm{Z}^-(P_n) = 0$ because of white vertex forcing. At the first time step, vertices $v_2$ and $v_{n-1}$ are forced blue by the endpoints of the path (via white vertex forcing). At the second time step, vertex $v_4$ is forced blue by $v_3$ and vertex $v_{n-3}$ is forced by $v_{n-2}$. The propagation process continues in this way until the two middle-most vertices $v_{\frac{n}{2}}, v_{\frac{n}{2}+1}$ are forced at time step $\lceil \frac{n}{4} \rceil$ by their neighbors to either side, respectively. Then the forcing proceeds out from the center of the path until both end vertices of the path are colored, which occurs at time step $\frac{n}{2}$. Since the zero forcing number is 0 for even length paths, there is no way to pick a different zero forcing set other than the efficient skew zero forcing set $\emptyset$. Thus $\mathrm{pt}^-(P_n) = \frac{n}{2}$, $\mathrm{PT}^-(P_n) = \frac{n}{2}$, and the skew propagation time interval is full for $n$ even.

Now suppose that $n \geq 3$ is odd, $P_n = v_1 v_2 \ldots v_{n-1} v_n$, $n$ is of the form $4k+1$ for some integer $k$. Then $\mathrm{Z}^-(P_n) = 1$, no even vertex is a skew zero forcing set, and every odd vertex is a skew zero forcing set. We show that the vertex $v_\ell$ where $\ell = \frac{n+1}{2}$ is an efficient skew zero forcing set. Coloring this vertex blue divides the path into two subpaths, each of even length $\frac{n-1}{2}$, that are simultaneously self-forcing (that is, white vertex forcing starts the propagation process on each subpath at time step 1, and vertices are forced in both subpaths at each additional time step until there are no vertices left to force), and the propagation time is $\frac{n-1}{4}$, by the first case. This shows that $\mathrm{pt}^-(P_n) \leq \frac{n-1}{4}$ for $n$ is of the form $4k+1$ for some integer $k$.

Finally, suppose that $n \geq 3$ is odd, $P_n = v_1 v_2 \ldots v_{n-1} v_n$, $n$ is of the form $n = 4k+3$. Again, we have that $\mathrm{Z}^-(P_n) = 1$, but the vertex $v_\ell$ where $\ell = \frac{n+1}{2}$ is no longer a zero forcing set because $\frac{n+1}{2}$ is even. However, the vertex $v_\ell$ where $\ell = \frac{n-1}{2}$ is a minimum zero forcing set that splits the graph into two even length paths that are simultaneously self-forcing (as in the previous case, except that the subpaths are no longer the same length). The longer of the two paths is length $\frac{n+1}{2}$, and since the propagation runs in parallel, the propagation time of the entire path is $\frac{n+1}{4}$, again by case for skew zero forcing of even length paths. This shows that $\mathrm{pt}^-(P_n) \leq \frac{n+1}{4}$ for $n$ of the form $n = 4k+3$.

We complete the argument by examining the skew propagation time of all possible skew zero forcing sets, meanwhile showing that the skew propagation time interval is full for any path. In the $n = 4k+1$ is odd case, if the skew zero forcing set is shifted by exactly 2 vertices to either side of the original set $\{v_\ell\}$, the skew propagation time with this set is $\frac{n+3}{4}$. Moving another 2 vertices in the same direction increases the propagation time to $\frac{n+7}{4}$, and so on, until the skew zero forcing set is an endpoint of the path, which has skew propagation time $\frac{n-1}{2}$. This is because when one end vertex is colored on a path on $4k+1$ vertices, what remains is a path on $4k$ vertices, and the forcing process on the half that did not have the blue endpoint occurs similarly to that case. Then the forcing process takes exactly the same time as for the path on $4k$ vertices, which takes

$\frac{4k}{2} = \frac{n-1}{2}$ time steps. For the final case, using the same method of moving 2 vertices away from the original efficient zero forcing set and repeating this process until the skew zero forcing set is an endpoint of the path also yields maximum skew propagation time $\frac{n-1}{2}$ and every skew propagation time between $\mathrm{pt}^-(P_n)$ and $\mathrm{PT}^-(P_n)$ is realized by some minimum zero forcing set. Thus the skew propagation time interval is full for odd paths as well. □

**Proposition 2.5.3.** *Let $C_n$ be a cycle on $n \geq 3$ vertices. Then for skew zero forcing $\mathrm{Z}^-(C_n) = 1$ if $n$ is odd and $\mathrm{Z}^-(C_n) = 2$ if $n$ is even, and*

$$\mathrm{pt}^-(C_n) = \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd} \\ \frac{n-2}{4} & \text{if } n \text{ is even of the form } 4k+2 \text{ for some integer } k \\ \frac{n}{4} & \text{if } n \text{ is even of the form } 4k \text{ for some integer } k \end{cases}$$

*Furthermore, $\mathrm{PT}^-(C_n) = \mathrm{pt}^-(C_n)$ and thus the skew propagation time discrepancy is 0 for any cycle. Thus skew propagation time interval is full for any cycle.*

*Proof.* Suppose $n \geq 3$ is odd. Then $\mathrm{Z}^-(C_n) = 1$, since if no vertices are colored blue then each vertex has more than one white neighbor and by coloring any one vertex we leave an even length path, which can self force. Hence any vertex $v$ is an efficient zero forcing set for $C_n$. So the propagation time for the path, $\mathrm{pt}^-(P_{n-1}) = \frac{n-1}{2}$, is also the propagation time of the cycle $C_n$ with skew zero forcing for $n$ odd. Then $\mathrm{PT}^-(C_n) = \frac{n-1}{2}$ as well and the skew propagation discrepancy is 0.

