

Positive semidefinite propagation time

by

Nathan Joel Warnberg

A thesis submitted to the graduate faculty
in partial fulfillment of the requirements for the degree of
DOCTOR OF PHILOSOPHY

Major: Mathematics

Program of Study Committee:

Leslie Hogben, Major Professor

Steve Butler

Domenico D'Allessandro

Justin Peters

Mayly Sanchez

Michael Young

Iowa State University

Ames, Iowa

2014

Copyright © Nathan Joel Warnberg, 2014. All rights reserved.

DEDICATION

I would like to dedicate this thesis to my wife Katey without whose support I would not have been able to complete this work. I would also like to thank my friends and family for their support and guidance during the writing of this work.

TABLE OF CONTENTS

LIST OF TABLES	v
LIST OF FIGURES	vi
ABSTRACT	ix
CHAPTER 1. INTRODUCTION	1
1.1 Overview	1
1.2 Literature review	2
1.2.1 Zero forcing	5
1.2.2 Standard propagation time	9
1.2.3 Positive semidefinite zero forcing	14
1.3 Organization	20
BIBLIOGRAPHY	21
CHAPTER 2. POSITIVE SEMIDEFINITE PROPAGATION TIME	25
2.1 Positive semidefinite propagation time	25
2.2 Propagation time interval	32
2.3 Graph families	40
2.3.1 Cartesian products	42
2.3.2 Summary and use of software	48
2.4 Graphs with extreme propagation time	48
2.4.1 High propagation time	49
2.4.2 Low propagation time	54
CHAPTER 3. COMPUTING POSITIVE SEMIDEFINITE MINIMUM RANK FOR SMALL GRAPHS	61

3.1	Introduction	61
3.2	Known results used by the program to establish positive semidefinite minimum rank/maximum nullity	64
3.3	Computation of positive semidefinite maximum nullity of graphs of order 7 or less	69
CHAPTER 4. GENERAL CONCLUSIONS		78
4.1	General discussion	78
4.2	Recommendations for future research	78
ACKNOWLEDGEMENTS		80

LIST OF TABLES

2.1	Summary of Graph Family Results this is filler so I know how wide I can make my table	49
-----	--	----

LIST OF FIGURES

1.1	Graph G	3
1.2	3
1.3	3
1.4	Graph H	3
1.5	G	4
1.6	$G - c$	4
1.7	$G - cd$	4
1.8	Path P_4 , cycle C_6 , tree and complete graph K_6 , respectively.	4
1.9	Wheel W_7 , complement of K_6 , complete multipartite $K_{3,2,2}$ and $C_3 \square P_2$, respectively.	4
1.10	$\text{pt}(G, B) = 6$	10
1.11	$\text{pt}(G, B) = 6$	10
1.12	$\text{pt}(G, B) = 4$	10
1.13	$\text{pt}(G, B) = 4$	10
1.14	Graphs on two parallel paths with minimum zero forcing set B and $\text{pt}(G, B) < G - 2$ where dashed vertices and edges may be absent or repeated.	11
1.15	V_8	17
1.16	Supertriangle T_3	19
2.1	Graph G	28
2.2	The original graph G	28
2.3	An optimal forcing tree cover \mathcal{T}	28

2.4	30
2.5	30
2.6	$\text{pt}_+(G) = 3 > 2 = \text{pt}(G)$ $Z(G) = 4$, $Z_+(G) = 1$	31
2.7	$\text{pt}_+(G) = 2 < 3 = \text{pt}(G)$ $Z(G) = 1 = Z_+(G)$	31
2.8	$\text{PT}_+(G) = 2 < 3 = \text{PT}(G)$ $Z(G) = 2 = Z_+(G)$	31
2.9	$\text{PT}_+(G) = 4 > 3 = \text{PT}(G)$ $Z(G) = 2 = Z_+(G)$	31
2.10	$\text{pt}(G) = 4$, $\text{PT}(G) = 6$, and there is no minimum ZFS B with $\text{pt}(G, B) =$ 5.	32
2.11	$K_{4,4}$	36
2.12	Graph G	40
2.13	$G - u$	40
2.14	$G - u - v$	40
2.15	$P_s \square P_2$	43
2.16	Path	51
2.17	Not a directed path	51
2.18	Directed path	51
2.19	A zigzag graph with P_1 , P_2 and Q in black. Gray edges and vertices are optional	52
3.1	A path.	62
3.2	A cycle.	62
3.3	Example of a tree.	62
3.4	The complete graph on 7 vertices.	62
3.5	63
3.6	63
3.7	63
3.8	63
3.9	63
3.10	Graph G551	66

3.11	Graph G128	66
3.12	Graph G1196	67
3.13	Graph G43	67
3.14	Graph G419	67
3.15	67
3.16	67
3.17	67
3.18	67
3.19	Graph G200	68
3.20	Graph G1090	69
3.21	Graph G17	71
3.22	A , a matrix representation of $G17$	71
3.23	Möbius ladder on 8 vertices	71

ABSTRACT

Let G be a simple, undirected graph. Positive semidefinite (PSD) zero forcing on G is based on the following color-change rule: Let W_1, W_2, \dots, W_k be the sets of vertices of the k connected components in $G - B$ (where B is a set of blue vertices). If $w \in W_i$ is the only white neighbor of some $b \in B$ in the graph $G[B \cup W_i]$, then we change w to blue. A positive semidefinite zero forcing set (PSDZFS) is a set of blue vertices that colors the entire graph blue. The positive semidefinite zero forcing number, denoted $Z_+(G)$, is the minimum cardinality of a positive semidefinite zero forcing set. The PSD propagation time of a PSDZFS B of graph G is the minimum number of iterations that it takes to color the entire graph blue, starting with B blue, such that at each iteration as many vertices are colored blue as allowed by the color-change rule. The minimum and maximum PSD propagation times are taken over all minimum PSD zero forcing sets of the graph. The PSD propagation time interval of a graph G is the set of integers $[\text{pt}_+(G), \text{pt}_+(G) + 1, \dots, \text{PT}_+(G)]$. It is believed that every integer in the interval is achievable by some minimum PSDZFS. This thesis develops tools to analyze the minimum and maximum PSD propagation time, tools for analyzing the PSD propagation time interval and applies these tools to study the PSD propagation time of many graph families.

CHAPTER 1. INTRODUCTION

1.1 Overview

In this thesis all graphs will be simple and undirected (precise definitions appear at the beginning of the Literature Review). Imagine that in a graph G some of the vertices are blue and the rest of the vertices are white. We say that a blue vertex b can force (or color) a white vertex w if w is the only white neighbor of b . We can think of this as the blue vertices having a virus or knowing a rumor and then spreading the virus or rumor to the white vertices. By iteratively applying this color change rule sometimes we are able to force the entire graph blue. If this happens we say that our initial set of blue vertices was a zero forcing set. An interesting question soon arises: what is smallest number of blue vertices that is needed to force the entire graph blue? This smallest number is called the zero forcing number and we denote it by $Z(G)$ [2]. A related graph parameter is propagation time. To compute propagation time we: assume an initial set B forces the entire graph G blue; examine the vertices of $V(G) - B$ and determine which of them are the only neighbor of some blue vertex, i.e. they can be forced; perform this initial set of forces simultaneously and say that they occurred at the first time step; now we repeat on the larger set of blue vertices and continue in this fashion until the entire graph has been forced blue while keeping track of how many time steps occurred. This number is called the propagation time (or iteration index) of vertex set B on graph G and is denoted $pt(G, B)$ ($I(G)$)[18] ([9]). On the surface computing these parameters seems like an interesting combinatorial optimization problem, which it is. However, they also have connections to linear algebra and physics. In linear algebra the zero forcing number gives us a tool to work on a minimum rank problem [23]. See [2], [4], and [15] for an overview of the minimum rank problem and its connection to the zero forcing number. In physics both the zero

forcing number and propagation time are used to study controllability of quantum systems [6], [7], [8], [28]. This thesis focuses on similar graph parameters called the positive semidefinite zero forcing number [3] and the positive semidefinite propagation time. As was the case with the zero forcing number, the positive semidefinite zero forcing number is related to a minimum rank problem and is discussed in much detail in [3], [4], [14], [15] and [25].

1.2 Literature review

Basic graph theory

Graphs arise in many natural settings and can be used to model a range of situations. Some examples of a graphs that everyone is familiar with are the world wide web or any social network. Graphs also have a strong bond with linear algebra, which will be exploited throughout this thesis. We start with some basic definitions, examples and graph families.

A *graph* is a pair $G = (V, E)$ such that V is nonempty and finite and elements of E are unordered pairs of distinct elements of V . The elements of V form the *vertices* of G and the elements of E form the *edges* of G . The *order* of a graph is the number of vertices, written $|G|$. The vertex set and edge set of G are denoted $V(G)$ and $E(G)$. An edge can be represented as the set $\{u, v\}$ with $u, v \in V(G)$ but oftentimes we just write uv . If there is an edge between vertex v and vertex u then vertex u is a *neighbor* of vertex v and vice versa, these vertices are also said to be *adjacent* to one another. The set of all neighbors of vertex v in graph G is called the *neighborhood* of v and is denoted $N_G(v)$; when the context is clear we will use $N(v)$. The number of neighbors that a vertex v has is called the *degree* of v , denoted $\deg(v)$. A degree one vertex is called a *leaf*. The smallest degree of any vertex in a graph G is called the *minimum degree*, denoted $\delta(G)$. The *closed neighborhood* of vertex v is $N[v] = N(v) \cup \{v\}$. A vertex u is *universal* if $N[v] = V(G)$. Two vertices u and v are *duplicates* if $N[u] = N[v]$. A *subgraph* H of G is a graph with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The idea of an *induced subgraph* is also important. Given a subset S of $V(G)$ the subgraph induced by S , denoted $G[S]$, has vertex set S and if, for any $u, v \in S$ we have $uv \in E(G)$, then $uv \in E(G[S])$.

Example 1.2.1. In Figure 1.1 we have an example of graph G with vertex set $\{a, b, c, d, e, f, g, h\}$. This graph has order 7. Notice that $ab \in E(G)$ but $ac \notin E(G)$. The neighborhood of vertex c is $N_G(c) = \{d, e, h\}$ (thus $\deg(c) = 3$) and the closed neighborhood is $N_G[c] = \{c, d, e, h\}$. Vertex b is an example of a leaf, and $\delta(G) = 1$. G has no universal vertex or duplicate vertex but H , in Figure 1.4, does have duplicate vertices b and d . In Figure 1.2 we have a subgraph of G that is not an induced subgraph because edge ch is missing; as another example, G is a subgraph of H that is not induced. In Figure 1.3 we have the subgraph in G that is induced by vertex set $\{c, e, f, g, h\}$. The graph in Figure 1.3 is also the graph $G - \{a, b, d\}$ (defined shortly).

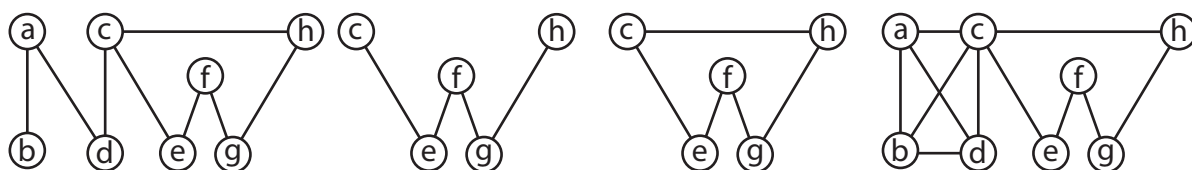
Figure 1.1: Graph G .

Figure 1.2

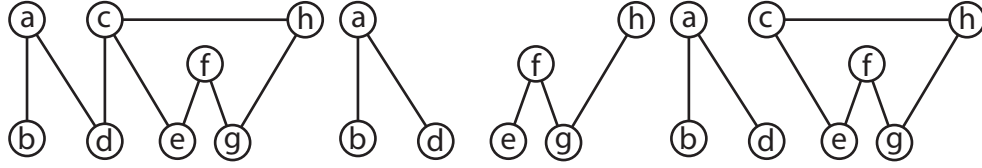
Figure 1.3

Figure 1.4: Graph H .

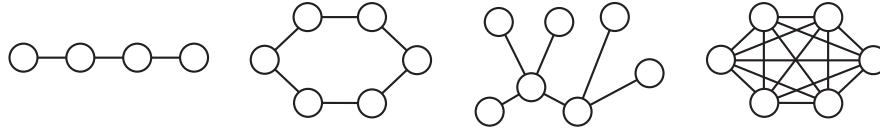
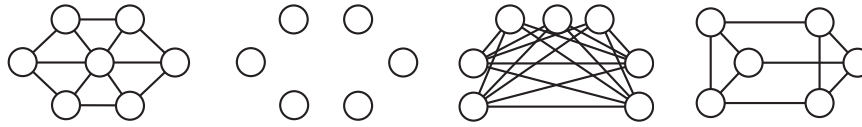
Sometimes we like to remove vertices from a graph. When this happens we also remove any edges that were adjacent to the removed vertices. More formally, if $S \subseteq V(G)$ then $G - S := G[V(G) \setminus S]$. If $S = \{v\}$ then we denote the resulting graph as $G - v$. Similarly, if e is an edge of graph G then the result of removing edge e is denoted $G - e$. A *path* is a sequence of vertices v_1, v_2, \dots, v_n such that edge $v_i v_{i+1}$ exists if $1 \leq i \leq n - 1$ and no vertex is repeated. If v_1, v_2, \dots, v_n is a path and $v_n v_1$ is also an edge, then we say that the graph has a *cycle*. A graph G is *connected* if there is a path between any $u, v \in V$ and is *disconnected* otherwise. A *connected component* W in a graph G is a maximal connected subgraph. A vertex $v \in V(G)$ is a *cut vertex* if $G - v$ has more connected components than G .

Example 1.2.2. In Figure 1.6 we have removed vertex c and its corresponding edges from graph G . In Figure 1.7 we have removed edge cd from G . Note that vertex c is a cut vertex because $G - c$ has two connected components, namely $G[\{a, b, d\}]$ and $G[\{e, f, g, h\}]$. In Figure 1.5 we also see that vertices c, e, f, g, h form a cycle in G and that a path exists between any two vertices so G is connected. Note that graph H in Figure 1.4 has several cycles.

Many common graph families will be discussed. If G has order n and consists only of a

Figure 1.5: G .Figure 1.6: $G - c$.Figure 1.7: $G - cd$

path it is denoted as P_n (Figure 1.8) and if G consists only of a cycle then it is denoted as C_n (Figure 1.8). A graph with exactly one cycle is *unicyclic*. If a graph G has no cycles then it is called a *forest* and if in addition it is connected it is called a *tree* (Figure 1.8). A graph G of order n is *complete* if every vertex is adjacent to every other vertex; we denote this by K_n (Figure 1.8).

Figure 1.8: Path P_4 , cycle C_6 , tree and complete graph K_6 , respectively.Figure 1.9: Wheel W_7 , complement of K_6 , complete multipartite $K_{3,2,2}$ and $C_3 \square P_2$, respectively.

The *union* of two graphs G and H is denoted $G \cup H$ and is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The *join* of two graphs G and H with disjoint vertex sets, denoted $G + H$, is $G \cup H$ and all edges between vertices of G and H . If u is a vertex that is not in $V(C_n)$ then we obtain the *wheel* graph by $C_n + u$ and denote it by W_{n+1} (Figure 1.9); note that u is an example of a universal vertex. The *complement* of a graph G , denoted \overline{G} , has vertex set $V(\overline{G}) = V(G)$ and edges $e \in E(\overline{G})$ only if $e \notin E(G)$ (Figure 1.9). A *complete bipartite* graph $K_{m,n}$ is $\overline{K_m} + \overline{K_n}$. A *complete multipartite* graph is $K_{p_1, p_2, \dots, p_n} = \overline{K_{p_1}} + \overline{K_{p_2}} + \dots + \overline{K_{p_n}}$ (Figure 1.9). The *Cartesian product* of graphs G and H , denoted $G \square H$, has vertex set $V(G) \times V(H)$ and has an edge between two vertices if they are identical in one coordinate and adjacent in the other (Figure 1.9). A *hypercube* is defined

iteratively where $Q_1 = K_2$ and $Q_n = Q_{n-1} \square K_2$.

1.2.1 Zero forcing

Now that we have established some basic terminology, we can discuss a graph parameter called the zero forcing number. Let G be a graph and $B \subseteq V(G)$ be a set of blue vertices and let the rest of the vertices be white. Then the *color change rule* is: if a blue vertex b has exactly one white neighbor w then vertex b forces vertex w to be colored blue, written $b \rightarrow w$. Given an initial set of blue vertices B we iteratively apply the color change rule until no more vertices can be forced blue and the set of blue vertices we end up with is called the *final coloring* (or *derived set*). Note that for a given graph G and an initial set of blue vertices B , the final coloring is unique. If the derived set of B is the entire vertex set of G then we say that B is a *zero forcing set*. The fewest number of initial blue vertices needed to force the entire graph blue is called the *zero forcing number* and is denoted $Z(G)$. The zero forcing number was introduced in [2] along with one of its most important relationships (Theorem 1.2.4). Physicists introduced the same parameter around the same time and referred to it as the *infection number* of a graph [7], [8], [28]. It was also shown in [6] that a zero forcing set of a graph also corresponds to control of a quantum system.

Next we exhibit a relationship between graph theory and linear algebra. Let $S_n(\mathbb{R})$ denote the real symmetric $n \times n$ matrices. For $A = [a_{ij}] \in S_n(\mathbb{R})$, the *graph of A* , denoted $\mathcal{G}(A)$, is the graph with vertices $\{1, 2, \dots, n\}$ and edges $\{ij : a_{ij} \neq 0 \text{ and } i \neq j\}$. The *maximum nullity* of a graph G of order n is

$$M(G) = \max\{\text{null}(A) : A \in S_n(\mathbb{R}) \text{ and } \mathcal{G}(A) = G\}.$$

The *minimum rank* of G is

$$\text{mr}(G) = \min\{\text{rank}(A) : A \in S_n(\mathbb{R}) \text{ and } \mathcal{G}(A) = G\}.$$

Example 1.2.3. To clarify the problem, consider the graph immediately below. The minimum rank question asks us to find the smallest rank over all matrices associated with the graph (the zero-nonzero pattern of the matrix immediately below) that are real and symmetric. Of course



checking the rank of all of the possible matrices would take forever, so over the years many techniques have been developed to help bound the minimum rank.

Notice that finding $\text{mr}(G)$ is equivalent to finding $M(G)$ as $\text{mr}(G) + M(G) = |G|$. We can also observe that upper or lower bounds on $M(G)$ yield lower or upper bounds on $\text{mr}(G)$ and vice versa. For a more thorough discussion of $\text{mr}(G)$ and $M(G)$ see [14] and [15]. These references also include much information on the relationship between the zero forcing number of a graph and minimum rank/maximum nullity. For an up to date list of graph parameters related to minimum rank see the American Institute of Mathematics Minimum Rank graph catalogue [1].

The following theorem was the driving force for the development of the zero forcing number.

Theorem 1.2.4. [2] *Let G be a graph. Then $M(G) \leq Z(G)$.*

Some lower bounds on the zero forcing number relate to other graph parameters, namely minimum degree, path cover number and clique cover number. The *path cover number* of a graph G , $P(G)$, is the fewest number of induced paths in G that cover all of the vertices of G . A *clique* in a graph G is a subgraph of G that is complete and the fewest number of cliques required to cover all of the edges of a graph G is called the *clique cover number*, denoted $cc(G)$.

Theorem 1.2.5. [5] *If G is a graph then $\delta(G) \leq Z(G)$.*

Theorem 1.2.6. [16] *If G is a graph then $P(G) \leq Z(G)$.*

Theorem 1.2.7. [14] $|G| - cc(G) \leq M(G) \leq Z(G)$.

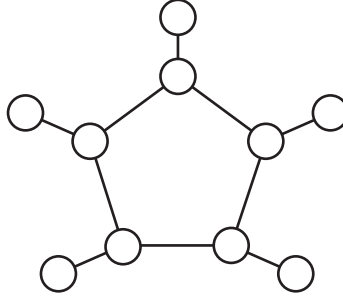
A Colin de Verdière-type parameter is more restrictive than the aforementioned maximum nullity parameter; in particular it requires that the matrix corresponding to the graph must

be real, symmetric and satisfy the Strong Arnold Hypothesis. For more about these types of parameters see [4], [10] and [14]. By using the zero forcing number as an upper bound many graph families were found to have $M(G) = Z(G)$ thus solving the minimum rank problem for those graph families. Techniques to find lower bounds on $M(G)$ included constructing a matrix that corresponded to the graph (sometimes using an orthogonal representation), using the Colin de Verdière-type parameter or using the clique cover number. The following is a non-exhaustive list of the graphs and graph families with $M(G) = Z(G)$ as established in [2]:

- If T is a tree then $M(T) = Z(T)$.
- $M(P_n) = Z(P_n) = 1$.
- $M(C_n) = Z(C_n) = 2$.
- $M(K_n) = Z(K_n) = n - 1$.
- The n th hypercube has $M(Q_n) = Z(Q_n) = 2^{n-1}$.
- The n th super triangle has $M(T_n) = Z(T_n) = n$.
- $M(K_s \square P_t) = Z(K_s \square P_t) = s$.
- $M(P_s \square P_t) = Z(P_s \square P_t) = \min\{s, t\}$.
- $M(C_s \square P_t) = Z(C_s \square P_t) = \min\{s, 2t\}$.
- $M(K_s \square K_t) = Z(K_s \square K_t) = st - s - t + 1$.
- $M(C_s \square K_t) = Z(C_s \square K_t) = 2t$ for $s \geq 4$.
- Any graph G with $|G| \leq 6$ has $M(G) = Z(G)$ (extended to $|G| \leq 7$ in [11]).

Note that there are graphs for which $M(G) < Z(G)$. For example, $C_5 \circ K_1$ (the pentasun), which is a 5-cycle with a leaf appended to each cycle vertex (see Figure immediately below). In this case we have $M(C_5 \circ K_1) = 2 < 3 = Z(C_5 \circ K_1)$ [4].

A *block* of a graph is a maximal connected subgraph that does not have a cut vertex. A *block-clique* graph is a graph whose blocks are all complete graphs. An *interval graph* is a



graph G for which we can associate with each vertex v an interval $I(v)$ in the real line such that two distinct vertices u and v are adjacent if and only if $I(u) \cap I(v) \neq \emptyset$. The set of intervals $\{I(v)\}_{v \in V(G)}$ is called the interval representation for G . A graph is a *unit interval graph* if it is an interval graph that has an interval representation in which all intervals have equal length.

Theorem 1.2.8. [20] *If G is a block-clique graph, then $Z(G) = M(G)$.*

Theorem 1.2.9. [20] *If G is a connected unit interval graph then $Z(G) = M(G)$.*

$P_m \square P_n$ is called a *grid graph* for obvious reasons. A graph is *chordal* if the largest induced cycle has length 3. A *triangular grid graph*, $P_m \boxtimes P_n$, is a grid graph that has been made chordal by adding a diagonal (in the same direction) of every square of the grid graph.

Theorem 1.2.10. [12] *If $m \leq n$ then $Z(P_m \boxtimes P_n) = M(P_m \boxtimes P_n) = m$.*

A characterization for $Z(G) = 1$ follows quickly from the fact that the minimum rank of a graph G of order n is $n - 1$ if and only if $G = P_n$ and that either leaf of the path is a zero forcing set [14]. Characterizations for $Z(G) = 2$ and $Z(G) = |G| - 1$ were discovered by Row.

