EQUIVARIANT GRÖBNER BASES AND THE GAUSSIAN TWO-FACTOR MODEL

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ABSTRACT. We show that the kernel I of the ring homomorphism $\mathbb{R}[y_{ij} \mid i, j \in \mathbb{N}, i > j] \to \mathbb{R}[s_i, t_i \mid i \in \mathbb{N}]$ determined by $y_{ij} \mapsto s_i s_j + t_i t_j$ is generated by two types of polynomials: *off-diagonal* 3×3-*minors* and *pentads*. This confirms a conjecture by Drton, Sturmfels, and Sullivant on the Gaussian two-factor model. Our proof is computational: inspired by work of Aschenbrenner and Hillar we introduce the concept of *G-Gröbner basis*, where G is a monoid acting on an infinite set of variables, and we report on a computation that yielded a finite G-Gröbner basis of I relative to the monoid G of strictly increasing functions $\mathbb{N} \to \mathbb{N}$.

1. Introduction and results

The *Gaussian k-factor model with n observed variables* consists of all covariance matrices of n jointly Gaussian random variables $X_1, \ldots, X_n$, the *observed variables*, consistent with the hypothesis that there exist k further variables $Z_1, \ldots, Z_k$, the *hidden variables*, such that the joint distribution of the X_i and the Z_j is Gaussian and such that the X_i are pairwise independent given all Z_j. This set of covariance matrices turns out to be

$$F_{k,n} := \{D + SS^T \mid D \in M_n(\mathbb{R}) \text{ diagonal and positive definite, and } S \in M_{n,k}(\mathbb{R})\},$$

where $M_{n,k}(\mathbb{R})$ is the space of real $n \times k$-matrices, and $M_n(\mathbb{R})$ is the space of real $n \times n$-matrices. In [7] this model is studied from an algebraic point of view. In particular, the ideal of polynomials vanishing on $F_{k,n}$ is determined for $k = 2, 3$ and $n \leq 9$. The case where $k = 1$ had already been done in [4]. The authors of [7] pose some very intriguing finiteness questions. In particular, one might hope that for fixed k the ideal of $F_{k,n}$ stabilises, as n grows, modulo its natural symmetries coming from simultaneously permuting rows and columns. For $k = 1$ this is indeed the case, and for arbitrary k it is true in a weaker, set-theoretic sense [5]. In this paper we prove that the ideals of $F_{2,n}$ stabilise at $n = 6$. To state our theorem we denote by y_{ij} the coordinates on the space of symmetric $n \times n$-matrices; we will identify y_{ji} with y_{ij}. Recall from [7] that the ideal of $F_{2,5}$ is generated by a single polynomial

$$P := \frac{1}{10} \sum_{\pi \in \mathrm{Sym}(5)} \mathrm{sgn}\,(\pi) y_{\pi(1),\pi(2)} y_{\pi(2),\pi(3)} y_{\pi(3),\pi(4)} y_{\pi(4),\pi(5)} y_{\pi(5),\pi(1)},$$

called the *pentad*. The normalisation factor is important only because it ensures that all coefficients are ± 1—indeed, the stabiliser in $\mathrm{Sym}(5)$ of each monomial in the

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pentad is the dihedral group of order 10. We consider P an element of $\mathbb{Z}[y_{ij} \mid i \geq j]$. The ideal of $F_{2,6}$ contains another type of equation: the *off-diagonal minor*

$$M := \det(y[\{4,5,6\},\{1,2,3\}]) \in \mathbb{Z}[y_{ij} \mid i \geq j]$$

the determinant of the square submatrix of y sitting in the lower left corner of y. If f is any polynomial in $\mathbb{R}[y_{ij} \mid i \geq j]$ that vanishes on $F_{2,n}$ and if we regard f as an element of $\mathbb{R}[y_{ij} \mid i > j][y_{11}, \ldots, y_{nn}]$, then each of the coefficients of the monomials in the diagonal variables y_{ii} is a polynomial in the off-diagonal variables that vanishes on $F_{2,n}$, as well. Therefore the following theorem settles the conjecture of Drton, Sturmfels, and Sullivant, that pentads and off-diagonal minors generate the ideal of $F_{2,n}$ for all n; see [7, Conjecture 26].