Now suppose that $n \geq 4$ is even. A set $B$ of 2 vertices is a skew zero forcing set if and only if the paths in $C_n - B$ are even. Then the zero forcing number $\mathrm{Z}^-(C_n) = 2$, since coloring one vertex is insufficient because $n-1$ is odd and $P_{n-1}$ is not self forcing, and any two adjacent vertices $u$ and $v$ make up a skew zero forcing set. For the case where $n$ is of the form $4k+2$, an efficient zero forcing set is any vertex $v$ and the vertex exactly $\frac{n}{2}$ vertices away from $v$ is a skew zero forcing set, and it takes $\frac{n-2}{4}$ time steps to force the graph (with the two even length paths self-forcing simultaneously). When $n$ is not of the form $4k+2$, a set consisting of any vertex $v$ and the vertex $\frac{n-2}{2}$ vertices away from $v$ is a skew zero forcing set, and it takes $\frac{n}{4}$ time steps to force the graph.

Let $B$ be a skew zero forcing set for $C_n$. Then $B$ consists of 2 blue vertices that break the graph into two even length paths when $C_n - B$. Otherwise as the skew zero forcing process proceeds, every other vertex of the cycle is forced to be blue and then no additional vertices can be forced.

Let $P_1$ and $P_2$ be the two even length paths in $C_n - B$. The skew zero forcing processes on the two paths run in parallel. At time step 1, 4 vertices are forced blue. Then at each additional time step 4 vertices are forced except possibly at the last step, when there are only two vertices left if $n = 4k+2$. In particular, the longest path is forced in double time; that is, 4 vertex forces at a time until there are either no vertices left to be forced or two vertices left to be forced. Then $\mathrm{PT}^-(C_n) = \frac{n-2}{4}$ if $n$ is of the form $4k+2$ for some integer $k$ and $\mathrm{PT}^-(C_n) = \frac{n}{4}$ if $n$ is of the form $4k$ for some integer $k$, as needed. □

The $m, k$ pineapple, with $m \geq 3$ and $k \geq 1$, is denoted $P_{m,k}$ and defined as the union of the graphs $K_m$ and $K_{1,k}$ with the specification that $K_m \bigcap K_{1,k}$ is the vertex of $K_{1,k}$ that is of degree $k$.

**Proposition 2.5.4.** *The $m, k$ pineapple has $\mathrm{Z}^-(P_{m,k}) = m + k - 4$ and $\mathrm{pt}^-(P_{m,k}) = 3$ for $m \geq 3$. Also $\mathrm{PT}^-(P_{m,k}) = 3$ and $P_{m,k}$ has a skew propagation time discrepancy of 0.*

*Proof.* In order for all leaves to be forced in the skew zero forcing process, at least $k-1$ of the leaves must be blue. Regardless of what is chosen for the skew zero forcing set, the vertex of degree $m + k - 1$ can be forced by one of the leaves at the first time step. Once that vertex is forced, there must be at most 2 white vertices left in the induced subgraph of $K_m$, so in order for the forcing process to be carried out, at least $k - 1 + m - 3 = |P_{m,k}| - 4$ vertices must be in the skew zero forcing set. This number of vertices is also sufficient to force the entire graph blue, and since $\mathrm{Z}^{-}(P_{m,k}) = m + k - 4$, there are only 4 vertices that need to be forced.

As just described, any minimum zero forcing set consists of $k-1$ of the leaves, and all but $m-3$ of the vertices in the $K_m$ part, none of which is the vertex of degree $m+k-1$. Then at the first time step, the single white leaf forces its only white neighbor in the $K_m$ part. At the second time step the remaining two white vertices in the $K_m$ force each other. At the final time step, the remaining leaf is forced blue, and therefore $\mathrm{pt}^{-}(P_{m,k}) = 3$. Since any minimum skew zero forcing set is of the same form, $\mathrm{PT}^{-}(P_{m,k}) = 3$ as well. □

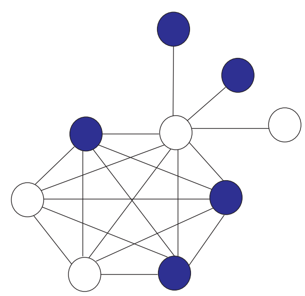


Figure 2.13 The 6, 3 pineapple has $\mathrm{Z}^{-}(P_{m,k}) = 5$ and $\mathrm{pt}^{-}(P_{m,k}) = 3$.

**Proposition 2.5.5.** *For a connected graph $G \neq K_1$, the skew zero forcing number of the corona $G \circ K_1$ is $\mathrm{Z}^{-}(G \circ K_1) = 0$ and $\mathrm{pt}^{-}(G \circ K_1) = \mathrm{PT}^{-}(G \circ K_1) = 2$. Thus the corona $G \circ K_1$ has a skew propagation time discrepancy of 0.*

*Proof.* The corona $G \circ K_1$ has $\mathrm{Z}^{-}(G \circ K_1) = 0$. This is illustrated by the skew zero forcing process on $G \circ K_1$: at the first time step, each leaf forces its only white neighbor in $G$, and at the second time step each vertex in $G$ forces its only white neighbor, which must be a leaf, and all remaining vertices are forced. □

**Proposition 2.5.6.** *The $n^{th}$ hypercube $Q_n$ for $n \geq 2$ has $\mathrm{Z}^{-}(Q_n) = 2^{n-1}$ and $\mathrm{pt}^{-}(Q_n) = 1$.*

*Proof.* Since $\mathrm{mr}^-(Q_n) = 2^{n-1}$ for $n \geq 2$ by [23], $\mathrm{M}^-(Q_n) = 2^{n-1}$ implies $\mathrm{Z}^-(Q_n) \geq 2^{n-1}$. Then since coloring one copy of $Q_{n-1}$ is sufficient to force the graph, $\mathrm{Z}^-(Q_n) = 2^{n-1}$ and each blue vertex forces its only white neighbor in the second copy of $Q_{n-1}$ in a single time step. Therefore $\mathrm{pt}^-(Q_{n-1}\square P_2) = 1$. □