Observation 1.2.11. *$Z(G) = 1$ if and only if $G = P_n$.*

A graph G is a *graph on two parallel paths* if there exist two independent induced paths of G that cover all the vertices of G and such that the graph can be drawn in the plane in such a way that the paths are parallel and edges (drawn as segments, not curves) between the two paths do not cross [21].

Theorem 1.2.12. [26] *$Z(G) = 2$ if and only if G is a graph on two parallel paths.*

Theorem 1.2.13. [26] *$Z(G) = |G| - 1$ if and only if $G = K_n$.*

Row also established the following results about cacti. A *cactus* is a graph such that every block is either a cycle or an edge. Another way to think of this is in a cactus graph any two cycles share at most one vertex.

Theorem 1.2.14. [27] *Let G be a cactus. Then $P(G) = Z(G)$.*

Theorem 1.2.15. [27] *Let G be a cactus in which each cycle has three vertices, an even number of vertices, or a vertex with only two neighbors. Then $Z(G) = M(G)$.*

1.2.2 Standard propagation time

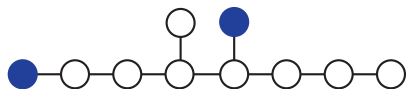
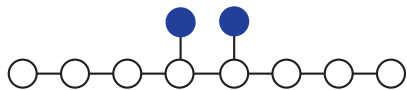
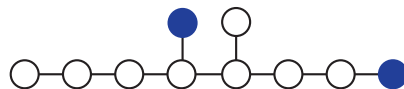
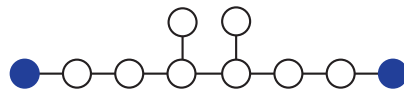
Recall that the propagation time is how long it takes an initial set of blue vertices to force (or color) the entire graph blue. The propagation time of a zero forcing set was introduced in [9] and [18]. The idea is that at each iteration of the zero forcing algorithm we look at the current set of blue vertices and determine which forces can occur independently and we say that they all occur at the same ‘time step.’ More precisely, let $G = (V, E)$ be a graph and B a zero forcing set of G . Define $B^{(0)} = B$, and for $t \geq 0$, $B^{(t+1)}$ is the set of vertices w for which there exists a vertex $b \in \bigcup_{s=0}^t B^{(s)}$ such that w is the only neighbor of b not in $\bigcup_{s=0}^t B^{(s)}$. The *propagation time* of B in G , denoted $\text{pt}(G, B)$ is the smallest integer t_0 such that $V = \bigcup_{t=0}^{t_0} B^{(t)}$.

The *minimum propagation time*, $\text{pt}(G)$, of a graph G is ‘fastest’ that G can be forced using a minimum zero forcing set and the *maximum propagation time*, $\text{PT}(G)$, is the ‘slowest’ that a graph can be forced using a minimum zero forcing set. Formally,

$$\text{pt}(G) = \min\{\text{pt}(G, B) \mid B \text{ is a min ZFS of } G\} \text{ and}$$

$$\text{PT}(G) = \max\{\text{pt}(G, B) \mid B \text{ is a min ZFS of } G\}.$$

The *propagation time interval* of a graph G is $[\text{pt}(G), \text{PT}(G)] = [\text{pt}(G), \text{pt}(G)+1, \dots, \text{PT}(G)-1, \text{PT}(G)]$. Based on a few examples, it was asked if this interval was always full, i.e. every integer in the interval could be realized as the propagation time of some minimum zero forcing set. An example was soon discovered that this was not the case. A smaller example, Figures 1.10 -

Figure 1.10: $\text{pt}(G, B) = 6$ Figure 1.12: $\text{pt}(G, B) = 4$ Figure 1.11: $\text{pt}(G, B) = 6$ Figure 1.13: $\text{pt}(G, B) = 4$

1.13, shows all of the minimum zero forcing sets for a graph and their associated propagation times. Note that we cannot achieve a propagation time of 5.

Some initial observations are based on the fact that at least one force must occur at each time step and that the maximum number of forces that can occur at any time step is based on the number of initial blue vertices. The propagation time of a graph is referred to as the *iteration index*, denoted $I(G)$, in [9]. In that paper the propagation time (iteration index) several graph families were analyzed.

Observation 1.2.16. [9],[18] *Let G be a graph. Then*

$$\frac{|G| - Z(G)}{Z(G)} \leq \text{pt}(G) \text{ and } \text{PT}(G) \leq |G| - Z(G).$$

An *efficient* zero forcing set B of a graph G has the property that B is a minimum zero forcing set and $\text{pt}(G, B) = \text{pt}(G)$. If we define $\text{Eff}(G)$ to be the set of all efficient zero forcing sets we have the following result, which says that we can always find more than one minimum zero forcing set that realizes the minimum propagation time.

Theorem 1.2.17. [18] *If G is a connected graph of order greater than one then $|\text{Eff}(G)| \geq 2$.*

Results on extreme propagation time have also been explored. Proposition 1.2.18, Lemma 1.2.19 and Lemma 1.2.20 use the characterization for $Z(G) = 1$ and the fact that at least one force must occur at each time step.

Proposition 1.2.18. [18] *For a graph G the following are equivalent:*

1. $\text{pt}(G) = |G| - 1$.
2. $\text{PT}(G) = |G| - 1$.

3. $Z(G) = 1$.
4. G is a path.

Lemma 1.2.19. [18] *Let G be a disconnected graph. Then the following are equivalent.*

1. $\text{pt}(G) = |G| - 2$.
2. $\text{PT}(G) = |G| - 2$.
3. $G = P_{n-1} \dot{\cup} P_1$.

Lemma 1.2.20. [18] *For a tree G , $\text{PT}(G) = |G| - 2$ if and only if G has one degree 3 vertex that has two leaves attached to it. The graph $K_{1,3}$ is the only tree with $\text{pt}(G) = |G| - 2$.*

The analysis of non-tree connected graphs with $\text{pt}(G) = |G| - 2$ was much more complicated. Any connected graph G that has a cycle and $\text{pt}(G) = |G| - 2$ must have $Z_+(G) = 2$ thus G is a graph on two parallel paths, but there are further restrictions.

Observation 1.2.21. [18] *If G is one of the graphs in Figure 1.14 then $\text{pt}(G) < |G| - 2$ because the black vertices are a minimum zero forcing set B with $\text{pt}(G, B) < |G| - 2$.*

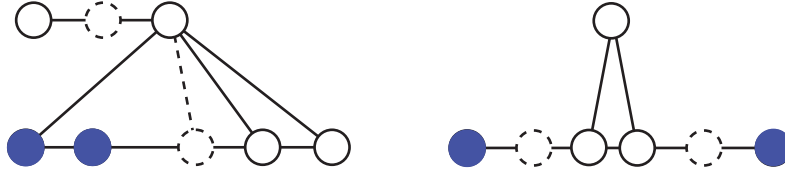


Figure 1.14: Graphs on two parallel paths with minimum zero forcing set B and $\text{pt}(G, B) < |G| - 2$ where dashed vertices and edges may be absent or repeated.

A graph on two parallel paths P_1 and P_2 is a *zigzag graph* if it satisfies the following conditions:

1. There is a path $Q = (z_1, z_2, \dots, z_\ell)$ that alternates between the two paths P_1 and P_2 such that:

- (a) $z_{2i-1} \in V(P_1)$ and $z_{2i} \in V(P_2)$ for $i = 1, \dots, \left\lfloor \frac{\ell+1}{2} \right\rfloor$;

- (b) z_j comes before z_{j+2} in their corresponding path
2. Every edge of G is from P_1 , P_2 , Q or is an edge of the form $z_j w$ where $1 < j < \ell$, w is in the opposite path as z_j , and z_{j-1} comes before w comes before z_{j+1} .

Theorem 1.2.22. [18] *Let G be a graph. Then $\text{pt}(G) = |G| - 2$ if and only if G is one of the following graphs.*

1. $P_{n-1} \dot{\cup} P_1$.
2. $K_{1,3}$.
3. A zigzag graph of order ℓ such that the following conditions are satisfied.
 - (a) G is not isomorphic to the graphs shown in Figure 1.14.
 - (b) $\deg(\text{first}(P_1)) > 1$ or $\deg(\text{first}(P_2)) > 1$ (both paths cannot begin with degree one vertices).
 - (c) $\deg(\text{last}(P_1)) > 1$ or $\deg(\text{last}(P_2)) > 1$ (both paths cannot end with degree one vertices).
 - (d) $z_2 \neq \text{first}(P_2)$ or $z_2 \sim \text{next}(z_1)$.
 - (e) $z_{\ell-1} \neq \text{last}(\text{path}(z_{\ell-1}))$ or $z_{\ell-1} \sim \text{prev}(z_\ell)$.

After studying high propagation time, looking at low propagation time is a natural step.

Observation 1.2.23. [18] *For a graph G the following are equivalent.*

1. $\text{pt}(G) = 0$.
2. $\text{PT}(G) = 0$.
3. $Z(G) = |G|$.
4. G has no edges.

Suppose $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ are graphs of equal order and $\mu : V_1 \rightarrow V_2$ is a bijection. Define the *matching graph* (H_1, H_2, μ) to be the graph constructed as the disjoint union of H_1 , H_2 and the perfect matching between V_1 and V_2 defined by μ .

Theorem 1.2.24. [18] *Any two of these conditions imply the third.*

1. $|G| = 2Z(G)$.
2. $\text{pt}(G) = 1$.
3. G is a matching graph.

The predictable structure of K_n was exploited for the next result.

Theorem 1.2.25. [18] *Let $|H| = n$ and μ be a bijection of vertices of H and K_n with μ acting on the vertices of H . Then $\text{pt}((H, K_n, \mu)) = 1$ if and only if H is connected.*

Next we see that a complete characterization of minimum zero forcing sets helps analyze the iteration index.

Observation 1.2.26. [9]

1. $\text{pt}(K_n) = 1$ for $n \geq 2$.
2. $\text{pt}(P_n) = n - 1$ for $n \geq 2$.
3. $\text{pt}(C_n) = \left\lceil \frac{n-2}{2} \right\rceil$ for $n \geq 3$.
4. $\text{pt}(K_{1,q}) = 2$ for $q \geq 2$, $\text{pt}(K_{p,q}) = 1$ for $p, q \geq 2$.

The next result takes advantage of the extreme bounds on the iteration index and finding an efficient minimum zero forcing set.

Theorem 1.2.27. [9]

1. For $t \geq s \geq 2$ $\text{pt}(P_s \square P_t) = t - 1$.
2. For $s, t \geq 2$ $\text{pt}(K_s \square P_t) = t - 1$.
3. For $s \geq 3$ and $t \geq 2$ $\text{pt}(C_s \square P_t) = \begin{cases} \left\lceil \frac{s-2}{s} \right\rceil & \text{if } s \geq 2t \\ t - 1 & \text{else} \end{cases}$.
4. For $s \geq 4$ and $t \geq 2$ $\text{pt}(C_s \square K_t) = \left\lceil \frac{s-2}{2} \right\rceil$.

Theorem 1.2.28. [9] For $s, t \geq 3$ $\text{pt}(K_s \square K_t) = 2$.

The next set of bounds are obtained by exhibiting a zero forcing sets that realize the given bound. The bounds are thought to be tight.

Theorem 1.2.29. [9] For $t \geq s \geq 2$ $\text{pt}(P_s \boxtimes P_t) \leq 2t + s - 4$.

Theorem 1.2.30. [9] For $s, t \geq 2$ $\text{pt}(P_s \boxtimes P_t) \leq s + t - 3$.

Here is an example of why nicely structured graphs are some of the first to be analyzed, namely their minimum zero forcing sets can be characterized and, usually, once you have a zero forcing set the propagation time is easily computed. For $2 \leq k_1 \leq k_2 \leq \dots \leq k_t$, let $B = (k_1, k_2, \dots, k_t)$ be a *bouquet* of $t \geq 2$ circles C^1, C^2, \dots, C^t with a cut vertex v where k_i is the number of vertices in $C^i - \{v\}$, so the i^{th} cycle is C_{k_i+1} .

Theorem 1.2.31. [9] For a bouquet of t circles $B = (k_1, k_2, \dots, k_t)$, $Z(B) = t + 1$.

Here a clever choice of zero forcing set yields the lowest propagation time.

Theorem 1.2.32. [9] For a bouquet of t circles $B = (k_1, k_2, \dots, k_t)$, $k_{i-1} \leq k_i$, $t \geq 2$ and $k_i \geq 2$ for all i then $\text{pt}(B) = \left\lceil \frac{k_t + k_{t-1}}{2} \right\rceil - 1$.

1.2.3 Positive semidefinite zero forcing

Like the standard zero forcing number, the positive semidefinite zero forcing number is a graph parameter that corresponds to a color change rule and was also developed to study minimum rank problems. We start with the positive semidefinite color change rule. In a graph G where some vertices are blue (call this set B) and the rest are white, the *positive semidefinite color change rule* is: Let W_1, W_2, \dots, W_k be the sets of vertices of the k connected components in $G - B$ (note we can have $k = 1$). If $w \in W_i$ is the only white neighbor of some $b \in B$ in the graph $G[B \cup W_i]$, then we change w to blue, say b forces w and write $b \rightarrow w$. Given an initial set of blue vertices B , we say the *final coloring* (or *derived set*) of B is the set of blue vertices that result from applying the positive semidefinite color change rule until no more forces are possible. Note that for a given graph G and an initial set of blue vertices B , the final coloring

is unique. A *positive semidefinite zero forcing set* (PSDZFS) of a graph G is a set of vertices B such that the final coloring of B is $V(G)$. The *positive semidefinite zero forcing number* of a graph G , denoted $Z_+(G)$, is the minimum of $|B|$ over all positive semidefinite zero forcing sets $B \subseteq V(G)$.

$$M_+(G) = \max\{\text{null}(A) : A \in S_n(\mathbb{R}) \text{ is positive semidefinite and } \mathcal{G}(A) = G\}.$$

The *positive semidefinite minimum rank* of G is

$$\text{mr}_+(G) = \min\{\text{rank}(A) : A \in S_n(\mathbb{R}) \text{ is positive semidefinite and } \mathcal{G}(A) = G\}.$$

One of main applications of the positive semidefinite zero forcing number is that it is an upper bound on the positive semidefinite maximum nullity of a graph [3]. Further, $\text{mr}_+(G) + M_+(G) = n$ so $n - Z_+(G)$ is a lower bound on positive semidefinite minimum rank. To establish this relationship we must first discuss orthogonal representations and the OS-number of a graph.

An *orthogonal representation* of a graph G with vertex set $\{v_1, v_2, \dots, v_n\}$ is a set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ in \mathbb{R}^n with the relationship: $\vec{v}_i \cdot \vec{v}_j = 0$ if $v_i v_j \notin E(G)$ and $\vec{v}_i \cdot \vec{v}_j \neq 0$ if $v_i v_j \in E(G)$. It is important to note that if we say matrix $X = [\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n]$, then X^*X is positive semidefinite, has rank equal to $\dim(\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\})$ and has $\mathcal{G}(X) = G$.

The OS number of a graph was developed in [17], using orthogonal representations of graphs, and it was shown to give a lower bound on the positive semidefinite minimum rank of a graph.

Let G be a graph and $S = \{v_1, v_2, \dots, v_m\}$ be an ordered set of vertices of G . Denote by G_k the subgraph of G induced by v_1, \dots, v_k for each k , $1 \leq k \leq m$. Let H_k be the connected component of G_k containing v_k . If for each k , $1 \leq k \leq m$ there exists $w_k \in V(G)$, $w_k \neq v_\ell$ for $\ell \leq k$, $w_k v_k \in E(G)$, $w_k v_s \notin E(G)$ for all $v_s \in V(H_k)$ with $s \neq k$, then S is called an *ordered subgraph OS-set* of vertices, or an *OS-set*. The *OS-number* of a graph G , denoted $\text{OS}(G)$, is defined to be the maximum of $|S|$ over all *OS-sets* S of G . The *OS-number* can be related to several well known graph parameters.

Theorem 1.2.33. [17] *Let G be a connected graph and let S be an OS-set in G . Then $|S| \leq \text{mr}_+(G)$. In particular, $\text{OS}(G) \leq \text{mr}_+(G)$.*

Theorem 1.2.34. [17] *If G is connected and chordal then $\text{OS}(G) = \text{mr}_+(G) = \text{cc}(G)$.*

Theorem 1.2.35. [22] *Let G be a connected graph. Then $OS(G) \leq |G| - \delta(G)$.*

There is also a strong connection between the OS-number and the positive semidefinite zero forcing number of a graph; in particular they complement one another.

Theorem 1.2.36. [3] *For any graph $G = (V, E)$ and any OS-set S , $V \setminus S$ is a positive semidefinite zero forcing set for G , and for any positive semidefinite zero forcing set X of G , there is an order that makes $V \setminus X$ an OS-set for G . Thus $Z_+(G) + OS(G) = |G|$.*

When we combine Theorem 1.2.35 and 1.2.36 we get Corollary 1.2.37.

Corollary 1.2.37. [3] *For every graph G , $\delta(G) \leq Z_+(G)$.*

Theorem 1.2.33 and 1.2.36 can be used to establish 1.2.38 but an alternate technique using the column inclusion principle is used. A matrix $A = [a_{ij}]$ has the *column inclusion principle*

if any vector column of the form $\begin{bmatrix} a_{i_1 j} \\ a_{i_2 j} \\ \vdots \\ a_{i_k j} \end{bmatrix}$ is in the column space of the principal submatrix $A[\{i_1, i_2, \dots, i_k\}]$ (which is the matrix obtained by keeping only the intersection of rows and columns $\{i_1, i_2, \dots, i_k\}$).

Theorem 1.2.38. [3] *For any graph G , $M_+(G) \leq Z_+(G)$.*

The positive semidefinite zero forcing number has been used to establish the maximum nullity of several graph families. One of the first applications of Theorem 1.2.38 was on graph products. It can be observed that a set of vertices associated with the same positive semidefinite zero forcing set in each copy of G or H is a positive semidefinite zero forcing set for $G \square H$. This result can be applied immediately and effectively to products with trees because it is readily shown that if T is a tree then $Z_+(T) = 1$.

Proposition 1.2.39. [3] *For all graphs G and H , $Z_+(G \square H) \leq \min\{Z_+(G)|H|, Z_+(H)|G|\}$.*

Corollary 1.2.40. [3] *If T is a tree and G is a graph then $Z_+(T \square G) \leq |G|$.*

In particular, this result is used to show that $M_+(T \square K_r) = Z_+(T \square K_r)$ by constructing a matrix that corresponds to $T \square K_r$ with nullity r .

Theorem 1.2.41. [3] *If T is a tree of order at least 2, then $M_+(T \square K_r) = Z_+(T \square K_r) = r$.*

Results from [22] tell us that the Möbius ladder on 8 vertices, V_8 , has $M_+(V_8) = 3 < 4 = Z_+(V_8)$ (see Figure 1.15).

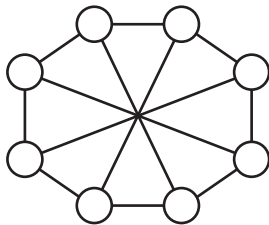


Figure 1.15: V_8

For the analysis of many more graph families and the development of more PSD zero forcing numbers see [25].

Some OS-number results from [22] can be combined with Theorem 1.2.36, to help establish corresponding results for Z_+ .

Corollary 1.2.42. [22] *Let G be a connected graph. For each $v \in V(G)$ there exist OS-sets S and S' such that $OS(G) = |S| = |S'|$ and $v \in S$ but $v \notin S'$.*

Theorem 1.2.43. [13] *If G is a graph and $v \in V(G)$, then there exist minimum positive semidefinite zero forcing sets B_1 and B_2 such that $v \in B_1$ and $v \notin B_2$.*

Theorem 1.2.44. [22] *If G is a connected graph with cut vertex v and G_1 and G_2 are the connected components of $G - v$ then $OS(G) = OS(G_1 \cup \{v\}) + OS(G_2 \cup \{v\})$.*

An induction proof shows that if G is connected with cut vertex v and G_1, G_2, \dots, G_k are the connected components of $G - v$ then $OS(G) = \sum_{i=1}^k OS(G_i \cup \{v\})$. This leads to the *cut-vertex reduction* formula for Z_+ . The cut vertex reduction formula is very useful for computing Z_+ since, like most graph parameters, Z_+ is much easier to compute on smaller graphs.

Theorem 1.2.45. [13] *If G is connected with cut vertex v and G_1, G_2, \dots, G_k are the connected components of $G - v$ then*

$$Z_+(G) = \sum_{i=1}^k Z_+(G_i \cup \{v\}) + k - 1.$$

Corollary 1.2.46. [13] *Suppose H is a graph, T is a tree and G is a graph in which H and T intersect at a single vertex, then $Z_+(G) = Z_+(H)$.*

The graph complement conjecture for graph parameter β , denoted GCC_β , is $\beta(G) + \beta(\overline{G}) \geq |G| - 2$. By first showing that the GCC for tree-width was true and noticing that tree-width is a lower bound on Z_+ [4] we get GCC_{Z_+} .

Corollary 1.2.47. [13] *For any graph G $Z_+(G) + Z_+(\overline{G}) \geq |G| - 2$.*

Graphs with very high and very low positive semidefinite zero forcing numbers were also characterized in [13]. Using $M_+(G) = 1$ if and only if G is a tree [19] if and only if $Z_+(G) = 1$, $M_+(G) \leq Z_+(G)$ and that any single vertex is a positive semidefinite zero forcing set for a tree we get Theorem 1.2.48.

Theorem 1.2.48. [3],[19] *Let G be a graph. The following are equivalent:*

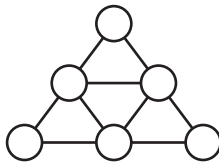
1. $M_+(G) = 1$.
2. $Z_+(G) = 1$.
3. G is a tree.

A *contraction* of edge $e = \{u, v\}$ is obtained by identifying vertices u and v , replacing any multiple edges by single edges, and deleting any loops that occur. The graph obtained by contracting edge e in graph G is denoted G/e . A *minor* of G is obtained by a series of edge deletions, vertex deletions and/or edge contractions.

Theorem 1.2.49. [13] *Let G be a graph. The following are equivalent.*

1. $M_+(G) = 2$.
2. $Z_+(G) = 2$.
3. *Either*

(a) G is a disjoint union of trees, or

Figure 1.16: Supertriangle T_3 .

(b) G is connected, exactly one block of G has a cycle, and G does not have a K_4 or T_3 minor. T_3 is shown in Figure 1.16.

Using the characterizations for $Z_+(G) = 1$ and $Z_+(G) = 2$ we readily have information for $Z_+(G) = 3$.

Corollary 1.2.50. [13] *If $Z_+(G) \leq 3$ then $Z_+(G) = M_+(G)$.*

The highest positive semidefinite zero forcing numbers are quite easy to characterize.

Observation 1.2.51. *Let G be a graph. The following are equivalent.*

1. $M_+(G) = |G|$.
2. $Z_+(G) = |G|$.
3. $G = \overline{K_n}$.

Observation 1.2.52. *Let G be a graph. The following are equivalent.*

1. $M_+(G) = |G| - 1$.
2. $Z_+(G) = |G| - 1$.
3. $G = K_n \cup \overline{K_s}$.

The next result uses the fact that for an induced subgraph H of G we have $|H| - Z_+(H) \leq |G| - Z_+(G)$ and that $|H| - Z_+(H) = 3$ for graphs P_4 , $K_{1,3}$, $P_3 \cup K_2$, $3K_2$, and the previous two results.

Theorem 1.2.53. [13] *Let G be a graph. The following are equivalent.*

1. $M_+(G) \geq |G| - 2$.