Theorem 1.1 (Main Theorem)**.** *For any field K and any natural number $n \geq 6$ the kernel $I_n(K)$ of the homomorphism $K[y_{ij} \mid 1 \leq j \leq i \leq n] \to K[s_1, \ldots, s_n, t_1, \ldots, t_n]$ determined by $y_{ij} \mapsto s_i s_j + t_i t_j$ is generated, as an ideal, by the orbits of P and M under the symmetric group* $\mathrm{Sym}(n)$.

Remark 1.2. In [8] it is proved that $F_{2,n}$ equals the set of all positive definite matrices with the property that every principal 6×6-minor lies in $F_{2,6}$. Our Main Theorem implies an analogous statement for the Zariski closures of $F_{2,n}$ and $F_{2,6}$.

We sketch the proof of the Main Theorem along with the organisation of the paper. In Section 3 we introduce *equivariant Gröbner bases*, which are a generalisation of Gröbner bases to a setting where a monoid G acts on the set of variables preserving the term order. Finite equivariant Gröbner bases do not always exist, even for ideals that are finitely generated modulo the action of G. Nevertheless, one can generalise the usual S-polynomial criterion to a finite test whether a given finite set of polynomials is an equivariant Gröbner basis. In Section 4 we put a suitable elimination order on the monomials in $y_{ij},\ i,j \in \mathbb{N},\ i \geq j$, and report on a computation that yields a finite G-Gröbner basis for the determinantal ideal generated by all 3×3-minors of y. Intersecting this G-Gröbner basis with the ring in the off-diagonal matrix entries gives the Main Theorem.

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3. Equivariant Gröbner bases

Consider a potentially infinite set X of variables. The free commutative monoid generated by X is denoted Mon; its elements are called *monomials*. Suppose that we have

(1) a monomial order, i.e., a well-order $\leq$ on Mon such that $m \leq m' \Rightarrow xm \leq xm'$ for all $x \in X,\ m, m' \in$ Mon; and
(2) a monoid G (i.e., a semigroup with identity) acting on X such that the induced action of G by homomorphisms on Mon preserves the strict order: $m < m' \Rightarrow gm < gm'$ for all $g \in G,\ m, m' \in$ Mon.

Example 3.1. The setting that Aschenbrenner and Hillar study in [1] fits into this framework, and indeed inspired our set-up. There $X = \{x_1, x_2, \ldots\}$ and G is the monoid $\mathrm{Inc}(\mathbb{N})$ of all increasing maps $\pi : \mathbb{N} \to \mathbb{N}$ acting on X by $\pi x_i = x_{\pi(i)}$.

As a monomial order one can choose the lexicographic order with $x_i > x_j$ if $i > j$. Aschenbrenner and Hillar have also turned their proof of finite generation of $\mathrm{Sym}(\mathbb{N})$-stable ideals in $K[x_1, x_2, \ldots]$ into an algorithm; see [2].

Remark 3.2. Note that G acts by injective maps on X (and on Mon) by the second requirement. It is essential that we allow G to be a monoid rather than a group. Indeed, the image of G in the monoid of injective maps $X \to X$ contains no other invertible elements than the identity: If $\pi : X \to X$ is an element in the image of G and if $\pi(x) \neq x$, then $\pi(x) > x$ since otherwise $x > \pi(x) > \pi^2(x) > \ldots$ would be an infinite strictly decreasing chain. But then, if π is invertible, we have $\pi(x) > x > \pi^{-1}(x) > \pi^{-2}(x) > \ldots$, another infinite decreasing chain.