**Proposition 2.5.7.** *For the cartesian product $K_3\square P_2$ the skew zero forcing number is $\mathrm{Z}^-(K_3\square P_2) = 2$, and for $s \geq 3$, $t \geq 3$, $\mathrm{Z}^-(K_s\square P_t) = s$. Then $\mathrm{pt}^-(K_3\square P_2) = 2$ and in general $\mathrm{pt}^-(K_s\square P_t) = t - 1$.*

*Proof.* The cartesian product $K_3\square P_2$ is as shown in Figure 2.14. Then at the first time step $c \to b$ and $u \to w$. At the second time step, $b \to c$ and $w \to u$.

In the general case, no fewer than $s$ vertices can be used to force the graph by the same argument as in Example 2.4.4; namely, a set of $s-1$ vertices can perform 1 force if all but one neighbor of a single vertex is blue, but then the forcing stops since every vertex either has all blue neighbors blue or 2 or more white neighbors. Then coloring the complete copy of $K_s$ on one of the ends of the path is the only possible skew zero forcing set (because otherwise no vertex forcing can occur at the outset), which means it must be efficient, and at each time step an additional $K_s$ is forced along the path $P_t$, until the entire graph is forced in $t-1$ time steps. □

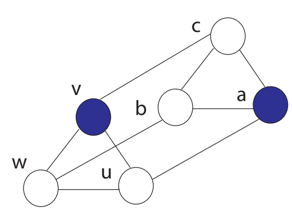



Figure 2.14 The cartesian product $K_3\square P_2$ has $\mathrm{Z}^-(K_3\square P_2) = 2$ and $\mathrm{pt}^-(K_3\square P_2) = 2$.

Recall that $W_n$ is used to denote the wheel on $n$ vertices, which is formed by joining a single isolated vertex $u$ to the cycle $C_{n-1}$.

**Proposition 2.5.8.** *For $n \geq 5$, the wheel $W_n$ has $\mathrm{Z}^-(W_n) = 2$ if $n$ is even, $\mathrm{Z}^-(W_n) = 3$ if $n$ is odd and*

$$\mathrm{pt}^-(W_n) = \begin{cases} \frac{n-2}{2} & \text{if } n \text{ is even} \\ \frac{n-3}{4} & \text{if } n \text{ is odd of the form } 4k+3 \text{ for some integer } k \\ \frac{n-1}{4} & \text{if } n \text{ is odd of the form } 4k+1 \text{ for some integer } k \end{cases}$$

*Proof.* By Proposition 2.2.28, $\mathrm{Z}^-(W_n) = \mathrm{Z}^-(C_n) + 1$. Therefore the wheel $W_n$ has $\mathrm{Z}^-(W_n) = 2$ if $n$ is even and $\mathrm{Z}^-(W_n) = 3$ if $n$ is odd. By Proposition 2.2.29, $\mathrm{pt}^-(W_n) = \mathrm{pt}^-(C_{n-1})$. Furthermore, if $n$ is odd and $n-1$ is even of the form $4k+2$ then $\mathrm{pt}^-(W_n) = \mathrm{pt}^-(C_{n-1} \vee K_1) = \frac{n-3}{4}$. If $n$ is odd and $n-1$ is even of the form $4k$, then $\mathrm{pt}^-(W_n) = \mathrm{pt}^-(C_{n-1} \vee K_1) = \frac{n-1}{4}$. If $n$ is even and $n-1$ is odd then $\mathrm{pt}^-(W_n) = \mathrm{pt}^-(C_{n-1} \vee K_1) = \frac{n-1-1}{2} = \frac{n-2}{2}$. □

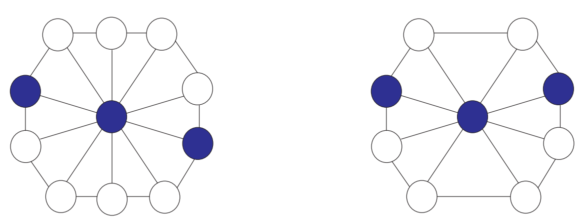


Figure 2.15 The wheel on 11 vertices has $\mathrm{Z}^-(W_{11}) = 3$, and $\mathrm{pt}^-(W_{11}) = 2$. The wheel on 9 vertices has $\mathrm{Z}^-(W_9) = 3$, and $\mathrm{pt}^-(W_9) = 2$.

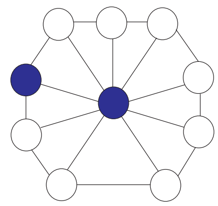


Figure 2.16 The wheel on 10 vertices has $\mathrm{Z}^-(W_{10}) = 2$, and $\mathrm{pt}^-(W_{10}) = 4$.

The last graphs we consider here are trees. The following leaf stripping algorithm for finding the skew zero forcing number of a tree was established in [22].

**Algorithm 2.5.9.** (Leaf-stripping)

**Input:** Graph $G$

**Output:** Graph $\hat{G}$ with $\delta(\hat{G}) \neq 1$, or $\hat{G} = \emptyset$

**Begin:**

1. $\hat{G} \leftarrow G$

2. **While** $\hat{G}$ has a leaf $u$ with neighbor $v$:

   $$\hat{G} \leftarrow \hat{G} - \{u, v\}$$

**Return:** $\hat{G}$

**Remark 2.5.10.** By [[23], Theorem 3.25], $\mathrm{Z}^-(G) = \mathrm{Z}^-(\hat{G})$ where $\hat{G}$ is produced by the leaf-stripping algorithm. In the case of a tree $T$, $\hat{T}$ will be a (possibly empty) set of isolated vertices, and thus $\mathrm{Z}^-(T) = |\hat{T}|$. Furthermore, $|\hat{T}| = |T| - 2{\cdot}\mathrm{match}(T)$.