2. $Z_+(G) \geq |G| - 2$.

3. G has no induced $P_4, K_{1,3}, P_3 \cup K_2, 3K_2$.

The effects of graph operations have also been studied. In particular, we define $z_v^+(G) = Z_+(G) - Z_+(G - v)$ for vertex $v \in V(G)$ and similarly $z_e^+(G) = Z_+(G) - Z_+(G - e)$ for edge $e \in E(G)$.

Theorem 1.2.54. [13] *Let G be a graph with vertex v . Then $Z_+(G - v) \geq Z_+(G) - 1$ so $z_v^+(G) \leq 1$.*

There is no lower bound on $z_v^+(G)$ as seen by letting s get large in the star $K_{1,s}$. There is a useful result about duplicate vertices, in particular the removal of one duplicate vertex always reduces the positive semidefinite zero forcing number by one.

Theorem 1.2.55. [13] *If v and w are duplicate vertices in connected graph G with $|G| \geq 3$, then $Z_+(G - v) = Z_+(G) - 1$.*

Theorem 1.2.56. [13] *Let G be a graph with edge e . Then $-1 \leq z_e^+(G) \leq 1$.*

Theorem 1.2.57. [13] *Let G be a graph with edge $e \in E(G)$. Then $Z_+(G) - 1 \leq Z_+(G/e)$.*

The *subdivision* of edge $e = uv$ of G is denoted G_e , is the graph formed by removing edge e and adding a new vertex w that is adjacent to both u and v .

Theorem 1.2.58. [13] *Let G be a graph and $e \in E(G)$. Then $Z_+(G_e) = Z_+(G)$ and any positive semidefinite zero forcing set for G is a positive semidefinite zero forcing set for G_e .*

1.3 Organization

This dissertation is organized in the format of a dissertation containing journal papers. In the Introduction basic terminology is discussed and then a literature review discusses the history of the problem and brings the reader up to date with the current state of knowledge.

Chapter 2 contains the paper ‘‘Positive semidefinite propagation time’’ [30] submitted to *Discrete Applied Mathematics*. Recall that in [9] and [18] the propagation time is related to the

standard zero forcing color change rule. In Chapter 2 we explore positive semidefinite propagation time, which is related to the PSD zero forcing number. Many tools are developed to find both minimum PSD propagation time, $\text{pt}_+(G)$, and maximum PSD propagation time, $\text{PT}_+(G)$. The paper first looks at the PSD propagation time interval $[\text{pt}_+(G), \text{pt}_+(G) + 1, \dots, \text{PT}_+(G)]$, which is of interest because it is believed that the interval is full, which is not true for standard propagation time. Next the paper analyzes some well known graph families and determines minimum and maximum propagation time for them. The last part of the paper starts to characterize graphs with extreme propagation times $0, 1, |G| - 1$ and $|G| - 2$.

Chapter 3 contains the paper “Computing positive semidefinite minimum rank for small graphs” [24], which has been accepted to *Involve, A Journal of Mathematics*. In the paper a survey of current graph parameters related to positive semidefinite minimum rank and positive semidefinite zero forcing number are discussed. These parameters are then implemented in the mathematical software SAGE [29] and the program is able to establish that $M_+(G) = Z_+(G)$ for all but 13 graphs of order 7 or less. Orthogonal representations were used on the remaining 13 graphs and it was established that all graphs of order 7 or less have $M_+(G) = Z_+(G)$. At the time of submission it was known that there was a graph on 8 vertices with $M_+(G) < Z_+(G)$ but it was not known if it was the smallest such graph. The paper was authored by Nathan Warnberg and fellow graduate student Steven Osborne. It is included since it was the author’s first independent research experience and helped introduce him to positive semidefinite zero forcing.

Chapter 4 is for general conclusions and summary of the dissertation. It also includes suggestions for areas of future research.

Bibliography

- [1] AIM Minimum Rank Graph Catalog. <http://www.aimath.org/data/paper/AIM08/>.
- [2] AIM Minimum Rank – Special Graphs Work Group (F. Barioli, W. Barrett, S. Butler, S. M. Cioabă, D. Cvetković, S. M. Fallat, C. Godsil, W. Haemers, L. Hogben, R. Mikkelsen, S. Narayan, O. Pryporova, I. Sciriha, W. So, D. Stevanović, H. van der Holst, K. Vander

- Meulen, and A. Wangsness). Zero forcing sets and the minimum rank of graphs. *Lin. Alg. Appl.*, 428: 1628–1648, 2008.
- [3] F. Barioli, W. Barrett, S. Fallat, H.T. Hall, L. Hogben, B. Shader, P. van den Driessche, and H. van der Holst. Zero forcing parameters and minimum rank problems. *Lin. Alg. Appl.*, 433: 401–411, 2010.
- [4] F. Barioli, W. Barrett, S. Fallat, H.T. Hall, L. Hogben, B. Shader, P. van den Driessche, and H. van der Holst. Parameters related to tree-width, zero forcing, and maximum nullity of a graph. *J. Graph Theory*, 72: 146–177, 2013.
- [5] A. Berman, S. Friedland, L. Hogben, U.G. Rothblum, and B. Shader. An upper bound for the minimum rank of a graph. *Lin. Alg. Appl.*, 429: 1629–1638, 2008.
- [6] D. Burgarth, D. D’Alessandro, L. Hogben, S. Severini, and M. Young. Zero forcing, linear and quantum controllability for systems evolving on networks. *IEEE Trans. Auto. Control*, 58: 2349–2354, 2013.
- [7] D. Burgarth and V. Giovannetti. Full control by locally induced relaxation. *Phys. Rev. Lett.* PRL 99, 100–501, 2007.
- [8] D. Burgarth and Koji Maruyama. Indirect Hamiltonian identification through a small gateway. arXiv:0903.0612v1 [quant-ph], 2009.
- [9] K.B. Chilakamarri, N. Dean, C.X. Kang, and E. Yi. Iteration Index of a Zero Forcing Set in a Graph. arXiv:1105.1492, 2011.
- [10] Y. Colin de Verdière. On a new graph invariant and a criterion for planarity. In *Graph Structure Theory*, pp. 137–147, American Mathematical Society, Providence, RI, 1993.
- [11] L. DeLoss, J. Grout, L. Hogben, T. McKay, J. Smith, and G. Tims. Techniques for determining the minimum rank of a small graph. *Lin. Alg. Appl.* 432: 2995–3001, 2010.
- [12] C.J. Edholm, L. Hogben, M. Huynh, J. Lagrange, and D.D. Row. Vertex and edge spread of zero forcing number, maximum nullity, and minimum rank of a graph. *Lin. Alg. Appl.*, 436: 4352–4372, 2012.

- [13] J. Ekstrand, C. Erickson, H.T. Hall, D. Hay, L. Hogben, R. Johnson, N. Kingsley, S. Osborne, T. Peters, J. Roat, A. Ross, D.D. Row, N. Warnberg, and M. Young. Positive semidefinite zero forcing. *Lin. Alg. Appl.*, 439: 1862–1874, 2013.
- [14] S. Fallat and L. Hogben. The minimum rank of symmetric matrices described by a graph: A survey. *Lin. Alg. Appl.* 426: 558–582, 2007.
- [15] S. Fallat and L. Hogben. Minimum Rank, Maximum Nullity, and Zero Forcing Number of Graphs. In *Handbook of Linear Algebra*, 2nd ed. CRC Press, Boca Raton, FL, 2014.
- [16] L. Hogben. Minimum rank problems. *Lin. Alg. Appl.*, 432: 1961–1974, 2010.
- [17] P. Hackney, B. Harris, M. Lay, L.H. Mitchell, S.K. Narayan, and A. Pascoe. Linearly independent vertices and minimum semidefinite rank. *Lin. Alg. Appl.*, 431: 1105–1115, 2009.
- [18] L. Hogben, M. Hyunh, N. Kingsley, S. Meyer, S. Walker, M. Young. Propagation time for zero forcing on a graph. *Disc. Appl. Math.*, 160: 1994-2005, 2012.
- [19] H. van der Holst. On the maximum positive semi-definite nullity and the cycle matroid of graphs. *Elec. J. Lin. Alg.*, 18: 192–201, 2009.
- [20] L.-H. Huang, G.J. Chang, and H.-G. Yeh. On minimum rank and zero forcing sets of a graph. *Lin. Alg. Appl.*, 432: 2961–2973, 2010.
- [21] C.R. Johnson and J. Link. The extent to which triangular sub-patterns explain minimum rank. *Disc. Appl. Math.*, 156: 1637–1631, 2008.
- [22] L.H. Mitchell, S.K. Narayan, A.M. Zimmer. Lower bounds on minimum rank problems. *Lin. Alg. Appl.*, 432: 430-440, 2010.
- [23] P.M. Nysten. Minimum-rank matrices with prescribed graph. *Lin. Alg. Appl.*, 248: 303–316, 1996.
- [24] S. Osborne, N. Warnberg. Computing positive semidefinite minimum rank for small graphs. *Involve*, In Press.

- [25] T. Peters. Positive semidefinite maximum nullity and zero forcing number. *Elec. J. Lin. Alg.*, 23: 815–830, 2012.
- [26] D.D. Row. A technique for computing the zero forcing number of a graph with a cut-vertex. *Lin. Alg. Appl.*, 436: 4423–4432, 2012.
- [27] D.D. Row. Zero forcing number, path cover number and maximum nullity of cacti. *Involve*, 4(3): 277–291, 2011.
- [28] S. Severini. Nondiscriminatory propagation on trees. *J. Physics A*, 41: 482–002 (Fast Track Communication), 2008.
- [29] W. Stein. *Sage: Open-source mathematical software (Version: 5.4)*, The Sage Group, 2013.
<http://sagemath.org>
- [30] N. Warnberg. Positive Semidefinite Propagation Time. Under Review.

CHAPTER 2. POSITIVE SEMIDEFINITE PROPAGATION TIME

A paper submitted to *Discrete Applied Mathematics*.

Nathan Warnberg

Abstract

Let G be a simple, undirected graph. Positive semidefinite (PSD) zero forcing on G is based on the following color-change rule: Let W_1, W_2, \dots, W_k be the sets of vertices of the k connected components in $G - B$ (where B is a set of blue vertices). If $w \in W_i$ is the only white neighbor of some $b \in B$ in the graph $G[B \cup W_i]$, then we change w to blue. A minimum positive semidefinite zero forcing set (PSDZFS) is a set of blue vertices that colors the entire graph blue and has minimum cardinality. The PSD propagation time of a PSDZFS B of graph G is the minimum number of iterations that it takes to color the entire graph blue, starting with B blue, such that at each iteration as many vertices are colored blue as allowed by the color-change rule. The minimum and maximum PSD propagation times are taken over all minimum PSD zero forcing sets of the graph. It is conjectured that every propagation time between the minimum and maximum propagation time is attainable by some minimum PSDZFS (this is not the case for the standard color-change rule). Tools are developed that aid in the computation of PSD propagation time. Several graph families and graphs with extreme PSD propagation times $(|G| - 2, |G| - 1, 1, 0)$ are analyzed.

2.1 Positive semidefinite propagation time

The standard zero forcing number was introduced in [1] to aid in the study of minimum rank/maximum nullity problems. It was also studied independently by physicists, where it is known as graph infection or graph propagation in [4] and [13] respectively. The propagation time of a zero forcing set is defined as the number of iterations of the color change rule, coloring as many vertices per iteration as possible, needed to force an entire graph to be blue. Propagation time is implicit in [4] and explicit in [13] and is used to measure the time needed to gain control of a quantum system. In [7] and [9] propagation time is explored for numerous graph families as well as graphs that realize high and low propagation time. A natural extension of zero forcing is positive semidefinite zero forcing. Similar to zero forcing, positive semidefinite zero forcing was introduced to aid in the study of minimum rank/maximum nullity problems [2]. This paper takes a natural step and explores positive semidefinite propagation time. The current section introduces basic definitions and tools, which we will use throughout the paper. Section 2.2 investigates the positive semidefinite propagation time interval, which we conjecture is full for all graphs even though this is not the case for standard propagation time. Section 2.3 determines the positive semidefinite propagation time for some well known graph families. Section 2.4 investigates high and low positive semidefinite propagation times.

In this paper a graph is simple (no loops or multiple edges), finite and undirected. In a graph G where some vertices are blue (call this set B) and the rest are white, the *positive semidefinite color change rule* is: Let W_1, W_2, \dots, W_k be the sets of vertices of the k connected components in $G - B$ (note we can have $k = 1$). If $w \in W_i$ is the only white neighbor of some $b \in B$ in the graph $G[B \cup W_i]$, then we change w to blue, say b forces w and write $b \rightarrow w$. Given an initial set of blue vertices B we say the *final coloring* (or *derived set*) of B is the set of blue vertices that result from applying the positive semidefinite color change rule until no more forces are possible. Note that for a given graph G and an initial set of blue vertices B , the final coloring is unique. A *positive semidefinite zero forcing set* (PSDZFS) of a graph G is a set of vertices B such that the final coloring of B is $V(G)$. The *positive semidefinite zero forcing number* of a graph G , denoted $Z_+(G)$, is the minimum $|B|$ over all positive semidefinite zero forcing sets $B \subseteq V(G)$.

Let $S_n(\mathbb{R})$ denote the real symmetric $n \times n$ matrices. For $A = [a_{ij}] \in S_n(\mathbb{R})$, the *graph of*

A , denoted $\mathcal{G}(A)$, is the graph with vertices $\{1, 2, \dots, n\}$ and edges $\{ij : a_{ij} \neq 0 \text{ and } i \neq j\}$. The *maximum positive semidefinite nullity* of a graph G of order n is

$$M_+(G) = \max\{\text{null}(A) : A \in S_n(\mathbb{R}) \text{ is positive semidefinite and } \mathcal{G}(A) = G\}.$$

The *positive semidefinite minimum rank* of G is

$$mr_+(G) = \min\{\text{rank}(A) : A \in S_n(\mathbb{R}) \text{ is positive semidefinite and } \mathcal{G}(A) = G\}.$$

One of the main applications of the positive semidefinite zero forcing number is that it is an upper bound on the positive semidefinite maximum nullity of a graph [2]. Further, $mr_+(G) + M_+(G) = n$ so $n - Z_+(G)$ is a lower bound on positive semidefinite minimum rank.

Definition 2.1.1. Let $G = (V, E)$ be a graph and B a PSDZFS for G . Define $B_+^{(0)} = B$. For $t \geq 0$, let $W_t = \{W_{t_1}, W_{t_2}, \dots, W_{t_m}\}$ be the set of vertex sets of connected components of $G - \bigcup_{s=0}^t B_+^{(s)}$. Then $B_+^{(t+1)}$ is the set of vertices w such that $w \in W_{t_i}$, for some $i \in \{1, 2, \dots, m\}$,

and w is the only white neighbor of some $b \in \bigcup_{s=0}^t B_+^{(s)}$ in the graph $G \left[W_{t_i} \cup \left(\bigcup_{s=0}^t B_+^{(s)} \right) \right]$. The *positive semidefinite propagation time* of B in G , denoted by $\text{pt}_+(G, B)$, is the smallest integer t_0 such that $V(G) = \bigcup_{t=0}^{t_0} B_+^{(t)}$.

Definition 2.1.2. The *minimum positive semidefinite (PSD) propagation time* of G is

$$\text{pt}_+(G) = \min\{\text{pt}_+(G, B) : B \text{ is a minimum PSDZFS of } G\}.$$

Definition 2.1.3. The *maximum PSD propagation time* of G is

$$\text{PT}_+(G) = \max\{\text{pt}_+(G, B) : B \text{ is a minimum PSDZFS of } G\}.$$

Definition 2.1.4. If G is a graph then we say a minimum PSDZFS B is *efficient* if $\text{pt}_+(G, B) = \text{pt}_+(G)$.

Notice that if B and B' are PSDZF sets of a graph G and $B \subseteq B'$ then $\bigcup_{i=0}^t B_+^{(i)} \subseteq \bigcup_{i=0}^t B'_+{}^{(i)}$ and thus $\text{pt}_+(G, B') \leq \text{pt}_+(G, B)$. However, the question was raised for standard zero forcing, as well as positive semidefinite zero forcing, as to whether there is a graph with a non-minimum PSDZFS that has slower propagation time than every minimum PSDZFS. The

question was answered positively for standard zero forcing in [?]. The graph in Figure 2.1 answers the question positively for positive semidefinite zero forcing. Observe that $Z_+(G) = 3$ and $\text{PT}_+(G) = 3$ e.g. using $B = \{a, f, g\}$. However $\{a, b, c, f\}$ has a propagation time of 4.

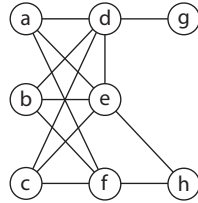


Figure 2.1: Graph G

As in [9] we will define a set of forces instead of the more often used chronological list of forces since when studying propagation time many forces occur simultaneously. Let G be a graph with PSDZFS B . At each time step perform as many forces as possible and as each force occurs put it into a *set of forces* \mathcal{F} . For each $b \in B$ define V_b to be the set of vertices y such that there is a sequence of forces $b = v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_k = y$ in \mathcal{F} (the empty sequence is permitted). The *forcing tree* T_b is the induced subgraph $G[V_b]$. The *forcing tree cover* (for the set of forces \mathcal{F}) is $\mathcal{T} = \{T_b \mid b \in B\}$. An *optimal forcing tree cover* is a forcing tree cover from a set of forces of a minimum PSDZFS [8].

Example 2.1.5. In this example we illustrate some of the previous definitions.

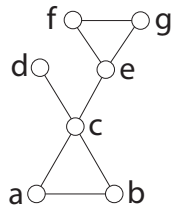


Figure 2.2: The original graph G .

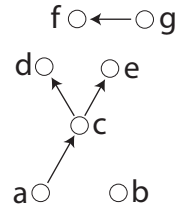


Figure 2.3: An optimal forcing tree cover \mathcal{T} .

First observe that $Z_+(G) = 3$ and $B = \{a, b, g\}$ is a minimum PSDZFS. Then $B_+^{(0)} = \{a, b, g\}$, $B_+^{(1)} = \{c\}$, $B_+^{(2)} = \{d, e\}$, $B_+^{(3)} = \{f\}$. Thus $\text{pt}_+(G, B) = 3$. One possible set of

forces is $\mathcal{F} = \{a \rightarrow c, c \rightarrow d, c \rightarrow e, g \rightarrow f\}$. We also have $\text{pt}_+(G) = 2$, e.g. using $\{a, c, g\}$, and $\text{PT}_+(G) = 3$.

The following observation allows us to concentrate, for the most part, on connected graphs.

Observation 2.1.6. *If G is disconnected with connected components C_1, C_2, \dots, C_k , then $\text{pt}(G) = \max\{\text{pt}(C_i)\}$, $\text{PT}(G) = \max\{\text{PT}(C_i)\}$, $\text{pt}_+(G) = \max\{\text{pt}_+(C_i)\}$ and $\text{PT}_+(G) = \max\{\text{PT}_+(C_i)\}$.*

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. If $U \subseteq V(G)$ then the *complement* of U is $\bar{U} = V(G) \setminus U$. A *trivial graph* has no edges, i.e. is a set of isolated vertices. The *degree* of a vertex v is the number of vertices that are adjacent to v , namely $\deg(v) = |\{u \in V(G) : uv \in E(G)\}|$. A vertex u is a *universal vertex* in a graph G if $\deg(u) = |G| - 1$. A *cut vertex* is a vertex whose removal would disconnect the graph. A *cut set* is a vertex set whose removal would disconnect the graph. The *neighborhood* of a vertex v in graph G , denoted $N_G(v)$ or $N(v)$, is the set of vertices adjacent to v . The *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. A subset of $D \subseteq V(G)$ is a *dominating set* if $N[D] := \bigcup_{v \in D} N[v] = V(G)$. The *distance* between vertices u and v , $d(u, v)$, is the length of the shortest path between u and v in G .

The *complete graph on n vertices* is denoted K_n and is the graph where $N[v] = V(G)$ for every vertex v in G . A graph is *bipartite* on $n + m$ vertices if $V(G)$ can be partitioned into two sets of cardinality n and m such that edges only occur between the two sets and not within the two sets. A bipartite graph $G = (V + W, E)$, $|V| = n$, $|W| = m$, is *complete bipartite*, denoted $K_{n,m}$, if edge vw exists for every $v \in V$ and $w \in W$. A *star* on $n + 1$ vertices is a complete bipartite graph of the form $K_{1,n}$.

Definition 2.1.7. Let G be a graph and B a PSDZFS with $\text{pt}_+(G, B) = t$. We will call the $t + 1$ -tuple $(|B_+^{(0)}|, |B_+^{(1)}|, \dots, |B_+^{(t)}|)$ the *forcing list sequence* of B and will denote it by $\text{FLS}(G, B)$.

In Example 2.1.5 $\text{FLS}(G, B) = (3, 1, 2, 1)$.

Remark 2.1.8. $\text{pt}_+(G, B)$ is equal to one less than the length of $\text{FLS}(G, B)$ so minimizing or maximizing one will minimize or maximize the other. Also, the sum of the elements in $\text{FLS}(G, B)$ is $|G|$.

Definition 2.1.9. Let G be a graph and $U, V \subseteq V(G)$ and $v \in V(G)$. Define the *distance from a set U to a vertex v* as $d(U, v) = \min\{d(u, v) \mid u \in U\}$ and the *distance between two sets $U, V \subseteq V(G)$* as $\max\{d(U, v) \mid v \in V\}$.

If B is a PSDZFS then $\text{dist}(B, \overline{B})$ is the distance between the farthest white vertex and its closest blue vertex. Using the graph and PSDZFS B from Example 2.1.5 we have $\text{dist}(B, \overline{B}) = \max\{\min\{d(a, c), d(b, c), d(g, c)\}, \min\{d(a, d), d(b, d), d(g, d)\}, \min\{d(a, e), d(b, e), d(g, e)\}, \min\{d(a, f), d(b, f), d(g, f)\}\} = \max\{1, 2, 1, 1\} = 2$.

Observation 2.1.10. If G is a graph and B is a PSDZFS then $\text{pt}_+(G, B) \geq \text{dist}(B, \overline{B})$.

Definition 2.1.11. We say a minimum PSDZFS B exhibits *full forcing* if for all $t = 0, 1, 2, \dots, \text{pt}_+(G, B) - 1$, $v \in B_+^{t'}$ implies $(N(v) - \bigcup_{t \leq t'} B_+^t) \in B_+^{t'+1}$ i.e. the white neighbors of the blue vertices are all forced at the next iteration of the forcing algorithm. A graph has *universal full forcing* if every minimum PSDZFS exhibits full forcing.

Example 2.1.12. The graph shown in Figure 2.4 has universal full forcing. In Figure 2.5 we have a graph where $B_1 = \{1, 6\}$ has full forcing but $B_2 = \{6, 9\}$ does not since 6 cannot force its neighbors at the first time step.

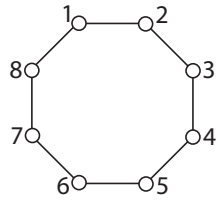


Figure 2.4

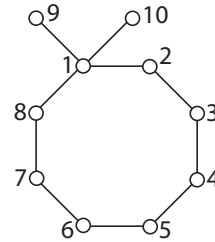


Figure 2.5

Remark 2.1.13. Let G be a graph. It is clear that if B is a PSDZFS with full forcing then $\text{pt}_+(G, B) = \text{dist}(B, \overline{B})$. Further, if G has universal full forcing then $\text{pt}_+(G) = \min\{\text{dist}(B, \overline{B}) : B \text{ is a min PSDZFS}\}$ and $\text{PT}_+(G) = \max\{\text{dist}(B, \overline{B}) : B \text{ is a min PSDZFS}\}$. Finally, if B is a minimum PSDZFS that exhibits full forcing and minimizes $\text{dist}(B, \overline{B})$, then $\text{pt}_+(G) = \text{pt}_+(G, B) = \text{dist}(B, \overline{B})$.