**Observation 2.5.11.** *Consider the output $\hat{T}$ of Algorithm 2.5.9 applied to some tree $T$. Then $T - V(\hat{T})$ is a forest with a perfect matching.*

Since we are interested in skew propagation time, and since in the skew case, multiple forces may occur at each time step, dealing with simultaneous leaf-stripping makes more sense. Also, we want to use the knowledge we have about blue vertices by removing them first. In skew propagation time, any blue vertex that does not perform a force can be deleted without affecting the propagation time.

**Algorithm 2.5.12.** (Simultaneous leaf-stripping for forests with a perfect matching)

**Input:** A forest $T$ that has a perfect matching

**Output:** A number of steps $s$

**Begin:**

1. $s \leftarrow 0$
2. $\hat{T} \leftarrow T$
3. **While** $V(\hat{T}) \neq \emptyset$:
   (a) $L \leftarrow$ the set of leaves of $\hat{T}$
   (b) $N \leftarrow$ the set of neighbors of leaves in $\hat{T}$
   (c) $\hat{T} \leftarrow \hat{T} - (L \cup N)$
   (d) $s \leftarrow s + 1$

**Return:** $s$

Suppose we run the leaf-stripping algorithm on a tree $T$. Then in the resulting minimum skew zero forcing set $V(\hat{T})$ for $T$, the vertices in this set do not need to perform a force in some skew forcing process on $T$. That is, $T - V(\hat{T})$ is self forcing. Hence, $\mathrm{pt}^-(T) \leq \mathrm{pt}^-(T - V(\hat{T})) \leq 2s$ where the number of steps $s$ is the output of Algorithm 2.5.12, because there are at most $s$ steps at which white vertex forcing occur, and then at most another $s$ additional time steps for blue vertex forces. This establishes the next result.

**Theorem 2.5.13.** *For any tree $T$, it is the case that $\mathrm{pt}^-(T) \leq 2s$ where $s$ is the result of Algorithm 2.5.12 applied to $T - V(\hat{T})$ with $\hat{T}$ the result of the leaf-stripping algorithm applied to $T$.*

Observe that there are often several choices involved in applying Algorithm 2.5.9 to a tree, and said choices may result in different values of $s$ returned by the Algorithm 2.5.12. Consider a path on an odd number of vertices to see this. Intuition and examples suggest that a good (i.e., small) value of $s$ is likely to be obtained by stripping all the original leaf vertices from $T$, then all the leaves in the next round, etc. This is illustrated in Example 2.5.16 below.

Now we show that this upper bound for the skew zero forcing number of a tree is tight in some examples, but also that there exist trees with skew propagation time strictly less than $2s$.

**Example 2.5.14.** Consider the star $S$ on 4 vertices. Then a minimum skew zero forcing set for $S$ is any 2 of the leaves of the graph, and $\mathrm{pt}^-(S) = 1 < 2$, where $s = 1$ the output from Algorithm 2.5.12 implies $\mathrm{pt}^-(S) < 2s$.

**Example 2.5.15.** Consider the comb $C$ on 6 vertices. The result of running Algorithm 2.5.9 on $C$ is the empty set. However, Algorithm 2.5.12 also yields $s = 1$ in the case of the comb. Hence $2s = 2 = \mathrm{pt}^-(C)$.

We illustrate better the processes and outputs of Algorithms 2.5.9 and 2.5.12 by applying these algorithms sequentially starting with a tree $T$ on a larger number of vertices and in Algorithm 2.5.9 first removing all original leaves, then removing the set of new leaves, and so on.

**Example 2.5.16.** Consider the following tree $T$ on 20 vertices:

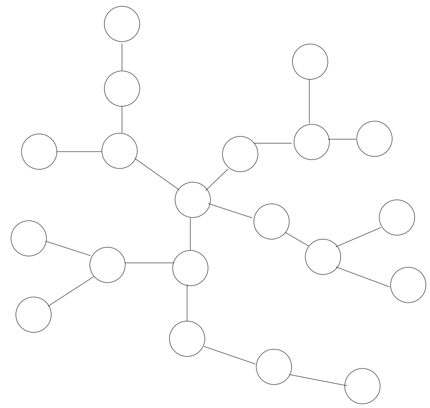


Figure 2.17 A tree $T$ on 20 vertices.

The first step is to run the tree in Figure 2.17 through Algorithm 2.5.9. The input is the tree $T$, and the final output is $\hat{T}$ such that $T - V(\hat{T})$ is a forest with a perfect matching. We can use the opportunity of having certain choices in Algorithm 2.5.9 to strip all of the original leaves of the tree first, as shown in Figure 2.19. Thus we choose to use the first 6 iterations in Algorithm 2.5.9 to remove all the original leaves of the graph (colored red) and the white neighbors of these leaves (colored black) as shown in Figure 2.18. The vertices to be removed in the $7^{th}$ and $8^{th}$ iterations in Algorithm 2.5.9 are colored red and black accordingly in Figure 2.20. The remaining set of white vertices $\hat{T}$ is the final output of Algorithm 2.5.9, and forms a skew zero forcing set for $T$. This set is shown alongside the original tree in Figure 2.21.