Remark 2.1.14. Let B be a PSDZFS for G . Let W_1, W_2, \dots, W_k be the sets of vertices of the k connected components of $G - \bigcup_{i=0}^t B_+^{(i)}$ for some t . Note that for $i \neq j$, $W_i \cap N_G(W_j) = \emptyset$. This means any forcing that happens in $G[W_i]$ does not induce any forcing in $G[W_j]$ for $i \neq j$. Therefore at least one vertex must be forced in each component at each time step, if not the forcing will have stalled in that component and B was not a PSDZFS.

Remark 2.1.15. In general, $\text{pt}_+(G)$ and $\text{pt}(G)$ are not comparable and neither are $\text{PT}_+(G)$ and $\text{PT}(G)$ as shown in Figures 2.6 - 2.9. Even if $Z_+(G) = Z(G)$ we cannot compare $\text{PT}_+(G)$ and $\text{PT}(G)$ (Figures 2.8 and 2.9). However, since a standard zero forcing set is also a PSDZFS, if $Z_+(G) = Z(G)$ we can say $\text{pt}_+(G) \leq \text{pt}(G)$.

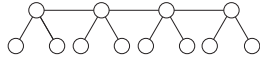


Figure 2.6:
 $\text{pt}_+(G) = 3 > 2 = \text{pt}(G)$
 $Z(G) = 4, Z_+(G) = 1$



Figure 2.7: $\text{pt}_+(G) = 2 < 3 = \text{pt}(G)$
 $Z(G) = 1 = Z_+(G)$

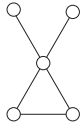


Figure 2.8:
 $\text{PT}_+(G) = 2 < 3 = \text{PT}(G)$
 $Z(G) = 2 = Z_+(G)$

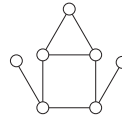


Figure 2.9: $\text{PT}_+(G) = 4 > 3 = \text{PT}(G)$
 $Z(G) = 2 = Z_+(G)$

The following is clear from Definitions 2.1.2, 2.1.3 and because the worst we can do is force exactly one vertex blue at each time step. Also, the only way for a graph to have PSD propagation time 0 is if it is a set of isolated vertices.

Observation 2.1.16. *Let G be a graph with at least one edge. Then*

$$1 \leq \text{pt}_+(G) \leq \text{PT}_+(G) \leq |G| - Z_+(G).$$

Furthermore, if B is a PSDZFS such that at least k vertices are forced at each time step

then

$$\text{pt}_+(G, B) \leq \frac{|G| - |B|}{k}.$$

2.2 Propagation time interval

Definition 2.2.1. The *PSD propagation time interval* of G is defined as

$$[\text{pt}_+(G), \text{PT}_+(G)] = \{\text{pt}_+(G), \text{pt}_+(G) + 1, \dots, \text{PT}_+(G) - 1, \text{PT}_+(G)\}.$$

If every integer in the PSD propagation time interval is achievable by some minimum PSDZFS we say G has a *full PSD propagation time interval*.

We conjecture that every graph has a full PSD propagation time interval. This section concentrates on results that support the claim. It was shown in [9] that this is not the case for standard propagation time interval. Figure 2.10 is an example of a graph that does not have a full standard propagation time interval.

Example 2.2.2. Notice that (up to symmetry) the graph in Figure 2.10 has only 3 types of minimum standard zero forcing sets; the two vertices at the ends, the two upper vertices and opposite upper and end vertices. The first two sets have propagation time 4 and the last zero forcing set has propagation time 6.

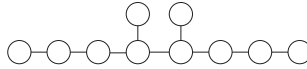


Figure 2.10: $\text{pt}(G) = 4$, $\text{PT}(G) = 6$, and there is no minimum ZFS B with $\text{pt}(G, B) = 5$.

Next we use some basic tools to answer questions about the propagation time interval for some common graph families. It is well known that if T is a tree then $Z_+(t) = 1$ [2].

Observation 2.2.3. *If T is a tree then T has universal full forcing.*

The *diameter* of a graph G is $\text{diam}(G) = \max\{d(u, v) \mid u, v \in V(G)\}$.

Proposition 2.2.4. *If T is tree, $\text{diam}(T) = d$, then*

$$\text{pt}_+(T) = \left\lceil \frac{d}{2} \right\rceil \text{ and } \text{PT}_+(T) = d$$

Further, every tree has a full PSD propagation time interval.

Proof. By Remark 2.1.13 $\text{pt}_+(T) = \min\{\text{dist}(B, \overline{B})\}$. This is achieved by choosing B_{min} to be the vertex in the middle of a maximum path, thus $\text{pt}_+(T) = \text{dist}(B_{min}, \overline{B_{min}}) = \left\lceil \frac{d}{2} \right\rceil$. Also by Remark 2.1.13 $\text{PT}_+(T) = \max\{\text{dist}(B, \overline{B})\}$. If we choose B_{max} to be a vertex at the end of a maximum path then $\text{PT}_+(T) = \text{dist}(B_{max}, \overline{B_{max}}) = d$. Finally, we fill the propagation time interval by moving our PSDZFS (in this case one vertex) along a maximum path. \square

Corollary 2.2.5. *If $G = P_n$ then $\text{pt}_+(G) = \left\lceil \frac{n-1}{2} \right\rceil$ and $\text{PT}_+(G) = n-1$.*

Corollary 2.2.6. *If $G = S(k_1, k_2, \dots, k_s)$ is a generalized star and $k_1 \geq k_2 \geq k_3 \geq \dots \geq k_s$, then*

$$\text{pt}_+(G) = \left\lceil \frac{k_1 + k_2}{2} \right\rceil \text{ and } \text{PT}_+(G) = k_1 + k_2.$$

It has been established that a cycle C_n has $Z_+(C_n) = 2$ [2].

Observation 2.2.7. *If C_n is a cycle on n vertices then C_n has universal full forcing*

Proposition 2.2.8. *Let C_n be the cycle on n vertices, then*

$$\text{pt}_+(C_n) = \left\lceil \frac{n-2}{4} \right\rceil \text{ and } \text{PT}_+(C_n) = \left\lceil \frac{n-2}{2} \right\rceil.$$

Further, C_n has a full PSD propagation time interval.

Proof. Since C_n has universal full forcing then by Remark 2.1.13 we have to minimize and maximize $\text{dist}(B, \overline{B})$ to find $\text{pt}_+(C_n)$ and $\text{PT}_+(C_n)$, respectively. If we label our vertices v_1, v_2, \dots, v_n with $d = \left\lceil \frac{n}{2} \right\rceil$, then $B_{min} = \{v_1, v_d\}$ gives $\text{dist}(B_{min}, \overline{B_{min}}) = \text{pt}_+(C_n) = \left\lceil \frac{n-2}{4} \right\rceil$. Choosing $B_{max} = \{v_1, v_2\}$ we have $\text{dist}(B_{max}, \overline{B_{max}}) = \text{PT}_+(C_n) = \left\lceil \frac{n-2}{2} \right\rceil$. To fill the propagation time interval we let $B_i = \{v_1, v_i\}$ for $i = 2, 3, \dots, d$. \square

The ability to swap one vertex into and one vertex out of a PSDZFS and not change the PSD propagation time by more than 1 is a valuable tool for establishing full PSD propagation time intervals. The proof of Lemma 2.2.10 is based on arguments in [10]. We state the result related to Lemma 2.2.10 as Lemma 2.2.9.

Lemma 2.2.9. [10] *Let G be a graph and B be a PSDZFS of G with $v \in B$ such that $v \rightarrow w$ at the first time step. Then $B' = (B \setminus \{v\}) \cup w$ is also a PSDZFS for G .*

Lemma 2.2.10. *Let G be a graph and B be a PSDZFS with $v \in B$ such that $v \rightarrow w$ at the first time step. Define $B' := (B \setminus \{v\}) \cup w$. Then $w \rightarrow v$ at the first time step and $\text{pt}_+(G, B) - 1 \leq \text{pt}_+(G, B') \leq \text{pt}_+(G, B) + 1$.*

Proof. Let G be a graph and B a PSDZFS for G with $v \rightarrow w$ at the first time step. Let W_w be the vertex set of the connected component of $G - B$ that contains w . Define $B' := (B \setminus \{v\}) \cup \{w\}$. Let W_v be the vertex set of the connected component of $G - B'$ containing v . If w cannot force v at the first time step then w has some other white neighbor, say x , in $G[B' \cup W_v]$. This means there is a path of white vertices from x to v in $G[W_v]$. However, since w is also a neighbor of x this means v will have two white neighbors in $G[B \cup W_w]$. This contradicts that $v \rightarrow w$ at the first time step when considering B . Thus $w \rightarrow v$ at the first time step when we force using B' . Now B has been forced blue so the rest of the graph can be forced.

Since $w \rightarrow v$ at the first time step we clearly have our upper bound. The way to achieve the lower bound is if the vertex set of the connected component of $G - B$ that w is in initially, W_w , took the longest to force by at least one time step. By exchanging v and w we have sped up the forcing of the component W_w by at most 1 and the other components have not slowed down by more than 1. \square

Definition 2.2.11. Let G be a graph with PSDZFS B and B' be another PSDZFS with the same cardinality. If there exists a sequence of PSDZF sets $B = B_1, B_2, \dots, B_k = B'$ that have the same cardinality and $(B_i \setminus \{v_i\}) \cup \{w_i\} = B_{i+1}$ such that $v_i \rightarrow w_i$ at the first time step when using B_i as the PSDZFS, then we say we say we can *migrate* from B to B' .

Corollary 2.2.12. *Let G be a graph and suppose we can migrate from one minimum PSDZFS, B , to another, B' . Then for every integer k between $\text{pt}_+(G, B)$ and $\text{pt}_+(G, B')$ there exists a minimum PSDZFS B'' such that $\text{pt}_+(G, B'') = k$.*

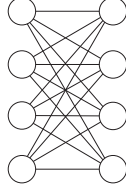
Corollary 2.2.13. *If G is a graph and B_{\min} is a minimum PSDZFS such that $\text{pt}_+(G) = \text{pt}_+(G, B_{\min})$, B_{\max} is a minimum PSDZFS such that $\text{PT}_+(G) = \text{pt}_+(G, B_{\max})$ and we can migrate from B_{\min} to B_{\max} then G has a full PSD propagation time interval.*

Corollary 2.2.13 gives us a method to establish full propagation time for unicyclic graphs and provides alternate proofs for trees and cycles.

Proposition 2.2.14. *Let G be a unicyclic graph. Then G has a full propagation time interval.*

Proof. Let G be a unicyclic graph with cycle C_n and label the cycle vertices r_1, r_2, \dots, r_n . Let T_i be a rooted tree with root r_i so $G = C_n \cup T_1 \cup \dots \cup T_n$. Observe that $Z_+(G) = 2$ and that minimum PSDZF sets are of the form $\{v_i, v_j\}$ with $v_i \in V(T_i)$ and $v_j \in V(T_j)$, $i \neq j$. Now we will show that we can migrate from $\{v_i, v_j\}$ to $\{v_i, v_k\}$. First we observe that if $v_j \neq r_j$ then v_j will force each of its neighbors at the first time step. Using this idea iteratively we can migrate from $\{v_i, v_j\}$ to $\{v_i, r_j\}$. Similarly, we can migrate from $\{v_i, r_j\}$ to $\{r_i, r_j\}$. Notice that r_j can force its cycle neighbors, which will allow us to migrate from $\{r_i, r_j\}$ to $\{r_i, r_k\}$. Now we can force into T_i and T_k at the first time step, which allows us to migrate from $\{r_i, r_k\}$ to $\{v_i, v_k\}$. Since v_i, v_j and v_k were arbitrary we have the ability to migrate from any minimum PSDZFS to another. Thus the propagation time interval is full for G by Corollary 2.2.13 \square

Example 2.2.15. Note that we are not always able to migrate from one minimum PSDZFS to another. Consider the complete bipartite graph $K_{4,4}$ in Figure 2.11 and note that $Z_+(K_{4,4}) = 4$ by [11]. Also note that minimum PSDZF sets are of the form: 4 vertices on the right, 4 vertices on the left, 3 vertices on the right and 1 on the left, 3 vertices on the left and 1 on the right. If migration were possible we would also need 2 vertices on each side to be a PSDZFS but they are not. Note, however, that the PSD propagation time interval for $K_{4,4}$ is the full interval $[1, 2]$ (see also Proposition 2.3.2).

Figure 2.11: $K_{4,4}$

Next we investigate how the addition and removal of duplicate vertices affect the PSD propagation time interval. Let G be a graph and $u, v \in V(G)$ such that $N[u] = N[v]$, i.e. their closed neighborhoods are the same. Then we say u and v are *duplicate vertices*.

Observation 2.2.16. *Let G be a graph and $u, v \in V(G)$. If u and v are duplicate vertices then at least one of them must be in every PSDZFS.*

Proposition 2.2.17. [8] *If u and v are duplicate vertices in a connected graph G with $|G| \geq 3$, then $Z_+(G - \{u\}) = Z_+(G) - 1$.*

Proposition 2.2.18. *Let G is a graph with duplicate vertices u and v that have at least one neighbor in common. If B is a PSDZFS with $u \in B$, then $\text{pt}_+(G, B) = \text{pt}_+(G - \{u\}, B \setminus \{u\})$.*

Proof. Let G be a graph and u and v be duplicate vertices. By Observation 2.2.16 at least one of u or v must be in any PSDZFS of G and by Proposition 2.2.17 $Z_+(G - \{v\}) = Z_+(G - \{u\}) = Z_+(G) - 1$. Let B be a PSDZFS for G and without loss of generality assume $u \in B$. If u forces a vertex $w \neq v$ at time t , then v must be blue so v can also force w at time t . If u forces v at time t then $N[u] \setminus \{v\} = N(v)$ is all blue and contains a vertex besides u . Thus v is isolated in $G - \bigcup_{i=0}^{t-1} B_+^{(i)}$ and can be forced by some vertex besides u at time t , which does not affect any other forcing that u performs. Therefore, the removal of u from G does not change the propagation time, thus $\text{pt}_+(G, B) = \text{pt}_+(G - \{u\}, B \setminus \{u\})$. \square

Corollary 2.2.19. *If G is a graph and u is a duplicate vertex in G then*

1. $\text{pt}_+(G) = \text{pt}_+(G - \{u\})$
2. $\text{PT}_+(G) = \text{PT}_+(G - \{u\})$

3. If the PSD propagation time interval is full for G , it is also full for $G - \{u\}$.

Remark 2.2.20. Let G be a nontrivial graph. Add a vertex u to G that is duplicate to some vertex $v \in V(G)$ such that they have at least one neighbor in common. Call this new graph G' . Let $B \subseteq V(G)$ be a set of blue vertices and define $B' := B \cup \{u\}$. Then the connected components of $G - B$ and the connected components of $G' - B'$ are the same. This means that if $w \in N_G[v]$ cannot force in $G - B$ then $w \in N_{G'}(u)$ cannot force in $G' - B'$.

Proposition 2.2.21. *Let G be a nontrivial graph with a full PSD propagation time interval. Add a vertex u to G that is a duplicate of some vertex $v \in V(G)$ such that they have at least one neighbor in common. Call this new graph G' . Then the PSD propagation time interval of G' is full.*

Proof. By Proposition 2.2.17 $Z_+(G) = Z_+(G') - 1$ so $Z_+(G') = Z_+(G) + 1$. Let B be a minimum PSDZFS of G with $\text{pt}_+(G, B) = j$. Note that $B' := B \cup \{u\}$ is a minimum PSDZFS for G' and that $\bigcup_{i=0}^t B_+^{(i)} \cup \{u\} \subseteq \bigcup_{i=0}^t B_+^{(i)}$ for all t . Now we will show the other containment.

First notice that $B_+^{(0)} = B \cup \{u\} = B_+^{(0)} \cup \{u\}$. Consider $w \in B_+^{(1)}$. If w was forced by u then the argument in Proposition 2.2.18 says that w can be forced by some other vertex at the first time step so $w \in B_+^{(1)}$. Remark 2.2.20 says that any any neighbors of u that force in G' at the first time step will also force in G at the first time step. Finally, note that u has no affect on any forcing beyond $N_{G'}[u]$ and since the set of blue vertices and the graphs G' and G are the same beyond $N_{G'}[u]$ all of the forcing must be the same. Thus $B_+^{(1)} \subseteq B_+^{(1)}$. We can extend this argument in a similar way at each time step thus $\bigcup_{i=0}^t B_+^{(i)} \subseteq \bigcup_{i=0}^t B_+^{(i)} \cup \{u\}$ for all t . This gives us containment in both directions thus $\text{pt}_+(G, B) = \text{pt}_+(G', B') = j$. So we have shown that the PSD propagation time interval of G' contains the PSD propagation time interval of G . However, Corollary 2.2.19 tells us that the PSD propagation time intervals are actually the same so G' has a full PSD propagation time interval. \square

Corollary 2.2.22. *If G is a graph with a full PSD propagation time interval adding or removing a duplicate vertex does not change the fullness of the interval.*

Next we discuss graphs that have a universal vertex. Of particular interest is the effect on the propagation time interval by adding a universal vertex.

Remark 2.2.23. Let G be a graph with no isolated vertices and add a universal vertex u to get G' . First notice that once u is blue it does not affect the number or composition of any connected components. Second, if u forces some vertex v at time t then v is an isolated vertex in $G - \bigcup_{i=0}^{t-1} B_+^{(i)}$ and, since G has no isolated vertices, there exists some vertex besides u that can force v at time t . Together this means that once u is blue its removal from G' neither inhibits or assists the PSD propagation zero forcing algorithm.

In [9] it was shown that there is no connected graph of order at least two that has a unique efficient standard zero forcing set. This was proved by showing that every graph of order at least two has at least two efficient zero forcing sets. The next theorem allows us to easily show these results do not hold for efficient PSD zero forcing sets.

Theorem 2.2.24. *If $G' \neq K_n$ is a graph with a universal vertex u , then u is in every efficient PSDZFS.*

Proof. Let B' be an efficient PSDZFS for G' , that is $\text{pt}_+(G', B') = \text{pt}_+(G')$. If $u \notin B'$ then $B_+^{(1)} = \{u\}$. If no more forcing occurs then $Z_+(G') = n - 1$ and G' is a K_n . Since we are assuming $G' \neq K_n$ we know $B_+^{(2)}$ is not empty. Now, there is some $v \in B'$ with $N(v) \setminus \{u\} \subseteq B'$ and $v \rightarrow u$. However, $B'' = (B' \setminus \{v\}) \cup \{u\}$ is also a minimum PSDZFS and $B_+^{(1)} = \{v\} \cup B_+^{(2)}$ since v is isolated in $G' - B''$ and u is blue. In other words, $\bigcup_{i=0}^1 B_+^{(i)} = \bigcup_{i=0}^2 B_+^{(i)}$, which implies $\bigcup_{i=0}^{j-1} B_+^{(i)} = \bigcup_{i=0}^j B_+^{(i)}$ for all $j \geq 2$. In particular, this is true for $j = \text{pt}_+(G')$, thus $\text{pt}_+(G', B'') < \text{pt}_+(G')$, which is a contradiction. Therefore, u is in every efficient PSDZFS of G' when G' is not complete. \square

Theorem 2.2.24 confirms that a star $K_{1,n}$, $n > 1$, has a unique efficient PSD zero forcing set. We should also note that the converse of Theorem 2.2.24 is not true as the middle vertex of P_5 is in every efficient PSDZFS but is not universal. Further, this property of universal vertices is not analogous in standard propagation time since the middle vertex of P_3 is universal but is not in any minimum zero forcing set, let alone an efficient one.

Theorem 2.2.25. *Let G be a graph with no isolated vertices and add a universal vertex u to get G' . Then:*

1. $Z_+(G') = Z_+(G) + 1$
2. *If B is a PSDZFS for G then $B' = B \cup \{u\}$ is a PSDZFS for G' and $\text{pt}_+(G, B) = \text{pt}_+(G', B')$.*
3. *Any minimum PSDZFS for G' is of the form $B \cup \{u\}$ or $B \cup \{v\}$ where $N_G(v) \subseteq B$ and B is a minimum PSDZFS for G .*
4. $\text{pt}_+(G) = \text{pt}_+(G')$
5. $\text{PT}_+(G) \leq \text{PT}_+(G') \leq \text{PT}_+(G) + 1$
6. *If the PSD propagation time interval is full for G it is also full for G' .*

Proof. Let G be a graph with no isolated vertices and add universal vertex u to get G' . By [8] the addition of a single vertex increases the PSD zero forcing number by at most one, so $Z_+(G') \leq Z_+(G) + 1$. This bound can be achieved by simply finding a minimum PSDZFS for G and adding u to it. Now, let B' be a PSDZFS for G' . If $u \in B'$ then, by Remark 2.2.23, $B' \setminus \{u\}$ is a PSDZFS for G so $Z_+(G) + 1 \leq |B'| \leq Z_+(G')$. If $u \notin B'$, then u must be the only vertex forced at the first time step. Let's say $v \in B'$ forces u . Then $N_G(v) \subseteq B'$. Once u is forced we can remove it without affecting future forces, again by Remark 2.2.23. This means B' is a PSDZFS for $G = G' - u$. If this is the case then we can change v from blue to white and still have a PSDZFS for G since v will be isolated when we remove the blue vertices. Thus $B' \setminus \{v\}$ is a PSDZFS for G so $Z_+(G) + 1 \leq |B'| \leq Z_+(G')$. This establishes (1), (2), (3) and (4).

Let B be a minimum PSDZFS for G , then $B \cup \{u\}$ is a minimum PSDZFS for G' and $\text{pt}_+(G, B) = \text{pt}_+(G', B')$. This means the PSD propagation time interval of G' contains the PSD propagation time interval of G .

Now assume B' is a minimum PSDZFS for G' of the form $B \cup \{v\}$, $N_G(v) \subseteq B$, B a minimum PSDZFS for G . Then $v \rightarrow u$ at the first time step. If the graph has been completely forced after

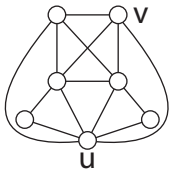
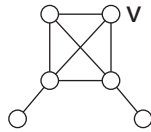
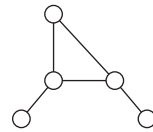
one step then G and G' are complete graphs and $\text{pt}_+(G) = \text{pt}_+(G') = \text{PT}_+(G) = \text{PT}_+(G') = 1$. If more forcing occurs observe that $B'_+(2) = B_+^{(1)} \setminus \{v\}$, thus $\bigcup_{i=0}^2 B'_+^{(i)} = \bigcup_{i=0}^1 B_+^{(i)}$. This pattern

continues and we see that $\bigcup_{i=0}^{j+1} B'_+^{(i)} = \bigcup_{i=0}^j B_+^{(i)}$. So unless $G = K_n$, $\text{pt}_+(G, B) + 1 = \text{pt}_+(G', B')$.

This means that adding the universal vertex u to G adds at most one to $\text{PT}_+(G)$. Thus $\text{pt}_+(G) = \text{pt}_+(G')$, $\text{PT}_+(G) \leq \text{PT}_+(G') \leq \text{PT}_+(G) + 1$, and if the PSD propagation time interval is full for G , it is also full for G' . \square

Corollary 2.2.26. *If G' is a graph with universal vertex u and $G' - u$ has a full PSD propagation time interval, then so does G' .*

Example 2.2.27. Note that Corollaries 2.2.22 and 2.2.26 allow us to simplify the process of determining whether or not a graph has a full PSD propagation time interval. Starting with graph G , Figure 2.12, we observe that G has a universal vertex u . When we remove u from G we notice in Figure 2.13 that the top two vertices are duplicates, so we remove the top right one, v , and end up with a unicyclic graph, Figure 2.14. Note that Proposition 2.2.14 tells us that $G - u - v$ has a full PSD propagation time interval. Adding duplicate and universal vertices does not change the fullness of the interval so G also has a full PSD propagation time interval.

Figure 2.12: Graph G Figure 2.13: $G - u$ Figure 2.14: $G - u - v$

2.3 Graph families

Here we study some well known graph families. For most of these families we determine the minimum and maximum PSD propagation time and show that they have full PSD propagation

time intervals. This adds more evidence for Conjecture 2.3.15 that all PSD propagation time intervals are full.

Proposition 2.3.1. *If $G = W_n$, the wheel on n vertices, $n \geq 4$, then $\text{pt}_+(G) = \left\lceil \frac{n-3}{4} \right\rceil$ and $\text{PT}_+(G) = \left\lceil \frac{n-2}{2} \right\rceil$. Furthermore, G has a full PSD propagation time interval.*

Proof. Let $G = W_n$, then by [11], $Z_+(G) = 3$. Also notice that the hub of the wheel is a universal vertex. In particular, adding a universal vertex to C_{n-1} gives W_n . By Theorem 2.2.25, $\text{pt}_+(C_{n-1}) = \text{pt}_+(W_n)$. Therefore, by Proposition 2.2.8, $\text{pt}_+(W_n) = \left\lceil \frac{(n-1)-2}{4} \right\rceil = \left\lceil \frac{n-3}{4} \right\rceil$.

Let B consist of three consecutive cycle vertices. Observe that we force the hub at the first time step, and at every other step we force at most two vertices, thus $\text{pt}_+(G, B) = 1 + \left\lceil \frac{n-4}{2} \right\rceil = \left\lceil \frac{n-2}{2} \right\rceil$ so $\text{PT}_+(G) \geq \left\lceil \frac{n-2}{2} \right\rceil$. Notice that every other minimum PSDZFS contains u so using Theorem 2.2.25(2) and Proposition 2.2.8 their corresponding propagation times are bound above by $\left\lceil \frac{n-3}{2} \right\rceil$. Therefore $\text{PT}_+(G) = \left\lceil \frac{n-2}{2} \right\rceil$.

The PSD propagation time interval is full by Theorem 2.2.25. \square

We define K_{n_1, n_2, \dots, n_k} to be the *complete multipartite* graph on $\sum_{i=1}^k n_i$ vertices where we partition the vertices into sets $V_{n_1}, V_{n_2}, \dots, V_{n_k}$ such that:

- for $1 \leq i \leq k$, $|V_{n_i}| = n_i$
- $v, w \in V_{n_i}$ means $vw \notin E(G)$
- for $1 \leq i < j \leq k$, $v \in V_{n_i}$ and $w \in V_{n_j}$ means $vw \in E(G)$.

If $k = 2$ we say G is complete bipartite.

Proposition 2.3.2. *If $n_1 \geq n_2 \geq \dots \geq n_k$, $k \geq 2$, $n_1 \geq 2$, and $G = K_{n_1, n_2, \dots, n_k}$ then $\text{pt}_+(G) = 1$ and $\text{PT}_+(G) = 2$. (Note: if $G = K_{1,1, \dots, 1}$ then G is complete so $\text{pt}_+(G) = \text{PT}_+(G) = 1$.) Further, the PSD propagation time interval is full.*

Proof. Let G be as hypothesized. Then $Z_+(G) = n_2 + n_3 + \dots + n_k$ by [11]. Clearly G is not a set of isolated vertices so $\text{pt}_+(G) \geq 1$. As in the definition, we denote the vertex partition sets by $V_{n_1}, V_{n_2}, \dots, V_{n_k}$. Note that if $B = V_{n_2} \cup V_{n_3} \cup \dots \cup V_{n_k}$ then $G - B$ is a set of n_1 isolated vertices. This gives $\text{pt}_+(G, B) = 1$ so $\text{pt}_+(G) \leq 1$, therefore $\text{pt}_+(G) = 1$.

Now let B be a minimum PSDZFS. Suppose first that there are distinct i, j, k such that $V_{n_i} - B, V_{n_j} - B$ and $V_{n_k} - B$ have at least one element. Then $G - B$ is connected so we apply standard zero forcing. However, this tells us that every vertex of B has at least two white neighbors so no forcing occurs and B is not a minimum PSD zero forcing set.

So if B is a minimum PSDZFS at most two vertex partition sets, say V_{n_i} and V_{n_j} , initially have white vertices. If both V_{n_i} and V_{n_j} have at least two white vertices then $G - B$ is connected and every vertex in B has at least two white neighbors so B is not a minimum PSDZFS.

Thus the only possibilities are: $V_{n_i} \subseteq B, V_{n_j} \subseteq B, |V_{n_i} - B| = 1$ or $|V_{n_j} - B| = 1$. In the former two cases $\text{pt}_+(G, B) = 1$. In the latter cases we only have one connected component at the first iteration of our algorithm so we apply standard zero forcing. Without loss of generality assume $|V_{n_i} - B| = 1$, then V_{n_j} has one blue vertex initially and that blue vertex has one white neighbor in V_{n_i} so that is the only force we perform. Now, at the second iteration when we remove the blue vertices all of the remaining white vertices are isolated thus $\text{pt}_+(G, B) = 2$.

Obviously the PSD propagation time interval is full. \square

2.3.1 Cartesian products

The *Cartesian product* of two graphs H and G , denoted $H \square G$, has vertex set $V(H) \times V(G)$ with vertices (u, v) and (u', v') adjacent if and only if (1) $u = u'$ and $vv' \in E(G)$ or (2) $v = v'$ and $uu' \in E(H)$. This section concentrates on the PSD propagation times and PSD zero forcing numbers of some Cartesian products.

Proposition 2.3.3. *For $s \geq 2$, let $G = P_s \square P_2$, then $\text{pt}_+(G) = \left\lceil \frac{s-1}{2} \right\rceil$ and $\text{PT}_+(G) = s$. Further, G has a full PSD propagation time interval.*

Proof. Let $s \geq 2$ and $G = P_s \square P_2$. Then, by [11], $Z_+(G) = 2$. Number the vertices as in Figure

2.15. Let $m = \left\lceil \frac{s}{2} \right\rceil$ and $B = \{v_{m1}, v_{m2}\}$. Then, B exhibits full forcing and $d = \text{dist}(B, \overline{B}) =$

$$\begin{cases} m-1 & \text{if } s \text{ is odd} \\ m & \text{if } s \text{ is even} \end{cases}$$

is minimal. By Remark 2.1.13 $\text{pt}_+(G) = d$. Finally, observe that

$d = \left\lceil \frac{s-1}{2} \right\rceil$ and we have our first result.

Recall that the forcing list sequence of a PSDZFS B , $\text{FLS}(G, B)$, is the tuple whose $(k+1)$ th entry is the number of vertices forced at the k th time step in the forcing algorithm. Notice that for almost every minimum PSDZFS the forcing list sequence is comprised of 2's and 4's. If $B = \{v, w\}$ is a cut set we will force at least two vertices at every time step. If B is not a cut set then initially we implement standard zero forcing. This means B must contain a degree two vertex, v , and one of its neighbors, w . If w also has degree two we will force two vertices at the first time step. If w has degree three then we only force one vertex at the first time step. In fact, up to symmetry, there is only one minimum PSDZFS, $B_{max} = \{v_{11}, v_{21}\}$, that has any 1's and it has two of them. Clearly, B_{max} has maximized the length of $\text{FLS}(G, B_{max})$ so $\text{PT}_+(G) = \text{pt}_+(G, B_{max}) = s$. Finally, we can achieve all other integers in the PSD propagation time interval by letting $B_i = \{v_{i1}, v_{i2}\}$ for $i = 1, 2, 3, \dots, m$. \square

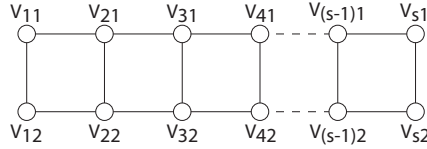


Figure 2.15: $P_s \square P_2$

A *contraction* of edge $e = \{u, v\}$ is obtained by identifying vertices u and v , replacing any multiple edges by single edges, and deleting any loops that occur. A *minor* of G is obtained by a series of edge deletions, vertex deletions and/or edge contractions. The *Hadwiger number*, $h(G)$, is the largest r such that K_r is a minor of G .

Theorem 2.3.4. [3] For graph G , $h(G) - 1 \leq M_+(G) \leq Z_+(G)$.

Proposition 2.3.5. Let $G = P_s \square P_3$, $s \geq 3$, then $Z_+(G) = 3$ and $\text{pt}_+(G) = \left\lceil \frac{s-1}{2} \right\rceil$.

Proof. Let G be as in the theorem. Note that $H = P_3 \square P_3$ is a minor of G . Label the vertices of H $\{1, 2, \dots, 9\}$ starting in the top left corner then proceed left to right, top to bottom. Contract vertices 3, 6, 9, 8 into a single vertex and 1, 4, 7 into a single vertex to obtain a K_4 . Thus G has a K_4 minor so the Hadwiger number of G is 4 and Theorem 2.3.4 gives $3 \leq Z_+(G)$.

Now, orient the graph so that there are three rows and s columns and note that any column forms a PSDZFS thus $Z_+(G) = 3$. Now let B consist of the vertices in the $\left\lceil \frac{s}{2} \right\rceil$ column, then $\text{pt}_+(G, B) = \left\lceil \frac{s-1}{2} \right\rceil$. Also notice that B has full forcing and minimizes $\text{dist}(B, \overline{B})$ so Remark 2.1.13 gives $\text{pt}_+(G) = \left\lceil \frac{s-1}{2} \right\rceil$. \square

Proposition 2.3.6. *Let $G = C_s \square P_2$, $s \geq 4$, then $\text{pt}_+(G) = \left\lceil \frac{s-2}{4} \right\rceil$.*

Proof. Let $G = C_s \square P_2$. Then $Z_+(G) = 4$ by [11]. Label the vertices of each cycle $c_{1i}, c_{2i}, \dots, c_{si}$ for $i = 1, 2$. Let $m = \left\lceil \frac{s}{2} \right\rceil$ and $B = \{c_{11}, c_{12}, c_{m1}, c_{m2}\}$. Observe that B is a minimum PSDZFS, minimizes $\text{dist}(B, \overline{B})$, exhibits full forcing and $\text{pt}_+(G, B) = \left\lceil \frac{s-2}{4} \right\rceil$. By Remark 2.1.13, $\text{pt}_+(G, B) = \text{pt}_+(G)$. \square

The c -cube Q_c , $c \geq 1$, is defined as repeated Cartesian products of K_2 . In particular, $Q_1 = K_2$, $Q_c = Q_{c-1} \square K_2$ for $c \geq 2$. The c -cube is also known as the c th hypercube.

Proposition 2.3.7. *Let $G = Q_c$, then $\text{pt}_+(G) = 1$.*

Proof. Let $G = Q_c$, then $Z_+(G) = 2^{c-1}$ by [11]. Notice that $B = V(Q_{c-1})$ is a minimum PSDZFS and $\text{pt}_+(G, B) = 1$. Thus by Observation 2.1.16 $\text{pt}_+(G) = 1$. \square

Proposition 2.3.8. *Let $G = C_4 \square K_3$, then $\text{pt}_+(G) = 1$.*

Proof. Let $G = C_4 \square K_3$ and note that by [11] $Z_+(G) = 6$. Observe that choosing the vertex sets of any two copies of K_3 produces a minimum PSDZFS. Further, each of these sets has a PSD propagation time of 1. By Observation 2.1.16 $\text{pt}_+(G) = 1$. \square

Of course we can always find pt_+ and PT_+ by identifying all possible minimum PSDZF sets and their corresponding propagation times. We do this for the cartesian products $K_s \square P_t$, $s \geq 3$ and $t \geq 2$, and $K_3 \square K_3$.

Lemma 2.3.9. *Let $G = K_s \square P_t$, $s \geq 3$ and $t \geq 2$, where $K_s^{(i)}$ is the i th copy of K_s . Further, label the vertices of $K_s^{(i)}$ by $v_{1i}, v_{2i}, \dots, v_{si}$ where edges not within copies of K_s are of the form*

$v_{ji} \sim v_{j(i+1)}$ for $1 \leq j \leq s$ and $1 \leq i \leq t-1$. Then the only vertex cut sets of G with s vertices are made up of sets of vertices in neighboring copies of K_s 's such that the set of first indices is $\{1, 2, \dots, s\}$.

Proof. Observe first that if C is a set of vertices from neighboring copies of K_s whose set of first indices is $\{1, 2, 3, \dots, s\}$ then C is a cut set. Now assume that C is contained in two neighboring copies of K_s , namely $K_s^{(i)}$ and $K_s^{(i+1)}$, and assume that the first set of the indices of the vertices in C are missing j , $1 \leq j \leq s$. This means v_{ji} and $v_{j(i+1)}$ are not contained in C . Let $v, w \in V(G) \setminus C$ with $v \in K_s^{(a)}$, $w \in K_s^{(b)}$ and without loss of generality let $a \leq b$. If $i+1 \leq a$ or $b \leq i$ we can easily find a path from v to w that contains no vertices of C . If $a \leq i$ and $i+1 \leq b$ then we can find a path, P_v , from v to v_{ji} that has no vertices of C . We can also find a path, P_w , from $v_{j(i+1)}$ to w that contains no vertices of C . Then the path consisting of P_v , P_w and the edge between v_{ji} and $v_{j(i+1)}$ is a path from v to w that contains no vertices of C . Since v and w were arbitrary C is not a cut set. We can use a similar argument if we assume C is spread out between more copies of K_s , or nonadjacent copies of K_s . \square

Lemma 2.3.10. Let $G = K_s \square P_t$, $s \geq 3$ and $t \geq 2$, where $K_s^{(i)}$ is the i^{th} copy of K_s . Further, label the vertices of $K_s^{(i)}$ by $v_{1i}, v_{2i}, \dots, v_{si}$ where edges not within copies of K_s are of the form $v_{ji} \sim v_{j(i+1)}$ for $1 \leq j \leq s$ and $1 \leq i \leq t-1$. Then there are only three types of minimum positive semidefinite zero forcing sets:

1. vertex cut sets
2. end sets, e.g. $V(K_s^{(1)})$
3. almost end sets i.e. $\{v_{11}, v_{21}, \dots, v_{(j-1)1}, v_{(j+1)1}, \dots, v_{s1}, v_{k2}\}$ where $k \neq j$.

Proof. Let $G = K_s \square P_t$ as above. Note that $Z_+(G) = s$ by [11] and observe that if B is one of the above three sets then B is a minimum PSDZFS. Also note that the result is obvious for $s = 2$. Now we suppose that $s \geq 3$, $t \geq 3$ and that B is a vertex subset of G , $|B| = 2$, and B is not one of (1), (2) or (3).

This means that B is not a cut set so we apply standard zero forcing at the first time step. We will argue that no forcing occurs. Clearly B is not contained in one copy of K_s and if B is

contained in three or more copies of K_s then $|B \cap K_s^{(i)}| \leq s - 2$ for $1 \leq i \leq t$. Therefore each vertex in B has at least two white neighbors within the copy of K_s that it is contained in so no forcing occurs.

Now we suppose that B is contained in two copies of K_s , say $K_s^{(i)}$ and $K_s^{(j)}$, $i < j$. If $K_s^{(i)}$ and $K_s^{(j)}$ are not adjacent then clearly all blue vertices have two or more white neighbors and no zero forcing occurs. Next we suppose that $K_s^{(i)}$ and $K_s^{(j)}$ are adjacent and without loss of generality that $j = i + 1$. Let $B_1 = B \cap K_s^{(i)}$ and $B_2 = B \cap K_s^{(i+1)}$.

Case 1: $i = 1$

Since B is not an almost end set $|B_1| \leq s - 2$ so every vertex in B_1 is adjacent to at least two white vertices in $K_s^{(1)}$. Further, we note that $|B_2| \leq s - 1$ so each vertex in B_2 is adjacent to one white vertex in $K_s^{(2)}$. However, since $t \geq 3$ each vertex in B_2 is also adjacent to a white vertex in $K_s^{(3)}$ so no forcing occurs.

Case 2: $1 < i < t - 1$

Then $|B_1| \leq s - 1$. Then every vertex in B_1 has at least one white neighbor in B_1 and one white neighbor not in B_1 so no vertices in B_1 do any forcing. A similar argument can be made for B_2 so no forcing occurs.

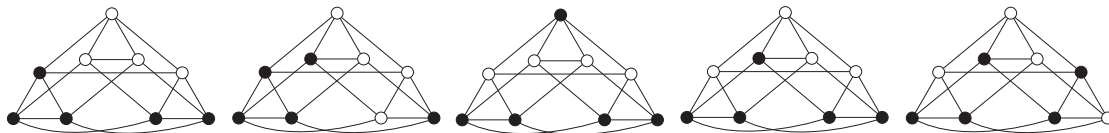
Therefore, B must be one of the above three forms. \square

Proposition 2.3.11. *If $G = K_s \square P_t$, $s \geq 3$ and $t \geq 2$, then $\text{pt}_+(G) = \left\lceil \frac{t-1}{2} \right\rceil$ and $\text{PT}_+(G) = t$. Further, G has a full PSD propagation time interval.*

Proof. Let $G = K_s \square P_t$ and B be the set of vertices of the the $\left\lceil \frac{t}{2} \right\rceil^{\text{th}}$ copy of K_s . Then B exhibits full forcing and minimizes $\text{dist}(B, \bar{B})$ so by Remark 2.1.13 $\text{pt}_+(G) = \text{pt}_+(G, B) = \left\lceil \frac{t-1}{2} \right\rceil$.

Note that the first two types of minimum PSDZF sets in Lemma 2.3.10 all force at least s vertices at each time step, so for such a B Observation 2.1.16 gives $\text{pt}_+(G, B) \leq \frac{st-s}{s} = t-1$. If B is of form (3) then only one vertex is forced at the first time step, $s-1$ are forced at the second times step and thereafter s vertices are forced at each time step. Thus $\text{pt}_+(G, B) = t$ and $\text{PT}_+(G) = t$.

To fill the PSD propagation time interval we let $B = V(K_s^{(i)})$ for $i = 1, 2, \dots, t$. \square



Proposition 2.3.12. *Let $G = K_3 \square K_3$, then $\text{pt}_+(G) = 1$ and $\text{PT}_+(G) = 2$. Further, G has a full PSD propagation time interval.*

Proof. Let $G = K_3 \square K_3$, then by [11] $Z_+(G) = 5$. By Observation 2.1.16 we know that $\text{pt}_+(G) \geq 1$. However, up to symmetry there are only 5 minimum PSDZFS sets, as analyzed below. Let B be a minimum PSDZFS.

Case 1: One copy of K_3 is entirely contained in B .

Subcase 1: One of the remaining K_3 's has two blue vertices (see Figure immediately above).

Up to symmetry these are all the same PSDZFS and $\text{pt}_+(G, B) = 2$.

Subcase 2: Each remaining K_3 has one blue vertex.

If they are in the same position (e.g. both the top vertex) then B is not a PSDZFS.

If they are in different positions then, up to symmetry, they are all the same and $\text{pt}_+(G, B) = 2$ (see Figure immediately above).

Case 2: Two K_3 's have two blue vertices.

Subcase 1: The pairs of blue vertices are in the same position.

If the other blue vertex is in the other position of the K_3 (see Figure immediately above), then $\text{pt}_+(G, B) = 1$. If the other blue vertex is in the same position as one of the others (see Figure immediately above), then $\text{pt}_+(G, B) = 2$.

Subcase 2: The pairs of blue vertices only match up in one position.

If the other blue vertex matches up with two of the other blue vertices (see Figure

immediately above), then $\text{pt}_+(G, B) = 2$. If the other blue vertex only matches up with one of the other blue vertices then we do not have a PSDZFS.

Therefore $\text{pt}_+(G) = 1$ and $\text{PT}_+(G) = 2$ and G has a full propagation time interval.

□

2.3.2 Summary and use of software

The minimum and maximum propagation times for various graph families are summarized in Table 2.1, which also includes the PSD zero forcing number and information about whether the PSD propagation time interval is full.

Notice that Proposition 2.3.8 is only a partial result in the sense that it does not establish $\text{PT}_+(C_4 \square K_3)$. Using the mathematical computer software SAGE [14] we were able write a brute force algorithm to calculate the PSD zero forcing number of a given graph and determine whether the PSD propagation time interval is full for a given graph. The URL in [5] has instructions on how to access the software in SAGE and a link to the actual code with comments (`psd_prop_time_interval.py`). This software was used to obtain the next two results.

Proposition 2.3.13. *Let $G = C_4 \square K_3$, then $\text{PT}_+(G) = 3$ and G has a full PSD propagation time interval.*

Proposition 2.3.14. *If $|G| \leq 10$ then G has a full PSD propagation time interval.*

Proposition 2.3.14, together with the results in Table 2.1, lead to the following conjecture:

Conjecture 2.3.15. *The PSD propagation time interval is full for all graphs.*

2.4 Graphs with extreme propagation time

Extreme propagation time was studied in [9]. In this section we investigate extreme minimum and maximum PSD propagation times. In particular we investigate the values $|G| - 1$, $|G| - 2$, 0 and 1 for both minimum and maximum PSD propagation time.

Table 2.1: Summary of Graph Family Results

Graph G	$Z_+(G)$ [reference]	$\text{pt}_+(G)$	$\text{PT}_+(G)$	Full interval	Prop Time Result
Tree, diam. d	1 [2]	$\left\lceil \frac{d}{2} \right\rceil$	d	Yes	[2.2.4]
P_n	1 [2]	$\left\lceil \frac{n-1}{2} \right\rceil$	$n-1$	Yes	[2.2.5]
C_n	2 [2]	$\left\lceil \frac{n-2}{4} \right\rceil$	$\left\lceil \frac{n-2}{2} \right\rceil$	Yes	[2.2.8]
W_n	3 [11]	$\left\lceil \frac{n-3}{4} \right\rceil$	$\left\lceil \frac{n-2}{2} \right\rceil$	Yes	[2.3.1]
K_{n_1, \dots, n_k}	$n_2 + \dots + n_k$ [11]	1	2	Yes	[2.3.2]
Q_n	2^{n-1} [11]	1	?	?	[2.3.7]
$K_s \square P_t$	s [11]	$\left\lceil \frac{t-1}{2} \right\rceil$	t	Yes	[2.3.11]
$P_s \square P_2, s \geq 2$	2 [11]	$\left\lceil \frac{s-1}{2} \right\rceil$	s	Yes	[2.3.3]
$P_s \square P_3, s \geq 3$	3 [2.3.5]	$\left\lceil \frac{s-1}{2} \right\rceil$?	?	[2.3.5]
$C_s \square P_2, s \geq 4$	4 [11]	$\left\lceil \frac{s-2}{4} \right\rceil$?	?	[2.3.6]
$K_3 \square K_3$	5 [11]	1	2	Yes	[2.3.12]
$C_4 \square K_3$	6 [11]	1	3	Yes	[2.3.8], [2.3.13]

2.4.1 High propagation time

It is well known that $Z_+(G) = 1$ if and only if G is a tree [2].