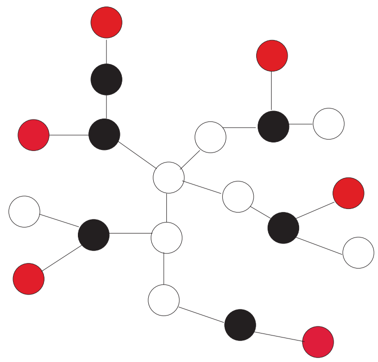


Figure 2.18 The tree $\hat{T}$ with leaves to be removed by 6 iterations of Algorithm 2.5.9 colored red, and the neighbors of the leaves to be removed from the graph colored black.

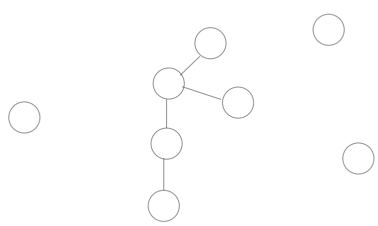


Figure 2.19 The forest $\hat{T}$ with the colored vertices in Figure 2.18 removed.

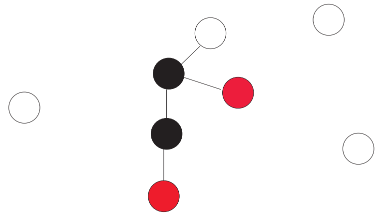


Figure 2.20 The forest $\hat{T}$ with 2 leaves colored red and their white neighbors, as per the $7^{th}$ and $8^{th}$ iterations of Algorithm 2.5.9, colored black.

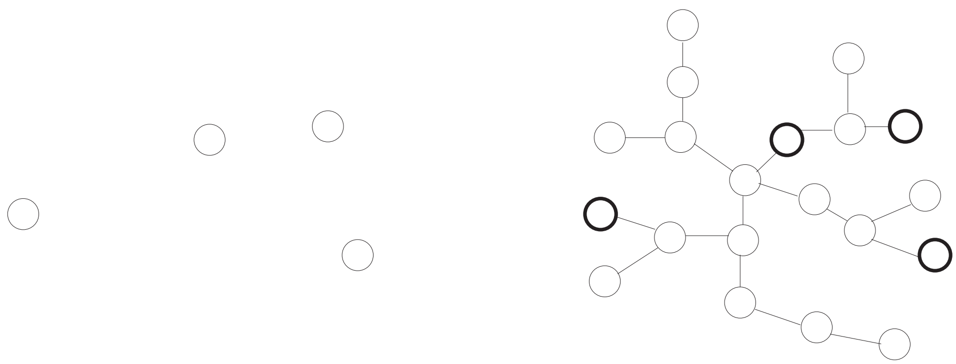


Figure 2.21 The forest $\hat{T}$ with the vertices colored during the $7^{th}$ and $8^{th}$ iterations of the Algorithm 2.5.9 removed from the graph and the resulting skew zero forcing set $B$ shown in bold on the original tree $T$.

We take $T - V(\hat{T})$, shown in Figure 2.22, and use that as the input for Algorithm 2.5.12. The first step of Algorithm 2.5.12 results in the tree shown in Figure 2.24, and $s = 1$. The leaves and vertices removed during the first six steps of the algorithm were colored red for the leaves and black for the leaf neighbors in Figure 2.23. The vertices colored red for the leaves and black for the leaf neighbors at the second step of the algorithm are shown in Figure 2.25. Thus the output of Algorithm 2.5.12 is the number $s = 2$, so that $2s = 4$ is an upper bound for the number of skew time steps that it takes to force the original tree $T$.

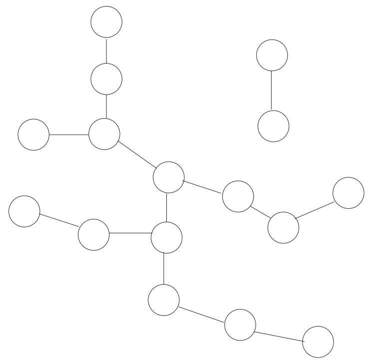


Figure 2.22 The forest $T - V(\hat{T})$, which is the input for Algorithm 2.5.12: At this point $\hat{T} \leftarrow (T - V(\hat{T}))$ and $s = 0$.

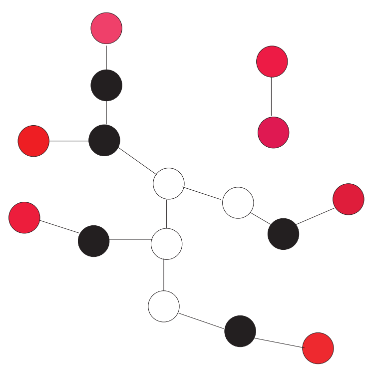


Figure 2.23 The graph $\hat{T}$ such that the set of leaves $L$ is made up of the red vertices, and the set $N$ is made up of the black vertices.

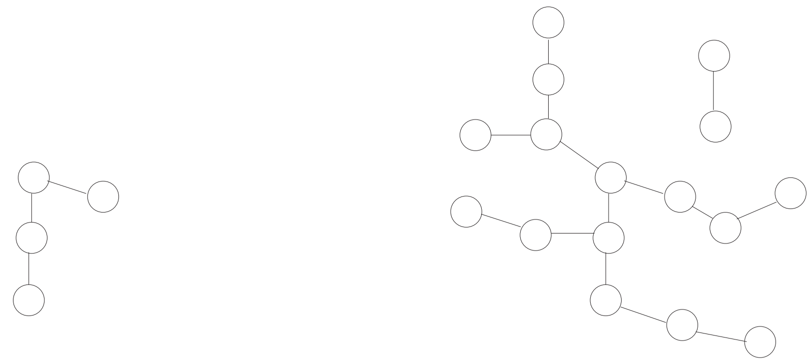


Figure 2.24 The graph $\hat{T} \leftarrow \hat{T} - (L \cup N)$ and $s = 1$ and the original forest $\hat{T}$.