Proposition 2.4.1. *If G is a graph then:*

1. $\text{pt}_+(G) = |G| - 1$ if and only if $G = K_2$.
2. $\text{PT}_+(G) = |G| - 1$ if and only if G is a path.

Proof. If $G = K_2$ then it is clear that $\text{pt}_+(G) = |G| - 1$. Now assume that $\text{pt}_+(G) = |G| - 1$. Then, by Observation 2.1.16, $Z_+(G) = 1$ and at each iteration of the positive semidefinite zero

forcing algorithm exactly one white vertex gets forced white. Since $Z_+(G) = 1$ we also know that G is a tree so every vertex is itself a minimum PSDZFS. Thus, by Observation 3.3, G cannot have any vertices of degree greater than one, so $G = K_2$.

If $G = P_n$ is the path on n vertices then, choosing either end of the path as our minimum PSDZFS gives $\text{PT}_+(G) = |G| - 1$. Now we assume $\text{PT}_+(G) = |G| - 1$. This means $Z_+(G) = 1$, G is a tree, and exactly one vertex is forced at each iteration of the PSD zero forcing algorithm. However, by Observation 3.3, we cannot have any vertices of degree greater than 2, so G must be a path. \square

Corollary 2.4.2. *If $\text{PT}_+(G) = |G| - 1$ then G has a full PSD propagation time interval.*

Proposition 2.4.3. *Let G be a disconnected graph, then:*

1. $\text{pt}_+(G) = |G| - 2$ if and only if $G = P_2 \dot{\cup} P_1$.
2. $\text{PT}_+(G) = |G| - 2$ if and only if $G = P_{n-1} \dot{\cup} P_1$.

Proof. For the first statement: If $G = P_2 \dot{\cup} P_1$ then clearly $\text{pt}_+(G) = |G| - 2$. Now assume that G is disconnected and $\text{pt}_+(G) = |G| - 2$. This means that $Z_+(G) = 2$ and exactly one vertex is forced during each iteration of the PSD zero forcing algorithm. Since G is disconnected this means that one component is a P_1 and the other component, H , has $\text{pt}_+(H) = |H| - 1$ so $H = P_2$. The proof of the second statement is similar. \square

Proposition 2.4.4. *If T is a tree, $\text{PT}_+(T) = |T| - 2$ if and only if $T = S(k, l, 1)$.*

Proof. \Leftarrow This direction is shown by applying Corollary 2.2.6.

\Rightarrow Assume T is a tree and $\text{PT}_+(T) = |T| - 2$. Clearly T is not a path else $\text{PT}_+(T) = |T| - 1$. We also know that $Z_+(T) = 1$ so in order to get $\text{PT}_+(T) = |T| - 2$, there is a single time step such that we force 2 vertices. As soon as a degree k vertex is blue it will force $k - 1$ vertices. This means exactly one vertex has degree 3 and one of its neighbors has degree 1, thus $T = S(k, l, 1)$. \square

Proposition 2.4.5. *If T is a tree, $\text{pt}_+(T) = |T| - 2$ if and only if $T = P_3$ or $T = P_4$.*

Proof. If $T = P_3$ or P_4 we observe that $\text{pt}_+(T) = |T| - 2$. Now assume T is a tree, so $Z_+(T) = 1$, and $\text{pt}_+(T) = |T| - 2$. Then we can force two vertices at only one time step of the PSD zero forcing algorithm. By Remark 2.1.13, all the vertices must be degree two or less, so T is a path. This in turn limits us to P_3 or P_4 . \square

Given graph G and PSDZFS B let T_b be the forcing tree induced by $b \in B$. Orient the edges of T_b according to the forcing order to get the *directed forcing tree* \vec{T}_b . This definition will be used In Proposition 2.4.6, Corollary 2.4.7 and Proposition 2.4.8. We are trying to preserve information, e.g. if $v \rightarrow w$ we keep the direction of the forcing arrow so we can distinguish between Figures 2.17 and 2.18.

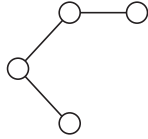


Figure 2.16: Path

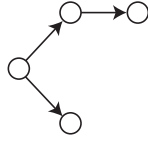


Figure 2.17: Not a directed path

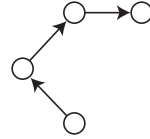


Figure 2.18: Directed path

Proposition 2.4.6. *Let B be a PSDZFS for G . If some directed forcing tree induced by B is not a directed path then there is some time step at which multiple vertices are forced.*

Proof. Let B be a PSDZFS for G such that some directed forcing tree induced by B is not a directed path, i.e. $v \rightarrow x$ and $v \rightarrow w$ at time t_1 and t_2 respectively. Without loss of generality let $t_1 \leq t_2$. We have at least two of connected components W_w containing w and W_x containing x in $G - \bigcup_{i=0}^{t_1-1} B_+^{(i)}$. By Remark 2.1.14 at least two vertices are forced at time t_1 . \square

Corollary 2.4.7. *If B is a PSDZFS for G such that there is not a time step at which multiple vertices are forced then every oriented forcing tree induced by B is a directed path and we are performing standard zero forcing.*

A graph G is a *graph of two parallel paths* if there exist two independent induced paths that cover all the vertices of G and such that any edges between the two paths can be drawn

as straight lines that do not cross [12]. A simple path is not considered to be such a graph and two disjoint paths not connected is considered to be such a path. Let G be a graph of two parallel paths P_1 and P_2 . If $v \in V(G)$ then $\text{path}(v)$ is the parallel path that contains vertex v and $\overline{\text{path}}(v)$ denotes the parallel path that does not contain v . By $\text{first}(P_i)$ and $\text{last}(P_i)$ we mean the first and last vertex of path P_i , $i = 1, 2$. If $v, w \in V(P_i)$ then $v \prec w$ means v precedes w in path P_i . Further, if $v \in V(P_i)$ and $v \neq \text{last}(P_i)$ then $\text{next}(v)$ is the vertex such that $v \prec \text{next}(v)$; $\text{prev}(v)$ is defined similarly. A *zigzag graph* is a special graph of two parallel paths and is found in [9], Definition 3.6.

A graph G on two parallel P_1 and P_2 is a zigzag graph if it satisfies the following conditions:

1. There exists a path $Q = (z_1, z_2, \dots, z_l)$ that alternates between two paths P_1 and P_2 such that:

- (a) $z_{2i-1} \in V(P_1)$ and $z_{2i} \in V(P_2)$ for $i = 1, 2, \dots, \lfloor \frac{l+1}{2} \rfloor$;
- (b) $z_j \prec z_{j+2}$ for $j = 1, 2, \dots, l-2$.

2. Every edge of G is an edge of P_1 , P_2 , or Q or is of the form

$$z_j w \text{ where } 1 < j < l, w \in \overline{\text{path}}(z_j), \text{ and } z_{j-1} \prec w \prec z_{j+1}.$$

The number l in Q is called the *zigzag order*.

An example of a zigzag graph is shown in Figure 2.19.

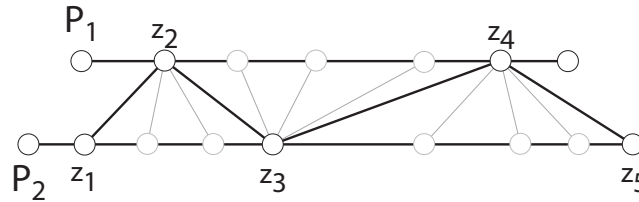


Figure 2.19: A zigzag graph with P_1 , P_2 and Q in black. Gray edges and vertices are optional

Proposition 2.4.8. *Let G be a connected graph that contains a cycle. Then G has $\text{PT}_+(G) = |G| - 2$ if and only if G is a zigzag graph such that*

1. $\text{deg}(\text{first}(P_1)) > 1$ or $\text{deg}(\text{first}(P_2)) > 1$ (both paths cannot begin with a degree one vertex)

2. $(z_{l-1}) \text{prev}(z_l) \in E(G)$
3. $z_{l-1} = \text{last}(\text{path}(z_{l-1}))$

Proof. Let G be connected, $|G| = n$, contain a cycle with $\text{PT}_+(G) = n - 2$. Then we note that $Z_+(G) = 2$ and there exists some PSDZFS such that exactly one vertex is forced at each time. By Corollary 2.4.7 our two maximal oriented forcing trees from B are directed paths and we are performing standard zero forcing; we call these forcing paths.

To show that G is a zigzag we follow the analysis from Theorem 3.7 of [9]. Let B be a minimum PSDZFS such that exactly one force is performed at each time step. Relabel the vertices of G as $V(G) = \{-1, 0, 1, 2, \dots, n - 2\}$, $B = \{-1, 0\}$, $0 \rightarrow 1$ and vertex t is forced at time t . Then G is a graph on two parallel paths P_1 and P_2 , which are the two forcing paths with the order being the forcing order. Observe that $\deg(0) \leq 2$ and $\deg(-1) \geq 2$ since 0 forces at the first time step and -1 does not. If $\deg(-1) = 2$ and $|G| > 3$ then let $P_1 = \text{path}(-1)$, $z_1 = -1$ and $z_2 = N(-1) \cap P_2$. Otherwise, let P_1 be $\text{path}(0)$, $z_2 = -1$ and $z_1 = \min\{N(-1)\}$. For $j \geq 2$ define z_{j+1} by $\max\{N(z_j) \cap \overline{\text{path}}(z_j)\}$ until $N(z_j) \cap \overline{\text{path}}(z_j) = \emptyset$. Define $Q = (z_1, z_2, \dots, z_l)$. With this labeling G is a zigzag graph.

Now we show that conditions (1) - (3) are satisfied. Since $\deg(-1) \geq 2$ one of the paths does not start with a degree one vertex so condition (1) is satisfied. If $(z_{l-1}) \text{prev}(z_l) \notin E(G)$ then when z_{l-1} is forced blue $z_{l-1} \rightarrow z_l$ and $z_{l-2} \rightarrow \text{next}(z_{l-2})$ at the next time step, which violates the one force per time step so condition (2) is satisfied. If $z_{l-1} \neq \text{last}(\text{path}(z_{l-1}))$ then when z_{l-1} is blue, $z_{l-1} \rightarrow \text{next}(z_{l-1})$ and $z_{l-2} \rightarrow \text{next}(z_{l-2})$ at the next time step, which violates the one force per time step so condition (3) is satisfied.

For the converse assume G is a zigzag graph that satisfies (1) - (3). Then $B = \{\text{first}(P_1), \text{first}(P_2)\}$ is a minimum PSDZFS such that $\text{pt}_+(G, B) = |G| - 2 \leq \text{PT}_+(G)$. However, Observation 2.1.16 gives $\text{PT}_+(G) \leq |G| - 2$. Therefore $\text{PT}_+(G) = |G| - 2$. \square

Corollary 2.4.9. *If G is a graph with $\text{PT}_+(G) = |G| - 2$ then G is one of the following graphs:*

- $P_{n-1} \dot{\cup} P_1$
- $S(k, l, 1)$

• A zigzag graph such that

1. $\deg(\text{first}(P_1)) > 1$ or $\deg(\text{first}(P_2)) > 1$ (both paths cannot begin with a degree one vertex)
2. $(z_{l-1}) \text{prev}(z_l) \in E(G)$
3. $z_{l-1} = \text{last}(\text{path}(z_{l-1}))$

Theorem 2.4.10. *If G is connected, not a tree and $\text{pt}_+(G) = |G| - 2$ then $G = C_3$.*

Proof. Assume G is connected, not a tree and $\text{pt}_+(G) = |G| - 2$. Then by Observation 2.1.16 $Z_+(G) = 2$, $\text{pt}_+(G) = \text{PT}_+(G) = |G| - 2$ and for every minimum PSDZFS exactly one vertex is forced at each time step. By Proposition 2.4.8 G is a zigzag graph. Further, [8] tells us that any vertex in G can be in a minimum PSDZFS. Then if G has any appended trees we have a cut vertex that is in a minimum PSDZFS. However, Remark 2.1.14 says that if we have a cut vertex in our PSDZFS we force multiple vertices at the first time step. Since exactly one vertex is forced at each time step G can have no appended trees. For similar reasons G cannot have any minimum PSDZF sets that are a cut set. This means that the zigzag order of G is 3, so G is a cycle, and in particular $G = C_3$. \square

Corollary 2.4.11. *If G is a graph with $\text{pt}_+(G) = |G| - 2$ then G is a P_3 , P_4 , C_3 or $P_2 \dot{\cup} P_1$. Furthermore, each of these graphs has a full PSD propagation time interval.*

2.4.2 Low propagation time

Low propagation time

Observation 2.4.12. *Let G be a graph. Then the following are equivalent:*

1. $\text{pt}_+(G) = 0$
2. $\text{PT}_+(G) = 0$
3. G is a trivial graph.

Observation 2.4.12 along with Observation 2.1.16 give the following:

Observation 2.4.13. *If $\text{PT}_+(G) = 1$ then $\text{pt}_+(G) = 1$ and G has a full PSD propagation time interval.*

Note that the converse of Observation 2.4.13 is not true since $G = P_3$ has $\text{pt}_+(G) = 1$ but $\text{PT}_+(G) = 2$.

Observation 2.4.14. *Let G be a graph. Then $\text{pt}_+(G) = 1$ if and only if there exists a minimum PSDZFS for G that is a dominating set and exhibits full forcing. Also, $\text{PT}_+(G) = 1$ if and only if every minimum PSDZFS is a dominating set with full forcing.*

The next proposition is a consequence of Proposition 2.2.4.

Proposition 2.4.15. *Let G be a tree. Then $\text{pt}_+(G) = 1$ if and only if G is a star and $\text{PT}_+(G) = 1$ if and only if $G = K_2$.*

Remark 2.4.16. If G is a graph and we append a tree to get G' , then, by cut vertex reduction [8], $Z_+(G) = Z_+(G')$.

Proposition 2.4.17. *Let G be a unicyclic graph with cycle vertices labelled in order around the cycle $\{1, 2, \dots, n\}$. Then $\text{pt}_+(G) = 1$ if and only if G is one of the following:*

1. *A C_3 or C_4 with stars attached to at most a pair of cycle vertices.*
2. *A C_5 with stars attached to cycle vertex 1 and/or cycle vertex 3.*
3. *A C_6 with stars attached to cycle vertex 1 and/or 4.*

Proof. Let G be one of the above graphs. Note that $Z_+(G) = 2$ by Remark 2.4.16 and choose our minimum PSDZFS, namely B , to be the pair of cycle vertices that (could) have stars attached to them. Then our B is a dominating set that exhibits full forcing so by Observation 2.4.14 $\text{pt}_+(G) = 1$.

Now assume G is unicyclic and $\text{pt}_+(G) = 1$. Then $Z_+(G) = 2$ and by Observation 2.4.14 there must exist a minimum PSDZFS, B , that is a dominating set. Label the vertices of the cycle in order by $\{1, 2, \dots, n\}$. Since B has to be a dominating set $n \leq 6$. Further, any trees

appended to the cycle must be stars and must be in $N(B)$. Thus at most two of the cycle vertices have appended stars. If $n = 3$ or $n = 4$ then any pair of vertices on the cycle dominates the cycle so stars can be appended to at most two cycle vertices. If $n = 5$ then, up to symmetry, only $\{1, 3\}$ dominates the cycle so the stars can be appended to this pair of cycle vertices. If $n = 6$ then, up to symmetry, only $\{1, 4\}$ dominates the cycle so the appended stars can only occur at those vertices. \square

Proposition 2.4.18. *G is unicyclic and $\text{PT}_+(G) = 1$ if and only if $G = C_3$ or C_4 .*

Proof. If $G = C_3$ or C_4 then $Z_+(G) = 2$, G has universal full forcing and every pair of vertices is a dominating set thus $\text{PT}_+(G) = 1$.

Now assume G is unicyclic and $\text{PT}_+(G) = 1$. Then $Z_+(G) = 2$, $\text{pt}_+(G) = 1$, G is one of the forms from Proposition 2.4.17, and every minimum PSDZFS must be a dominating set and exhibit full forcing. For the forms from Proposition 2.4.17, any set of two vertices that is not a set consisting of a leaf and its neighbor is a PSDZFS. This means our cycle must have 3 or 4 vertices. This also eliminates any leaves appended to our cycle. Therefore $G = C_3$ or C_4 . \square

The following definition is from [9]. Suppose $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ are graphs of the same order and $\mu : V_1 \rightarrow V_2$ is a bijection. Define the *matching graph* (H_1, H_2, μ) to be the graph constructed as the disjoint union of H_1, H_2 and the perfect matching between V_1 and V_2 defined by μ . From the same paper we also have the following theorem about standard zero forcing.

Theorem 2.4.19. [9] *Let $G = (V, E)$ be a graph. Then any two of the following conditions imply the third.*

1. $|G| = 2Z(G)$
2. $\text{pt}(G) = 1$
3. G is a matching graph.

This theorem does not extend to positive semidefinite zero forcing as the next example illustrates.

Example 2.4.20. If we try to extend Theorem 2.4.19 to positive semidefinite zero forcing our conditions would be:

1. $|G| = 2Z_+(G)$
2. $\text{pt}_+(G) = 1$
3. G is a matching graph.

If $G = P_3 \square P_2$ then G is a matching graph, $\text{pt}_+(G) = 1$ but $Z_+(G) = 2$ so $2Z_+(G) \neq |G|$. Also, if H is K_4 without one edge then $\text{pt}_+(H) = 1$, $|G| = 2Z_+(H)$ but H is not a matching graph

Observation 2.4.21. *If $|G| = 2Z_+(G)$ and G is a matching graph then $\text{pt}_+(G) = 1$.*

Proposition 2.4.22. *If H is a graph of order n and $G = (H, K_n, \mu)$ is a matching graph, then*

1. $Z_+(G) = n - 1$ if H is disconnected
2. $Z_+(G) = n$ and $\text{pt}_+(G) = 1$ if H is connected.

Proof. First assume H is disconnected with vertices $\{1, 2, \dots, n\}$ and let $G = (H, K_n, \mu)$ be a matching graph. K_n is a subgraph so by Theorem 2.3.4, $n - 1 \leq Z_+(G)$. Now let the connected components of H be C_1, C_2, \dots, C_k and label the vertices of C_k as $\{1, 2, \dots, l\}$. Define $B = V(C_1) \cup V(C_2) \cup \dots \cup V(C_{k-1}) \cup \{\mu(1), \mu(2), \dots, \mu(l-1)\}$. Note that $|B| = n - 1$ and $l = 1$ is allowed. Then $B_+^{(1)} = \mu(V(C_1)) \cup \mu(V(C_2)) \cup \dots \cup \mu(V(C_{k-1}))$, $B_+^{(2)} = \mu(l)$ and $B_+^{(3)} = \mu(V(C_k))$. Therefore B is a PSDZFS and $Z_+(G) = n - 1$.

If H is connected we contract all of the vertices of H into one vertex and we will have created a K_{n+1} minor, so $n \leq Z_+(G)$. If we choose our initial blue set of vertices to be $V(K_n)$, we will force the graph in one time step, thus $Z_+(G) = n$ and $\text{pt}_+(G) = 1$. \square

Corollary 2.4.23. *Let H be a graph of order n and $G = (H, K_n, \mu)$ be a matching graph.*

Then:

1. $Z_+(G) = n - 1$ if and only if H is disconnected

2. $Z_+(G) = n$ if and only if H is connected

Proposition 2.4.24. *If H is disconnected with order n and $G = (H, K_n, \mu)$ is a matching graph then $\text{pt}_+(G) \geq 2$.*

Proof. Since H is disconnected Proposition 2.4.22 gives $Z_+(G) = n - 1$. Let B be a minimum PSDZFS for G and define $X = V(K_n)$ and $Y = V(H)$. If $B \subseteq X$ or $B \subseteq Y$ then there is a white vertex in G that has no blue neighbor so Observation 2.1.10 gives $\text{pt}_+(G, B) \geq 2$. Now assume that initially K_n has at least two white vertices, w_1 and w_2 . Without loss of generality the pigeonhole principle says that w_1 has a neighbor in H , $\mu^{-1}(w_1)$, that is also white. Let W be the vertex set of the connected component in $G - B$ that contains w_1 , w_2 and $\mu^{-1}(w_1)$. Notice that any blue neighbor of w_1 in $G[B \cup W]$ is also a neighbor of w_2 . This means w_1 does not get forced at the first time step so $\text{pt}_+(G) \geq 2$. \square

The following corollary follows from Proposition 2.4.22 and the contrapositive of Proposition 2.4.24.

Corollary 2.4.25. *Let H be a graph of order n and $G = (K_n, H, \mu)$ be a matching graph, then $\text{pt}_+(G) = 1$ if and only if H is connected.*

Note that the arguments from Theorem 2.4.22 to Corollary 2.4.25 are valid for standard zero forcing sets and $h(G) - 1 \leq M(G) \leq Z(G)$ so the arguments here would simplify some of the arguments in [9] that dealt with matching graphs.

Bibliography

- [1] AIM Minimum Rank – Special Graphs Work Group (F. Barioli, W. Barrett, S. Butler, S. M. Cioabă, D. Cvetković, S. M. Fallat, C. Godsil, W. Haemers, L. Hogben, R. Mikkelsen, S. Narayan, O. Pryporova, I. Sciriha, W. So, D. Stevanović, H. van der Holst, K. Vander Meulen, A. Wangsness). Zero forcing sets and the minimum rank of graphs. *Linear Algebra and its Applications*, 428: 1628–1648, 2008.

- [2] F. Barioli, W. Barrett, S. Fallat, H. T. Hall, L. Hogben, B. Shader, P. van den Driessche, H. van der Holst. Zero forcing parameters and minimum rank problems. *Linear Algebra and its Applications*, 433: 401–411, 2010.
- [3] F. Barioli, W. Barrett, S. Fallat, H. T. Hall, L. Hogben, B. Shader, P. van den Driessche, H. van der Holst. Parameters related to tree-width, zero forcing, and maximum nullity of a graph. *Journal of Graph Theory*, 72: 146–177, 2013.
- [4] D. Burgarth and V. Giovannetti. Full control by locally induced relaxation. *Physical Review Letters* PRL 99, 100-501, 2007.
- [5] S. Butler, N. Warnberg. psd_prop_time_interval.py. https://github.com/warnberg/zero_forcing.
- [31] S. Butler, M. Young. Throttling zero forcing propagation speed on graphs. *Australasian Journal of Combinatorics*, 57: 65-71, 2013.
- [7] K. B. Chilakamarri, N. Dean, C. X. Kang, E. Yi. Iteration index of a zero forcing set in a graph. *Bulletin of the Institute of Combinatorics and its Applications*, 64: 57-72, 2012.
- [8] J. Ekstrand, C. Erickson, H. T. Hall, D. Hay, L. Hogben, R. Johnson, N. Kingsley, S. Osborne, T. Peters, J. Roat, A. Ross, D. Row, N. Warnberg, M. Young. Positive semidefinite zero forcing. *Linear Algebra and its Applications*, 439(7): 1862-1874, 2013.
- [9] L. Hogben, M. Huynh, N. Kingsley, S. Meyer, S. Walker, M. Young. Propagation time for zero forcing on a graph. *Discrete Applied Math*, 160: 1994-2005, 2012.
- [10] T. Peters. Positive semidefinite maximum nullity and zero forcing number. Doctoral dissertation, Iowa State University, 2012. <http://lib.dr.iastate.edu/etd/12432/>.
- [11] T. Peters. Semidefinite maximum nullity and zero forcing number. *Electronic Journal of Linear Algebra*, 23: 815 - 830, 2012.
- [12] D. D. Row. A technique for computing the zero forcing number of a graph with a cut-vertex. *Linear Algebra and its Applications*, 436: 4423-4432, 2012.