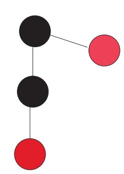


Figure 2.25 The graph $\hat{T}$ with new sets with vertices $N$ and $L$ colored black and red, respectively.

We now illustrate the skew zero forcing process on this tree. We start with the skew zero forcing set found using Algorithm 2.5.9 and proceed by forcing the graph as follows in the subsequent figures. Figure 2.26 shows the vertices forced at time step 1 labeled. Figure 2.27 shows the vertices forced at time step 2 labeled. Figure 2.28 shows the vertices forced at time step 3 labeled, and Figure 2.29 shows the vertices forced at time step 4 labeled.

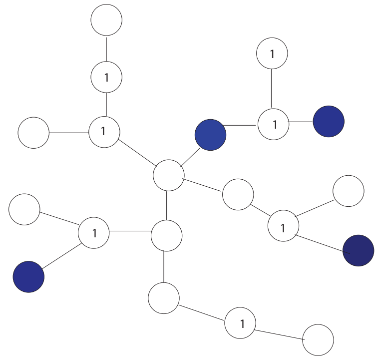



Figure 2.26 The vertices in $V(T)$ to be colored at skew time step 1 labeled.

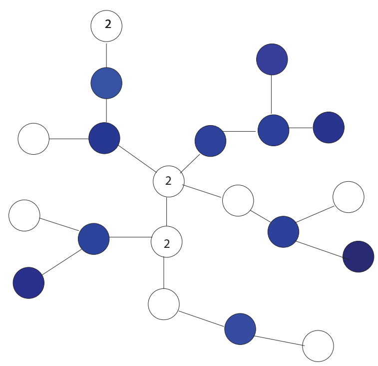



Figure 2.27 The vertices in $V(T)$ to be colored at skew time step 2 labeled.

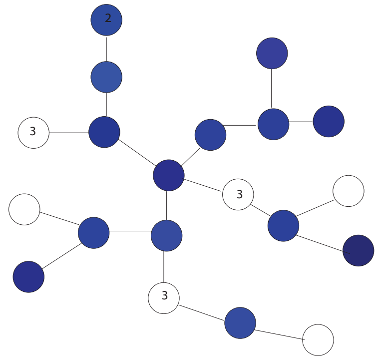



Figure 2.28  The vertices in $V(T)$ to be colored at skew time step 3 labeled.

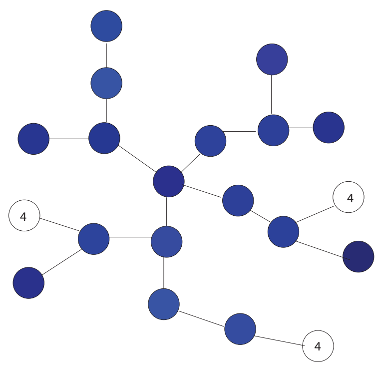



Figure 2.29  The vertices in $V(T)$ to be colored at skew time step 4 labeled.

Thus the propagation time for the original tree $T$ with skew zero forcing set $B$ as shown in Figure 2.21 is $\mathrm{pt}(T, B) = 4$. Thus $\mathrm{pt}^{-}(T) \leq 4$.

## 2.6 Summary table and use of the open source software SAGE

The table at the end of this section gives a summary of the minimum and maximum skew propagation times established in the previous section on select graphs and graph families.

Although the results in the previous section were obtained using standard combinatorial graph theory techniques, use of the open source software program SAGE was invaluable in verifying the suggesting values of certain results and performing additional computations.

For example, the following result was obtained by using a brute force technique in the software SAGE and checking each graph on 6 or fewer vertices for any graph without a full skew propagation time interval.

**Proposition 2.6.1.** *If* $|G| \leq 6$*, then* $G$ *has a full skew propagation time interval.*

Table 2.1 Summary table of skew propagation time of select graphs.

| Graph $G$ | $\mathrm{Z}^-(G)$ | $\mathrm{pt}^-(G)$ | $\mathrm{PT}^-(G)$ | Full Interval | Result |
|---|---|---|---|---|---|
| $H_s$ | 0 | $2s-1$ | $2s-1$ | yes | 2.4.20 |
| $K_{n_1,n_2,\ldots,n_s}$ | $n_1+n_2+\cdots+n_s-2$ | 1 | 1 | yes | 2.5.1 |
| $P_n$ | 0<br>1 | $\begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \lfloor\frac{n+1}{4}\rfloor & \text{if } n \text{ is odd} \end{cases}$ | $\frac{n}{2}$<br>$\frac{n-1}{2}$ | yes | 2.5.2 |
| $C_n$ | 1<br>2<br>2 | $\begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd} \\ \frac{n}{4} & \text{if } n \text{ is even and } 4k \text{ for some integer } k \\ \frac{n-2}{4} & \text{if } n \text{ is even and } 4k+2 \text{ for some integer } k \end{cases}$ | $\frac{n-1}{2}$<br>$\frac{n}{4}$<br>$\frac{n-2}{4}$ | yes | 2.5.3 |
| $P_{m,k}$ | $m+k-4$ | 3 | 3 | yes | 2.5.4 |
| $G \circ K_1$ | 0 | 2 | 2 | yes | 2.5.5 |
| $Q_n$ | $2^{n-1}$ | 1 | ? | ? | 2.5.6 |
| $K_3 \square P_2$ | 2 | 2 | ? | ? | 2.5.7 |
| $K_s \square P_t, s \geq 3, t \geq 3$ | s | $t-1$ | ? | ? | 2.5.7 |
| $W_n$ | 2<br>3<br>3 | $\begin{cases} \frac{n-2}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{4} & \text{if } n \text{ is odd and } 4k+1 \text{ for some integer } k \\ \frac{n-3}{4} & \text{if } n \text{ is odd and } 4k+3 \text{ for some integer } k \end{cases}$ | ? | ? | 2.5.8 |

# CHAPTER 3. SOME RESULTS ON THE PROPAGATION TIME OF LOOP GRAPHS THAT ALLOW LOOPS

## 3.1 Loop graph zero forcing and propagation time

In this section we consider briefly the propagation time of certain graphs that allow loops. This is a more complicated problem than the study of skew propagation time because the number of graphs that allow loops over a given simple labeled graph on $n$ vertices is $2^n$, since each vertex may or may not contain a loop.