- [13] S. Severini. Nondiscriminatory propagation on trees. *Journal of Physics A*, 41: 482-002 (Fast Track Communication), 2008.
- [14] W. Stein. *Sage: Open-source mathematical software (Version: 5.4)*, The Sage Group, 2013.
<http://sagemath.org>

CHAPTER 3. COMPUTING POSITIVE SEMIDEFINITE MINIMUM RANK FOR SMALL GRAPHS

A paper to appear in *Involve: A journal of mathematics* in 2014, published by Mathematical Sciences Publishers. © 2014 Mathematical Sciences Publishers

Steven Osborne and Nathan Warnberg

Abstract

The positive semidefinite minimum rank of a simple graph G is defined to be the smallest possible rank over all positive semidefinite real symmetric matrices whose ij th entry (for $i \neq j$) is nonzero whenever $\{i, j\}$ is an edge in G and is zero otherwise. The computation of this parameter directly is difficult. However, there are a number of known bounding parameters and techniques, which can be calculated and performed on a computer. We programmed an implementation of these bounds and techniques in the open-source mathematical software *Sage*. The program, in conjunction with the orthogonal representation method, establishes the positive semidefinite minimum rank for all graphs of order 7 or less.

3.1 Introduction

Define a *graph* $G = (V, E)$ with *vertex set* $V = V(G)$ and *edge set* $E = E(G)$. The graphs discussed herein are simple (no loops or multiple edges) and undirected. The *order* of G , $|G|$, is the cardinality of $V(G)$. Two vertices v and w of a graph G are *neighbors* if $\{v, w\} \in E(G)$. If H is a graph with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ we call H a *subgraph* of G . H is an

induced subgraph of G if H is a subgraph of G and if for all pairs $v, w \in V(H)$, $\{v, w\} \in E(H)$ if $\{v, w\} \in E(G)$. Given a set of vertices $S \subseteq V(G)$, $G - S$ is the induced subgraph of G with vertices $V(G) \setminus S$.

A graph $P = (V, E)$, where $V(P) = \{v_1, v_2, \dots, v_n\}$, is called a *path* if the edges of the graph are exactly $\{v_i, v_{i+1}\}$ for $i = 1, 2, \dots, n - 1$ (see Figure 3.1). A *cycle* is a path that also has the edge $\{v_n, v_1\}$ (see Figure 3.2). A graph G is *chordal* if every induced cycle has length no greater than 3. A graph is *connected* if for any two vertices, v_1, v_2 , there exists a path with endpoints v_1 and v_2 . A connected graph with no cycles is a *tree* (see Figure 3.3). An induced graph that is a tree is an *induced tree*. A graph with n vertices in which there is an edge between every vertex is called a *complete* graph and is denoted K_n .



Figure 3.1: A path.

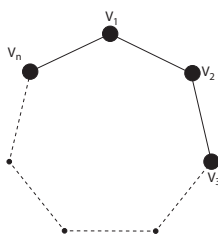


Figure 3.2: A cycle.

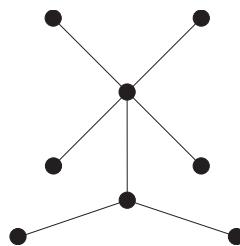


Figure 3.3: Example of a tree.

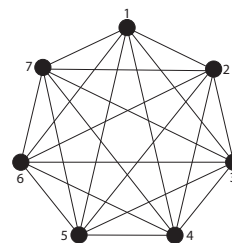


Figure 3.4: The complete graph on 7 vertices.

Let $S_n(\mathbb{R})$ denote the set of real symmetric $n \times n$ matrices. For $A = [a_{ij}] \in S_n(\mathbb{R})$, the *graph* of A , denoted $\mathcal{G}(A)$, is the graph with vertices $\{1, 2, \dots, n\}$ and edges $\{\{i, j\} : a_{ij} \neq 0 \text{ and } i \neq j\}$.

The *positive semidefinite maximum nullity* of G is

$$M_+(G) = \max\{\text{null}A : A \in S_n(\mathbb{R}) \text{ is positive semidefinite and } \mathcal{G}(A) = G\}$$

and the *positive semidefinite minimum rank* of G is

$$\text{mr}_+(G) = \min\{\text{rank}A : A \in S_n(\mathbb{R}) \text{ is positive semidefinite and } \mathcal{G}(A) = G\}.$$

Clearly $\text{mr}_+(G) + M_+(G) = |G|$.

The following concept was introduced in [2]: In a graph G where all vertices in some vertex set $S \subseteq V(G)$ are colored black and the remaining vertices are colored white, the *positive*

semidefinite color change rule is: If W_1, W_2, \dots, W_k are the sets of vertices of the k connected components of $G - S$ ($k = 1$ is a possibility), $w \in W_i$, $u \in S$, and w is the only white neighbor of u in the subgraph of G induced by $V(W_i \cup S)$, then change the color of w to black, written as $u \rightarrow w$. Given an initial set B of black vertices, the *final coloring* of B is the set of vertices colored black as result of applying the positive semidefinite color change rule iteratively until no more vertices may be colored black. If the final coloring of B is $V(G)$, B is called a *positive semidefinite zero forcing set* of G . The *positive semidefinite zero forcing number* of a graph G , denoted $Z_+(G)$, is the minimum of $|B|$ for all B positive semidefinite zero forcing sets of G . In [2] it was shown that if G is a graph then $M_+(G) \leq Z_+(G)$.

Example 3.1.1. Consider the graph G in Figure 3.5 with the set $B = \{v_4\}$ initially colored black. When the positive semidefinite color change rule is applied, the connected component of $G - B$, W_1 , is the induced subgraph of G on the vertices $\{v_1, v_2, v_3\}$. Since v_3 is the only white neighbor of v_4 in the subgraph of G induced by $W_1 \cup B$ (this is actually all of G), $v_4 \rightarrow v_3$ as demonstrated in Figure 3.6. For the next iteration, the set of black vertices is $B' = \{v_3, v_4\}$. The connected components of $G - B'$ are W'_1 , induced by $\{v_1\}$, and W'_2 , induced by $\{v_2\}$. Vertex v_1 is the only white neighbor of vertex v_3 in the subgraph of G induced by $W'_1 \cup B'$ and v_2 is the only white neighbor of vertex v_3 in the subgraph of G induced by $W'_2 \cup B'$. Therefore, $v_3 \rightarrow v_1$ (Figure 3.7) and $v_3 \rightarrow v_2$ (Figure 3.8). Now, the entire graph has been forced black (Figure 3.9) and since the process was started by a single black vertex, $Z_+(G) \leq 1$. However, at least one vertex must be colored to begin the zero forcing process. Therefore, $Z_+(G) = 1$.

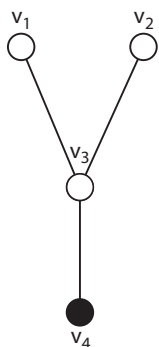


Figure 3.5

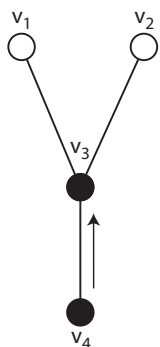


Figure 3.6

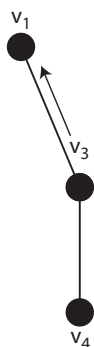


Figure 3.7

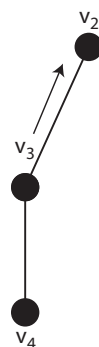


Figure 3.8

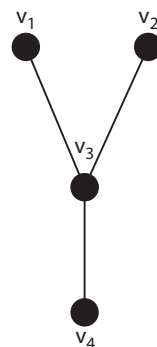


Figure 3.9

Let G be a graph and S the smallest subset of $V(G)$ such that $G - S$ is disconnected. Then $|S| = \kappa(G)$ is called the *vertex connectivity* of G . A *clique covering* of G is a set of induced subgraphs $\{S_i\}$ of G such that each S_i is complete and $E(G) = \bigcup E(S_i)$. The *clique cover number* of a graph G , denoted $\text{cc}(G)$, is the minimum of $|\{S_i\}|$ over all $\{S_i\}$ clique coverings of G .

In [3] $M_+(G)$ was determined for every graph G of order at most 6. Use of published software for computing $Z_+(G)$ [4], establishes $M_+(G) = Z_+(G)$ for $|G| \leq 6$. We developed a program [14] in the open-source computer mathematics software system Sage [17] to compute bounds for positive semidefinite maximum nullity. The program utilizes software for computing $Z_+(G)$ [4] and known results for computing positive semidefinite maximum nullity. These results are summarized in Section 3.2. A detailed description of the program may be found in Appendix A. Sections 3.2 and 3.3 provide a survey of techniques for computing positive semidefinite minimum rank.

In Section 3.3 we determine $M_+(G)$ for $|G| \leq 7$ and show $M_+(G) = Z_+(G)$ for all such graphs. For all but 13 graphs of order 7, $M_+(G)$ can be computed by the program. We then established $M_+(G)$ for the remaining 13 graphs by utilizing orthogonal representation to find a positive semidefinite matrix A with $\mathcal{G}(A) = G$ and nullity of $A = Z_+(G)$. This establishes that $M_+(G) = Z_+(G)$ for each graph G of order at most 7. These matrices are listed in Appendix B.

3.2 Known results used by the program to establish positive semidefinite minimum rank/maximum nullity

Note that all of our parameters sum over the connected components of a disconnected graph. Given its relation to the positive semidefinite zero forcing number, the following results are given in terms of positive semidefinite maximum nullity. However, given a graph G , $M_+(G) + \text{mr}_+(G) = |G|$, so all of the following results may easily be translated to positive semidefinite minimum rank.

Theorem 3.2.1. [8] *If G is a graph the following are true:*

1. $Z_+(G) = 1$ if and only if $M_+(G) = 1$.
2. $Z_+(G) = 2$ if and only if $M_+(G) = 2$.
3. $Z_+(G) = 3$ implies $M_+(G) = 3$.

Corollary 3.2.2. *If $Z_+(G) \geq 3$, then $M_+(G) \geq 3$.*

Observation 3.2.3. [8] $Z_+(G) = |G| - 1$ if and only if $M_+(G) = |G| - 1$.

Note that the only graph G having $Z_+(G) = |G| - 1$ is K_n , the complete graph on n vertices.

For a chordal graph G , it was shown in [3] that $cc(G) = mr_+(G)$, in [9] it was shown that $OS(G) = cc(G)$, and in [2] it was shown that $Z_+(G) + OS(G) = |G|$, where $OS(G)$ is the ordered subgraph number of G (see [13] for the definition of $OS(G)$). Thus $Z_+(G) = M_+(G)$, which gives the next theorem.

Theorem 3.2.4. [2][3][9] *If G is chordal, then $M_+(G) = Z_+(G)$.*

Example 3.2.5. Consider graph $G551$ in Figure 3.10. Sets of vertices of size 1 and 2 are clearly not positive semidefinite zero forcing sets, so $Z_+(G551) \geq 3$. Notice that choosing an initial set of 3 black vertices that are all non-adjacent does not force anything. By symmetry this reduces to two cases. In the first case we choose $\{1, 2\}$ as our adjacent black vertices and as our third we choose any of the remaining vertices and notice that the graph will not be forced. Similarly, choosing $\{1, 3\}$ as our adjacent black vertices and any of the remaining vertices as our third also fails to force the graph. Thus, $Z_+(G551) \geq 4$. Observe that $\{1, 3, 4, 5\}$ forms a positive semidefinite zero forcing set meaning $Z_+(G551) \leq 4$, hence $Z_+(G551) = 4$. However, $G551$ is chordal as its largest cycle is size 3. Therefore, by Theorem 3.2.4 $M_+(G551) = 4$.

Theorem 3.2.6. [11, 12] *For every graph G , $\kappa(G) \leq M_+(G)$.*

Example 3.2.7. By inspection, removing any one vertex from graph $G128$ (see Figure 3.11) will not result in a disconnected graph. Therefore, $\kappa(G) \geq 2$. Further, $\{3, 4\}$ forms a positive semidefinite zero forcing set for $G128$. Thus, $Z_+(G) \leq 2$. This gives $2 \leq \kappa(G) \leq M_+(G) \leq Z_+(G) \leq 2$.

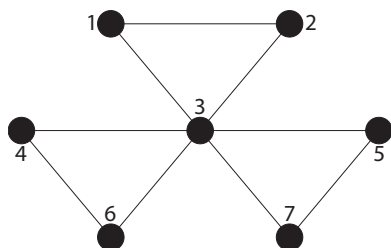


Figure 3.10: Graph G551

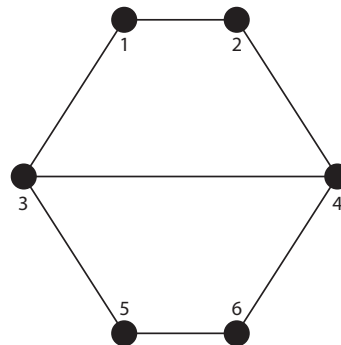


Figure 3.11: Graph G128

For a graph G the *neighborhood* of $v \in V(G)$ is $N_G(v) = \{w \in V(G) \mid v \text{ is adjacent to } w\}$. Vertices v and w are called *duplicate vertices* if $N_G(v) \cup \{v\} = N_G(w) \cup \{w\}$.

Proposition 3.2.8. [8] *If v and w are duplicate vertices in a connected graph G with $|G| \geq 3$, then $Z_+(G - v) = Z_+(G) - 1$.*

Proposition 3.2.9. [3] *If v and w are duplicate vertices in a connected graph G with $|G| \geq 3$, then $\text{mr}_+(G - v) = \text{mr}_+(G)$.*

Recall that for any graph G , $\text{mr}_+(G) + M_+(G) = |G|$, which gives the following corollary.

Corollary 3.2.10. *If v and w are duplicate vertices in a connected graph G with $|G| \geq 3$, then $M_+(G - v) = M_+(G) - 1$.*

Example 3.2.11. In graph $G1196$ (see Figure 3.12) notice that 2 and 4 are duplicate vertices, as are vertices 3 and 5. Removal of vertices 2 and 3 results in a graph that is isomorphic to graph $G43$ (see Figure 3.13). $Z_+(G43) = 2$ thus $M_+(G43) = 2$ by Theorem 3.2.1. Therefore, $M_+(G1196) = 4$ by Corollary 3.2.10.

Cut-vertex reduction is a standard technique in the study of minimum rank. A vertex v of a connected graph G is a *cut-vertex* if $G - v$ is disconnected. Suppose $G_i, i = 1, \dots, h$, are graphs of order at least two, there is a vertex v such that for all $i \neq j$, $G_i \cap G_j = \{v\}$, and $G = \cup_{i=1}^h G_i$ (if $h \geq 2$, then clearly v is a cut-vertex of G). Then it is observed in [10] that

$$\text{mr}_+(G) = \sum_{i=1}^h \text{mr}_+(G_i).$$

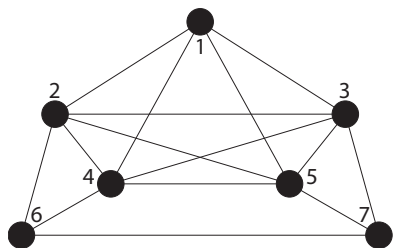


Figure 3.12: Graph G1196

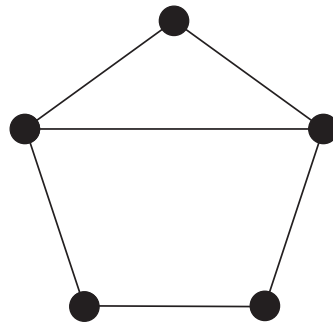


Figure 3.13: Graph G43

Because $\text{mr}_+(G) + M_+(G) = |G|$, this is equivalent to

$$M_+(G) = \left(\sum_{i=1}^h M_+(G_i) \right) - h + 1. \quad (3.1)$$

It is shown in [13] that

$$OS(G) = \sum_{i=1}^h OS(G_i)$$

Because $OS(G) + Z_+(G) = |G|$ (shown in [2]), this is equivalent to

$$Z_+(G) = \left(\sum_{i=1}^h Z_+(G_i) \right) - h + 1. \quad (3.2)$$

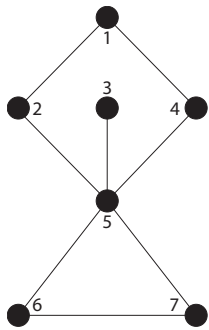
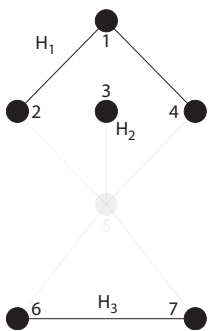
Figure 3.14:
Graph G419

Figure 3.15

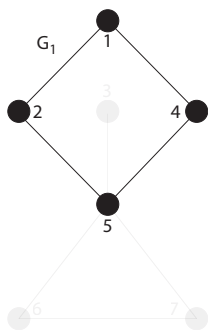


Figure 3.16

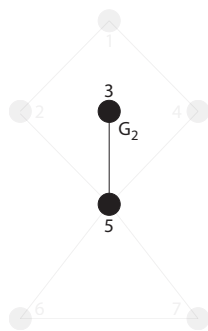


Figure 3.17

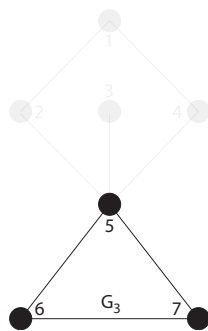


Figure 3.18

Example 3.2.12. Equation 3.2 can be used to compute $Z_+(G419)$ and $M_+(G419)$ (see Figure 3.14). Notice that vertex 5 is a cut vertex of the graph since removing it results in a disconnected graph with 3 components, namely H_1, H_2 and H_3 . When vertex 5 is reconnected to each of our components it is easy to see that $G_i \cap G_j = \{5\}$ for $i, j \in \{1, 2, 3\}$ with $i \neq j$, as illustrated by Figures 3.16, 3.17, and 3.18. It is also clear that $\cup_{i=1}^3 G_i = G419$, $Z_+(G_1) = 2$, $Z_+(G_2) = 1$,

and $Z_+(G_3) = 2$. Thus, by Equation 3.2, $Z_+(G_{419}) = 2 + 1 + 2 - 3 + 1 = 3$. A similar argument shows that $M_+(G_{419}) = 3$.

Observe that if $\kappa(G) = 1$, there exists a cut vertex. The next result is an immediate consequence of the cut-vertex reduction Equations 3.1 and 3.2.

Observation 3.2.13. [8] *Suppose G_i , $i = 1, \dots, h$ are graphs, there is a vertex v such that for all $i \neq j$, $G_i \cap G_j = \{v\}$, and $G = \bigcup_{i=1}^h G_i$. If $M_+(G_i) = Z_+(G_i)$ for all $i = 1, \dots, h$, then $M_+(G) = Z_+(G)$.*

Observation 3.2.14. [9] *If G is a graph then $cc(G) \geq mr_+(G)$.*

Corollary 3.2.15. $|G| - cc(G) \leq M_+(G)$.

Example 3.2.16. In Figure 3.19 notice that graph G_{200} is not complete so $mr_+(G_{200}) \geq 2$. Also, note that the subgraphs induced by $S_1 = \{1, 2, 3, 4, 5\}$ and $S_2 = \{4, 5, 6\}$ are complete and $E(G_{200}) = E(S_1) \cup E(S_2)$ so $cc(G_{200}) \leq 2$, hence $mr_+(G_{200}) = 2$.

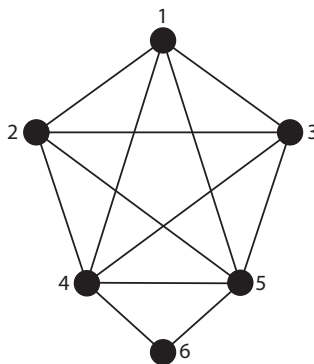


Figure 3.19: Graph G_{200}

In [3] the *tree size* of a graph G , denoted $ts(G)$, is defined to be the number of vertices in a maximum induced tree of G . Also from [3], if T is a maximum induced tree and w is a vertex not belonging to T , denote by $\mathcal{E}(w)$ the set of all edges of all paths in T between every pair of vertices of T that are adjacent to w . The following theorem was established by Booth et. al.

Theorem 3.2.17. [3] *For a connected graph G ,*

$$mr_+(G) = ts(G) - 1 \tag{3.3}$$

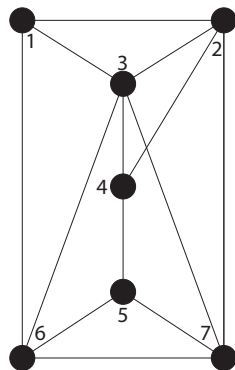


Figure 3.20: Graph G1090

if the following condition holds: there exists a maximum induced tree T such that for u and w not on T , $\mathcal{E}(u) \cap \mathcal{E}(w) \neq \emptyset$ if and only if u and w are adjacent in G .

Note that Equation 3.3 may be rewritten as $M_+(G) = |G| - \text{ts}(G) + 1$.

Example 3.2.18. To illustrate the previous theorem we consider graph $G1090$ (see Figure 3.20). To find $\text{ts}(G1090)$ notice that $G1090$ has two disjoint, induced K_3 's, namely the graphs induced by vertex sets $\{1, 2, 3\}$ and $\{5, 6, 7\}$. This means in order to find an induced tree, removal of one vertex from each K_3 is required. By inspection, removal of any of the nine pairs $\{\{1, 5\}, \{1, 6\}, \{1, 7\}, \{2, 5\}, \dots, \{3, 7\}\}$ results in a graph with a cycle, thus $\text{ts}(G1090) \leq 4$. However, the subgraph induced by $\{1, 4, 5, 6\}$ is a tree (call it T), hence $\text{ts}(G1090) = 4$. We show T satisfies the condition of Theorem 3.2.17. The vertices not in $G1090 - T$ are 2, 3, and 7, which are all adjacent in $G1090$. $\mathcal{E}(2) = \{(1, 6), (5, 6), (4, 5)\} = \mathcal{E}(3)$ and $\mathcal{E}(7) = \{(5, 6)\}$. Therefore, $\mathcal{E}(2) \cap \mathcal{E}(3) \cap \mathcal{E}(7) \neq \emptyset$ and the condition holds because $\{2, 3, 7\}$ are pairwise adjacent. Thus $M_+(G1090) = 4$.

3.3 Computation of positive semidefinite maximum nullity of graphs of order 7 or less

The program [14] implements the results from Section 3.2. Running the program on all graphs of order 7 or less yielded positive semidefinite maximum nullity for 1239 of 1252 graphs. It may be noted that the positive semidefinite maximum nullity was already known for the 208 graphs of order 6 or less (see [3]). However, the program was able to successfully compute the

positive semidefinite maximum nullity for these graphs without referencing this information. For the remaining 13 graphs, the method of orthogonal representations was used to construct a matrix representation exhibiting nullity equal to the positive semidefinite zero forcing number. These matrices are shown in Appendix B.

A set $\vec{V} = \{\vec{v}_1, \dots, \vec{v}_n\}$ in \mathbb{R}^d is an *orthogonal representation* of the graph G if for $i \neq j$, the dot product of \vec{v}_i with \vec{v}_j is nonzero if the vertices i and j are adjacent, and zero otherwise. If $\vec{V} = \{\vec{v}_1, \dots, \vec{v}_n\}$ is an orthogonal representation of the graph G in \mathbb{R}^d and $B = [\vec{v}_1 \dots \vec{v}_n]$, then $B^T B \in \mathcal{S}_+(G)$ and $\text{rank } B^T B \leq d$. Thus, if a representation is found in \mathbb{R}^d then $\text{mr}_+(G) \leq d$ and $M_+(G) \geq |G| - d$.