For a loop graph $\mathfrak{G} = (\mathfrak{V}, \mathfrak{E})$ we define the set

$$\mathcal{S}_\ell(\mathfrak{G}) = \{A | A^T = A \text{ and } a_{ij} \neq 0 \text{ if and only if} \{i,j\} \in E\},$$

the *loop minimum rank* of a graph $\mathfrak{G}$ to be

$$\operatorname{mr}_\ell(\mathfrak{G}) = \min\{\operatorname{rank} A | A \in \mathcal{S}_\ell(\mathfrak{G})\},$$

and the *loop maximum nullity* of $\mathfrak{G}$ to be

$$\operatorname{M}_\ell(\mathfrak{G}) = \max\{\operatorname{null} A | A \in \mathcal{S}_\ell(\mathfrak{G})\}.$$

Loop graph zero forcing is based on the following *color change rule*: for a loop graph $\mathfrak{G}$ in which some or none of the vertices are colored blue, if each vertex of $\mathfrak{G}$ is colored either white or blue, and vertex $v$ is a vertex with only one white neighbor $w$, then change the color of $w$ to blue. In fact, the color change rule remains the same as it was for skew zero forcing; the difference now is that a single vertex $v$ with a loop can force itself if $v$ is white and all the other neighbors of $v$ are blue. A *loop zero forcing set* for a graph $\mathfrak{G}$ is an initial set $B$ of blue vertices such that the set of blue vertices that results from applying the color change rule until no more changes are possible is the entire set of vertices of $\mathfrak{G}$. As with standard and skew zero forcing, applying the color change rule to a vertex $v$ with single white neighbor $w$ is called a *force*, and we write $v \to w$ to say that $v$ forces $w$. A *minimum loop zero forcing set* of a graph $\mathfrak{G}$ is a zero forcing set of the smallest possible cardinality, and the *loop zero forcing number* $\mathrm{Z}_\ell(\mathfrak{G})$ is $|B|$ where $B$ is a minimum loop zero forcing set.

The loop zero forcing number is an upper bound for loop maximum nullity, and $\mathrm{mr}_\ell(\mathfrak{G}) + \mathrm{M}_\ell(\mathfrak{G}) = |\mathfrak{G}|$ [20].

**Definition 3.1.1.** *Let $\mathfrak{G} = (\mathfrak{V}, \mathfrak{E})$ be a loop graph, and $B^\ell$ a loop zero forcing set of $\mathfrak{G}$. Define $B_0^\ell = B$, and for $t \geq 0$, let $B_{(t+1)}^\ell$ be the set of vertices $\{w\}$ for which there exists any vertex $v$ in the graph $G$, blue or white, such that $w$ is the only white neighbor of $v$ not in $\cup_{s=0}^{t} B_{(s)}^\ell$. The* loop propagation time *of $B^\ell$ in $\mathfrak{G}$, denoted* $\mathrm{pt}_\ell(\mathcal{G}, B^\ell)$, *is the smallest integer $t_0$ such that $V_G = \cup_{t=0}^{t_0} B_{(t)}^\ell$.*

**Definition 3.1.2.** The *minimum loop propagation time of* $\mathfrak{G}$ is

$$\mathrm{pt}_\ell(\mathfrak{G}) = \min\{\mathrm{pt}_\ell(\mathfrak{G}, B^\ell) | B^\ell \text{ is a minimum loop zero forcing set of } \mathfrak{G}\}.$$

Maximum loop propagation time can be defined similarly to the standard and skew maximum propagation time cases, but for the remainder of this paper we are only interested in minimum loop propagation time, and will refer to it simply as loop propagation time.

**Observation 3.1.3.** *Loop graphs that contain loops can still have minimum zero forcing sets of size 0.*

**Example 3.1.4.** A single vertex with a loop has minimum loop graph zero forcing set $\emptyset$, since the vertex has only one white neighbor, itself, before the first time step, and hence is completely forced after the single vertex forces itself at time step 1. See Figure 3.1.

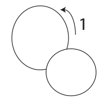



Figure 3.1 An isolated vertex with a loop, which has loop graph propagation time 1 and minimum loop zero forcing set $\emptyset$.

**Remark 3.1.5.** A set of isolated vertices $\mathfrak{G}$ with a loop on each vertex no longer has $\mathrm{Z}_\ell(\mathfrak{G}) = |\mathfrak{G}|$, since at at the first and only time step, each white vertex is its only white neighbor, and each vertex forces itself at time step 1 in the loop graph zero forcing process.