Example 3.3.1. Consider graph $G17$ in Figure 3.21. Note that when we refer to a graph in the form $G17$ we are using notation from [16]. To start constructing an orthogonal representation for $G17$ let $v_1, v_2, v_3, v_4 \in \mathbb{R}^2$ correspond to vertices 1, 2, 3 and 4 respectively. Choose as many

disjoint vertices as possible, say 1 and 4. By definition $v_1 \cdot v_4 = 0$ so let $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and

$v_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. To find v_2 and v_3 , set $v_2 = \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$ and $v_3 = \begin{bmatrix} a_3 \\ b_3 \end{bmatrix}$. Now, v_2 is adjacent to v_1 and v_4 so $v_1 \cdot v_2 \neq 0$ and $v_2 \cdot v_4 \neq 0$. Thus $a_2 \neq 0 \neq b_2$. Similarly, $a_3 \neq 0 \neq b_3$. Since

v_2 and v_3 are not adjacent, we know $v_2 \cdot v_3 = a_2 a_3 + b_2 b_3 = 0$. With these restrictions it is clear that $a_2 = a_3 = b_2 = 1$ and $b_3 = -1$ is a solution and an orthogonal representation

construction is complete. This gives $B = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix}$ and $B^T B = A$ (see Figure 3.22).

By construction, $\text{rank}(A) = 2$. Thus $\text{mr}_+(G17) \leq 2$ and $M_+(G17) \geq |G| - 2 = 2$. Observe that $\{1, 2\}$ forms a positive semidefinite zero forcing set for graph $G17$ hence $Z_+(G17) \leq 2$. Finally, $2 \leq M_+(G17) \leq Z_+(G17) \leq 2$.

In every case, positive semidefinite maximum nullity was found to equal the positive semidefinite zero forcing number. This has established the next result.

Theorem 3.3.2. *If G is a graph with 7 or fewer vertices, then $M_+(G) = Z_+(G)$.*

See [15] for a complete spreadsheet containing positive semidefinite maximum nullity and zero forcing number for all graphs with 7 or fewer vertices.

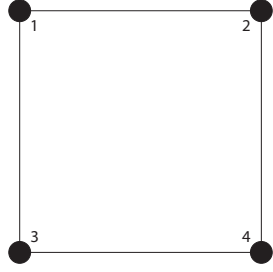


Figure 3.21: Graph G17

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{pmatrix}$$

Figure 3.22: A , a matrix representation of $G17$

Corollary 3.3.3. *Suppose $G_i, i = 1, \dots, h$, are graphs with $|G_i| \leq 7$, there is a vertex v such that for all $i \neq j$, $G_i \cap G_j = \{v\}$, and $G = \bigcup_{i=1}^h G_i$. Then $M_+(G) = Z_+(G)$.*

Proof. Apply Theorem 3.3.2 to Observation 3.2.13. □

Note that Theorem 3.3.2 cannot be extended to graphs with more than 7 vertices as $Z_+(V_8) = 4$ and $M_+(V_8) = 3$ (shown in [13]), where V_8 is the Möbius ladder on 8 vertices (see Figure 3.23).

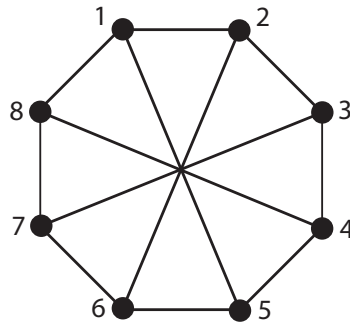


Figure 3.23: Möbius ladder on 8 vertices

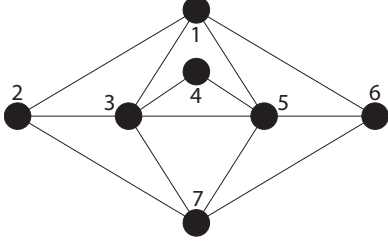
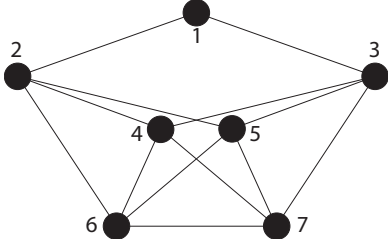
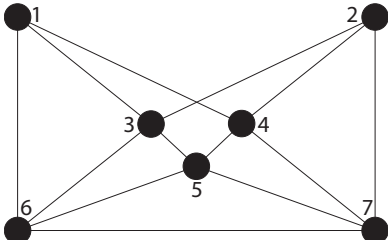
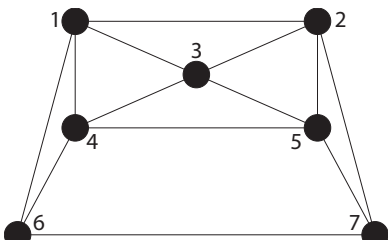
APPENDIX A: METHOD USED BY THE PROGRAM

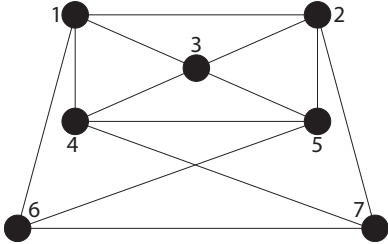
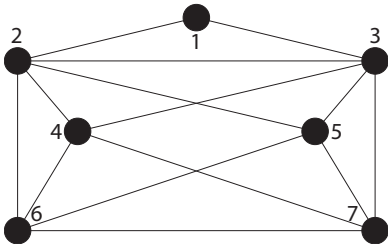
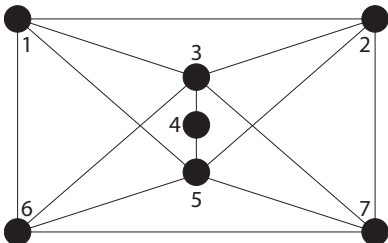
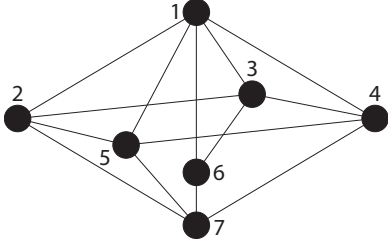
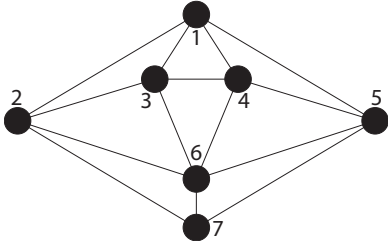
The program uses the following general method:

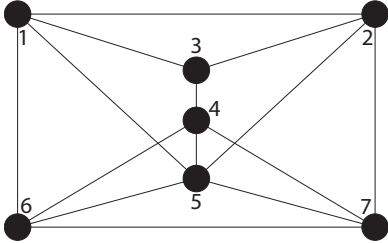
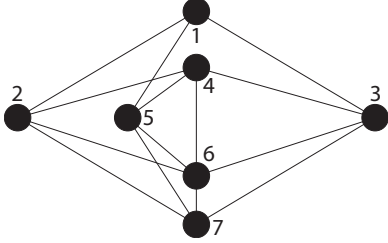
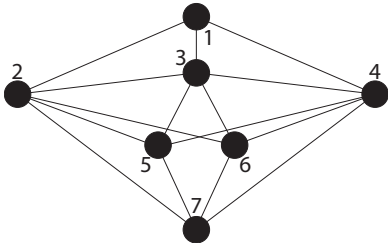
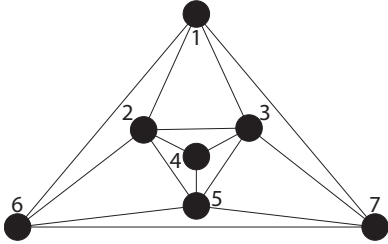
1. Separate the graph into its connected components and work on each component separately. Results will be summed before reporting.
2. Compute $Z_+(G)$.
 - (a) If $Z_+(G) \leq 3$, apply the results of Theorem 3.2.1.
 - (b) Else, use Corollary 3.2.2 to establish a lower bound for $M_+(G)$.
3. If $Z_+(G) = |G| - 1$, apply the results of Observation 3.2.3.
4. If G is chordal, apply Theorem 3.2.4.
5. Compute the vertex connectivity of G ($\kappa(G)$).
 - (a) If $\kappa(G) = Z_+(G)$, apply Theorem 3.2.6.
 - (b) Else, if $\kappa(G)$ is a tighter bound for $M_+(G)$, improve the lower bound.
6. If there are duplicate vertices in the graph, discard all but one copy by applying Corollary 3.2.10 and returning to step 2.
7. Apply the cut-vertex formula iteratively by applying Equation 3.1 and returning to step 2 for each component.
8. Compute the clique cover number of G .
 - (a) If $|G| - \text{cc}(G) = Z_+(G)$, apply Corollary 3.2.15.
 - (b) Else, if $\text{cc}(G)$ is a tighter bound for $M_+(G)$, improve the lower bound.
9. Apply Theorem 3.2.17 to determine if $M_+(G) = |G| - \text{ts}(G) + 1$.

APPENDIX B: MATRIX REPRESENTATIONS

For each of the following thirteen matrices $\text{null}(A) = 4 = Z_+(G)$.

$\begin{bmatrix} 2 & -1 & -1 & 0 & 1 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 2 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 2 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 2 \end{bmatrix}$	<p style="text-align: center;">G_{1060}</p> 
$\begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 3 & 0 & -1 & 3 & 1 & 0 \\ 1 & 0 & 2 & -2 & 1 & 0 & -1 \\ 0 & -1 & -2 & 5 & 0 & 1 & 3 \\ 0 & 3 & 1 & 0 & 5 & 2 & 1 \\ 0 & 1 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & -1 & 3 & 1 & 1 & 2 \end{bmatrix}$	<p style="text-align: center;">G_{1075}</p> 
$\begin{bmatrix} 1 & 0 & -1 & 4 & 0 & -1 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 1 \\ -1 & 4 & 33 & 0 & -4 & -15 & 0 \\ 4 & 2 & 0 & 21 & 1 & 0 & 3 \\ 0 & 0 & -4 & 1 & 1 & 4 & 1 \\ -1 & 0 & -15 & 0 & 4 & 17 & 4 \\ 0 & 1 & 0 & 3 & 1 & 4 & 2 \end{bmatrix}$	<p style="text-align: center;">G_{1100}</p> 
$\begin{bmatrix} 1 & 1 & 1 & 2 & 0 & 3 & 0 \\ 1 & 6 & 7 & 0 & -1 & 0 & 1 \\ 1 & 7 & 10 & -1 & -3 & 0 & 0 \\ 2 & 0 & -1 & 5 & 1 & 7 & 0 \\ 0 & -1 & -3 & 1 & 2 & 0 & 1 \\ 3 & 0 & 0 & 7 & 0 & 11 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 & 1 \end{bmatrix}$	<p style="text-align: center;">G_{1104}</p> 

$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & -1 & 0 \\ 1 & 3 & 2 & 0 & 1 & 0 & 1 \\ 1 & 2 & 2 & 2 & 1 & 0 & 0 \\ 1 & 0 & 2 & 6 & 1 & 0 & -2 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 & 1 \end{bmatrix}$	<p style="text-align: center;">G_{1105}</p> 
$\begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 3 & -2 & 1 & 1 & 3 & 0 \\ -1 & -2 & 6 & -2 & 1 & 0 & 3 \\ 0 & 1 & -2 & 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 2 & 1 \\ 0 & 3 & 0 & 1 & 2 & 5 & 1 \\ 0 & 0 & 3 & -1 & 1 & 1 & 2 \end{bmatrix}$	<p style="text-align: center;">G_{1135}</p> 
$\begin{bmatrix} 2 & 1 & -3 & 0 & 3 & -1 & 0 \\ 1 & 1 & -2 & 0 & 2 & 0 & 1 \\ -3 & -2 & 30 & 5 & 0 & 1 & -1 \\ 0 & 0 & 5 & 1 & 1 & 0 & 0 \\ 3 & 2 & 0 & 1 & 6 & -1 & 1 \\ -1 & 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 1 & 1 & 2 \end{bmatrix}$	<p style="text-align: center;">G_{1137}</p> 
$\begin{bmatrix} 3 & 1 & -3 & 1 & 3 & 1 & 0 \\ 1 & 1 & 2 & 0 & 2 & 0 & 1 \\ -3 & 2 & 21 & -4 & 0 & -1 & 0 \\ 1 & 0 & -4 & 1 & 1 & 0 & 1 \\ 3 & 2 & 0 & 1 & 5 & 0 & 3 \\ 1 & 0 & -1 & 0 & 0 & 1 & -2 \\ 0 & 1 & 0 & 1 & 3 & -2 & 6 \end{bmatrix}$	<p style="text-align: center;">G_{1165}</p> 
$\begin{bmatrix} 1 & 2 & 1 & 1 & 1 & 0 & 0 \\ 2 & 6 & 1 & 0 & 0 & 2 & 1 \\ 1 & 1 & 2 & 3 & 0 & -1 & 0 \\ 1 & 0 & 3 & 5 & -1 & -2 & 0 \\ 1 & 0 & 0 & -1 & 11 & -2 & -3 \\ 0 & 2 & -1 & -2 & -2 & 2 & 1 \\ 0 & 1 & 0 & 0 & -3 & 1 & 1 \end{bmatrix}$	<p style="text-align: center;">G_{1167}</p> 

$\begin{bmatrix} 2 & -3 & 1 & 0 & 1 & 1 & 0 \\ -3 & 6 & -1 & 0 & -1 & 0 & 1 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 & 2 & 3 & 1 \\ 1 & -1 & 0 & 2 & 2 & 3 & 1 \\ 1 & 0 & 0 & 3 & 3 & 5 & 2 \\ 0 & 1 & 0 & 1 & 1 & 2 & 1 \end{bmatrix}$	<p style="text-align: center;">G_{1168}</p> 
$\begin{bmatrix} 1 & 1 & 3 & 0 & 2 & 0 & 0 \\ 1 & 6 & 0 & -2 & 0 & -1 & 1 \\ 3 & 0 & 14 & 2 & 0 & 3 & 1 \\ 0 & -2 & 2 & 1 & -1 & 1 & 0 \\ 2 & 0 & 0 & -1 & 21 & -5 & -4 \\ 0 & -1 & 3 & 1 & -5 & 2 & 1 \\ 0 & 1 & 1 & 0 & -4 & 1 & 1 \end{bmatrix}$	<p style="text-align: center;">G_{1169}</p> 
$\begin{bmatrix} 1 & -4 & 1 & 1 & 0 & 0 & 0 \\ -4 & 21 & -2 & 0 & 1 & -3 & -1 \\ 1 & -2 & 2 & 2 & 1 & -1 & 0 \\ 1 & 0 & 2 & 6 & -1 & -3 & -2 \\ 0 & 1 & 1 & -1 & 2 & 0 & 1 \\ 0 & -3 & -1 & -3 & 0 & 2 & 1 \\ 0 & -1 & 0 & -2 & 1 & 1 & 1 \end{bmatrix}$	<p style="text-align: center;">G_{1202}</p> 
$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 & -3 \\ 1 & 3 & 1 & 1 & 1 & 4 & 0 \\ 1 & 1 & 3 & 3 & 1 & 0 & -4 \\ 0 & 1 & 3 & 5 & 2 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 & 1 & 6 & 2 \\ -3 & 0 & -4 & 0 & 1 & 2 & 14 \end{bmatrix}$	<p style="text-align: center;">G_{1205}</p> 

1

Bibliography

- [1] F. Barioli, W. Barrett, S. Fallat, H. T. Hall, L. Hogben, B. Shader, P. van den Driessche, H. van der Holst. Parameters related to tree-width, zero forcing, and maximum nullity of a graph. *Journal of Graph Theory*, 72: 146177, 2012.

- [2] F. Barioli, W. Barrett, S. Fallat, H. T. Hall, L. Hogben, B. Shader, P. van den Driessche, H. van der Holst. Zero forcing parameters and minimum rank problems. *Linear Algebra and its Applications*, 433: 401–411, 2010.
- [3] M. Booth, P. Hackney, B. Harris, C. R. Johnson, M. Lay, L. H. Mitchell, S. K. Narayan, A. Pascoe, K. Steinmetz, B. D. Sutton, W. Wang. On the minimum rank among positive semidefinite matrices with a given graph. *SIAM Journal of Matrix Analysis and Applications*, 30: 731–740, 2008.
- [4] S. Butler, J. Grout. Zq.py. https://github.com/jasongrout/minimum_rank/blob/master/Zq.py.
- [5] L. DeLoss, J. Grout, L. Hogben, T. McKay, J. Smith, G. Tims. Techniques for determining the minimum rank of a small graph. *Linear Algebra and its Applications*, 432: 2995 - 3001, 2010.
- [6] L. DeLoss, J. Grout, T. McKay, J. Smith, G. Tims. Program for calculating bounds on the minimum rank of a graph using Sage. <http://arxiv.org/abs/0812.1616>.
- [7] R. Diestel. *Graph Theory*, Electronic Edition, Springer-Verlag, NY, 2005.
- [8] J. Ekstrand, C. Erickson, H. T. Hall, D. Hay, L. Hogben, R. Johnson, N. Kingsley, S. Osborne, T. Peters, J. Roat, A. Ross, D. Row, N. Warnberg, M. Young. Positive Semidefinite Zero Forcing. *Linear Algebra and Applications* 439(7): 1862-1874, 2013.
- [9] P. Hackney, B. Harris, M. Lay, L. H. Mitchell, S. K. Narayan, A. Pascoe. Linearly independent vertices and minimum semidefinite rank. *Linear Algebra and its Applications*, 431: 1105–1115, 2009.
- [10] H. van der Holst. On the maximum positive semi-definite nullity and the cycle matroid of graphs. *Electronic Journal of Linear Algebra*, 18: 192–201, 2009.
- [11] L. Lovász, M. Saks and A. Schrijver. Orthogonal representations and connectivity of graphs. *Linear Algebra and its Applications*, 114/115: 439-454, 1989.

- [12] L. Lovász, M. Saks and A. Schrijver. A correction: "Orthogonal representations and connectivity of graphs." *Linear Algebra and its Applications*, 313: 101-105, 2000.
- [13] L. Mitchell, S. Narayan, and A. Zimmer. Lower bounds in minimum rank problems. *Linear Algebra and its Applications*, 432: 430-440, 2010.
- [14] S. Osborne, N. Warnberg. Program for calculating bounds of positive semidefinite maximum nullity of a graph using Sage. http://www.public.iastate.edu/~warnberg/msr_program.txt.
- [15] S. Osborne, N. Warnberg. Spreadsheet of positive semidefinite maximum nullity and zero forcing number of graphs with 7 or fewer vertices. <http://www.public.iastate.edu/~warnberg/MpZpSpreadsheet.csv>.
- [16] R.C. Read, R.J. Wilson. *An Atlas of Graphs*, Oxford University Press, Oxford,1998.
- [17] W. Stein. *Sage: Open-source mathematical software (Version 4.6.2)*, The Sage Group, 2011. <http://www.sagemath.org>.

CHAPTER 4. GENERAL CONCLUSIONS

4.1 General discussion

Previously known results on zero forcing, propagation time and positive semidefinite zero forcing were presented in Section 1.2. In Chapter 2 the ideas about propagation time based on the zero forcing number were extended to PSD propagation time using the PSD zero forcing number. In particular, the PSD propagation time interval, minimum and maximum propagation time for graph families and extreme propagation time were discussed.

In Chapter 3 current graph parameters that are related to PSD minimum rank and PSD zero forcing number are synthesized to create upper and lower bounds on PSD maximum nullity. These parameters were programmed using the mathematical software SAGE. From the program and the use of orthogonal representations it is established that $M_+(G) = Z_+(G)$ for all graphs on 7 or fewer vertices. This established that the Möbius ladder on 8 vertices (see Figure 1.15) is the smallest graph with $M_+(G) < Z_+(G)$.

4.2 Recommendations for future research

There are many questions yet to be answered about positive semidefinite propagation time. The most interesting question is whether or not the PSD propagation time interval is full for all graphs. To date it is known that all graphs of order 10 or less have a full propagation time interval, along with the graph families discussed in Chapter 2 and others that have been analyzed but not published. The migration tool has not been exploited as much as it could be, particularly on nicely structured graph families. There are also several other color change rules that have corresponding zero forcing numbers [1]; can the propagation time for these other rules be developed? It has also been established that the standard zero forcing number has

applications in physics [2]; can these connections be further strengthened and utilized? Zero forcing and positive semidefinite zero forcing also can be viewed as models for rumor spreading or virus spreading; can they be modified to investigate more complex models? The algorithms used for zero forcing and PSD zero forcing are also very similar to those used in the power domination [4] and generalized power domination [3] of graphs. Can known results for zero forcing algorithms help answer questions about graph domination problems?

Bibliography

- [1] F. Barioli, W. Barrett, S. Fallat, H.T. Hall, L. Hogben, B. Shader, P. van den Driessche, and H. van der Holst. Parameters related to tree-width, zero forcing, and maximum nullity of a graph. *J. Graph Theory*, 72: 146–177, 2013.
- [2] D. Burgarth, D. D’Alessandro, L. Hogben, S. Severini, and M. Young. Zero forcing, linear and quantum controllability for systems evolving on networks. *IEEE Trans. Auto. Control*, 58: 2349 - 2354, 2013.
- [3] G. Chang, P. Dorbec, M. Montassier, A. Raspaud. Generalized power domination in graphs. *Dis. Appl. Math.*, 160:1691-1698, 2012.
- [4] G. Xu, L. Kang, E. Shan, M. Zhao. Power domination in block graphs. *Theo. Comp. Sci.*, 359: 299 - 305, 2006

ACKNOWLEDGEMENTS

I would like to take this opportunity to express my thanks to those who helped me with various aspects of conducting research and the writing of this thesis. First I would like to thank my wife Katey for her constant support and my daughter Myla for bringing more joy than I ever could have hoped for. Of course this list is non-exhaustive and although I would like to thank everyone by name I am afraid some would be left out so I will start by thanking all of the graduate students, faculty and staff in the Mathematics Department at Iowa State University for both giving me the opportunity as a graduate student and making my experience enjoyable. For making the department feel like home I would like to thank Devin Bickner, Nicole Kingsley, Michelle Lastrina, Jake Manske, Jason Smith, Marshall Stuart, Chad Vidden, Andrew Wells and Andy Ylvisaker. I would also like to thank my research brothers Craig Erickson, Ching-Hung Lin, and Geoff Tims for many enlightening conversations and much support throughout my graduate career. To the many friends that I have made, in particular Ryan Johnson, Steven Osborne and Travis Peters who have always been there for me. A big thanks to Michael Young for his support and advice both personally and professionally; graduate school would have been much less fulfilling if he were not a part of it. Steve Butler has also been a source of inspiration for his wonderfully outgoing personality, excellent presentations, vast mathematical knowledge and his passion for helping students reach their potential. Thanks to Elgin Johnston who has organized many teaching seminars which have given me much food for thought about my pedagogy. Barb Licklider has inspired me to become a lifelong learner and to become the educator that I have always wanted to be. Last, but certainly not least, I would like to thank my advisor Leslie Hogben. Your passion for mathematics, your collaborative approach and your 'get down to business' attitude have molded me into a more successful mathematician than I could have ever imagined. I look forward to many more years of mathematics research with much credit given to you. I could not have done this without you. Thank you.