**Observation 3.1.6.** *Loop graphs still have propagation time 0 if and only if every vertex is in the loop zero forcing set, and so is initially colored blue. This holds for standard and skew propagation time as we have already seen. Therefore this is trivially low propagation time for loop graphs. This can only happen if the graph is a set of isolated vertices with no loops, which is the same as in the standard zero forcing case.*

**Remark 3.1.7.** For loop graph zero forcing, the loop propagation time can be equal to the order of the graph, unlike for standard zero forcing and skew zero forcing. Any path $\mathfrak{P}_\mathfrak{n}$ with a loop on exactly one of the endpoints has $\mathrm{pt}_\ell(\mathfrak{P}_\mathfrak{n}) = |\mathfrak{P}_\mathfrak{n}|$. It is easy to see that such a path has loop zero forcing number 0, because the loop zero forcing begins at the endpoint of the path that does not have a loop, and proceeds with every other vertex being forced by its only white neighbor, until the endpoint with the loop has itself as its only white neighbor if $n$ is odd, or the neighbor of the vertex with a loop forces it when $n$ is even. Then the loop zero forcing process proceeds with blue vertex forcing from the end of the path with the looped vertex to the other end of the path. Thus the empty set is an efficient loop zero forcing set, and it is the only minimum loop zero forcing set.

**Example 3.1.8.** Consider the family of loop leafed stars, which as the name suggests, is the graph on $n+1$ vertices formed by the join $K_1 \vee S$ with $S = \{s_1, s_2, \ldots, s_n\}$ a set of $n$ isolated vertices, each of which as a loop. We compare loop propagation time of the family of loop leafed stars and standard zero forcing with standard stars. In order for a star $K_{1,n}$ on $n+1$ vertices to be forced with standard zero forcing, at least $n-1$ vertices must be blue before any forcing can occur. This is because every leaf has the center vertex as a neighbor, but the center vertex can perform at most one force in the zero forcing process. Then at the first time step, any blue leaf can force the center vertex. Then at time step 2, the center vertex is blue and this vertex forces the remaining white leaf. So $\mathrm{Z}(K_{1,n}) = n-2$ and $\mathrm{pt}(K_{1,n}) = 2$.

Let $\mathfrak{G}$ be a loop leafed star on $n+1$ vertices for $n \geq 2$. Let the middle vertex be denoted by $v$, and the leaves of the star be denoted $s_i$ for $i = 1, \ldots, n$. A minimum loop zero forcing set for this loop leafed star is the vertex $v$, because it at least one vertex must be blue at the outset, and coloring $v$ is sufficient to force the graph. Thus $\mathrm{Z}_\ell(\mathfrak{G}) = 1$. The propagation process on this graph proceeds by each looped leaf $s_1$ through $s_n$, forcing itself in a single time step; in other words, $\mathrm{pt}_\ell(\mathfrak{G}) = 1$. Therefore the loop graph zero forcing set for $\mathfrak{G}$ is not a zero forcing set for the simple graph $G$ at all, let alone a minimum loop zero forcing set. In this example, using loop zero forcing has lowered the zero forcing number of the star and also decreased the propagation time for loop leafed stars.

As discussed in Chapter 2, the family of stars with no loops has skew zero forcing number $n-1$ in the same as in the standard zero forcing case, but the graph is forced in a single time step where the center vertex and the only white leaf force each other at time step 1.

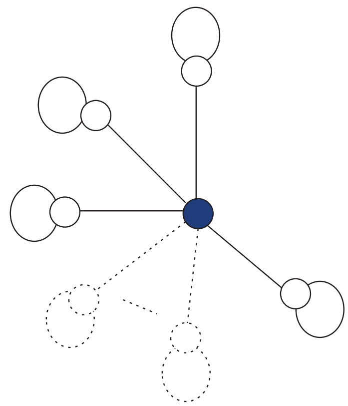


Figure 3.2 Loop leafed stars have propagation time 1, and coloring a single vertex is enough to propagate the entire graph.

# CHAPTER 4. SUMMARY AND CONCLUSIONS

## 4.1 General discussion

In this thesis we explored several topics regarding skew propagation time. We examined the history of standard and positive semidefinite propagation time, and how the skew propagation time problem came about. An interesting question regarding propagation time is whether or not the propagation time interval (of whatever type) is full for all graphs. That question was answered in the negative for skew propagation time in Example 2.3.2. Several tools for the study of skew propagation time were discussed in the early sections of this thesis.

Extremely high and low skew propagation time were studied. Low skew propagation time 1 is very difficult to categorize. It was proved in Theorem 2.4.20 that there is no graph $G$ such that the skew propagation time of $G$ is equal to $|G|$.

The last section on skew propagation time dealt with some common graph families. Their skew zero forcing numbers were given, the minimum skew propagation times were determined, and maximum skew propagation times were established for half graphs in Theorem 2.4.20, complete multipartite graphs in Proposition 2.5.1, paths in Proposition 2.5.2, cycles in Proposition 2.5.3, the $m, k$ pineapple graph in Proposition 2.5.4, and $G \circ K_1$ in Proposition 2.5.5. The skew zero forcing number for a tree is described using an algorithm from [22], and an upper bound for the skew propagation time of any tree is given in Theorem 2.5.13.

In the final section of the thesis, we introduced and discussed briefly loop graph zero forcing and loop propagation time.

## 4.2 Suggestions for additional research

There are many areas for future research in the study of skew propagation time, which include answering the following questions.

- Find more families of graphs with full skew propagation time intervals, or determine the minimum propagation time for more families.
- Do trees have a full skew propagation time interval?
- Can more graphs with high and low skew propagation times be categorized?
- Can trees with high or low propagation time be characterized?
- Can a trade off be established to optimize the minimum number of vertices it takes to achieve a certain skew propagation time?
- Are there applications of skew propagation time to current real life problems, for example, rumor spreading?

All of these are avenues for additional research that can and should be studied regarding skew propagation time. More work is being done on extreme skew propagation time, in particular Conjecture 2.4.22.

Finally, there are many open opportunities for further research into the propagation time of loop graphs where some, all, or none of the vertices allow loops, including extensions of loop leafed loop graphs. At the time of this writing, little is known about the propagation time of general loop graphs because of the vast number of loop graphs that can be constructed from a single underlying simple graph, each with their own properties, zero forcing numbers, and propagation times.

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