Let K be a field and let $K[X] = K\mathrm{Mon}$ be the polynomial K-algebra in the variables X, or, equivalently, the monoid K-algebra of Mon. Then G acts naturally on $K[X]$ by means of homomorphisms. A *G-orbit* is a set of the form $Gz = \{gz \mid g \in G\}$, where z is in a set on which G acts. Note that the ideal generated by the union of G-orbits in $K[X]$ is automatically *G-stable*, that is, closed under multiplication with elements from G.

We use the notation $\mathrm{lm}(f)$ for the *leading monomial* of f, i.e., the $\leq$-largest monomial having non-zero coefficient in f. The coeffient in f of that monomial, the *leading coefficient*, is denoted $\mathrm{lc}(f)$, and $\mathrm{lt}(f) = \mathrm{lc}(f)\mathrm{lm}(f)$ is the *leading term* of f. By the requirement that G preserve the order, we have $\mathrm{lm}(gf) = g\,\mathrm{lm}(f)$. Given an ideal I of $K[X]$, $\mathrm{lm}(I)$ is an ideal in the monoid Mon. If I is G-stable, then so is $\mathrm{lm}(I)$.

Definition 3.3 (Equivariant Gröbner basis)**.** Let I be a G-stable ideal in $K[X]$. A *G-Gröbner basis* of $I \subseteq K[X]$ is a subset B of I for which $\mathrm{lm}(GB)(= \{\mathrm{lm}(gb) \mid b \in B,\ g \in G\})$ generates the ideal $\mathrm{lm}(I)$ in Mon. If G is fixed in the context, we also call B an *equivariant Gröbner basis*.

Remark 3.4. At MEGA 2009, Viktor Levandovskyy pointed out to the second author that our equivariant Gröbner bases are in fact a special case of Gröbner S-bases in the sense of [6], which were invented for analysing certain two-sided ideals in free associative algebras.

Lemma 3.5. *If I is G-stable and B is a G-Gröbner basis of I, then $GB = \{gb \mid b \in B, g \in G\}$ generates the ideal I.*

Proof. If not, then take an $f \in I \setminus \langle GB \rangle$ with $\mathrm{lm}(f)$ minimal. Take $b \in B$ and $g \in G$ with $\mathrm{lm}(gb) | \mathrm{lm}(f)$. Subtracting a suitable multiple of gb from f yields an element in $I \setminus \langle GB \rangle$ with leading term strictly smaller than that of f, a contradiction. □

Algorithm 3.6 (Equivariant remainder)**.** Given $f \in K[X]$ and $B \subseteq K[X]$, proceed as follows: if $g\mathrm{lm}(b) | \mathrm{lm}(f)$ for some $g \in G$ and $b \in B$, then subtract the multiple of gb from f that lowers the latter's leading monomial. Do this until no such pair (g, b) exists anymore. The resulting polynomial is called a *G-remainder* (or an *equivariant remainder*, if G is fixed) of f modulo B.

This procedure is non-deterministic, but necessarily finishes after a finite number of steps, since $\leq$ is a well-order. Any potential outcome is called an equivariant remainder of f modulo B.

Definition 3.7 (Equivariant S-polynomials)**.** Consider two polynomials b_0, b_1 with leading monomials m_0, m_1, respectively. Let H be a set of pairs $(h_0, h_1) \in G \times G$

for which $Gb_0 \times Gb_1 = \bigcup_{(h_0,h_1)\in H}\{(gh_0b_0, gh_1b_1) \mid g \in G\}$. For every element $(h_0, h_1) \in H$ we consider the ordinary S-polynomial

$$S(h_0b_0, h_1b_1) := \mathrm{lc}(b_1)\frac{\mathrm{lcm}(h_0m_0, h_1m_1)}{h_0m_0}h_0b_0 - \mathrm{lc}(b_0)\frac{\mathrm{lcm}(h_0m_0, h_1m_1)}{h_1m_1}h_1b_1.$$

The set $\{S(h_0b_0, h_1b_1) \mid (h_0, h_1) \in H\}$ is called a *complete set of equivariant S-polynomials* for b_0, b_1. It depends on the choice of H. In our applications, H can be chosen finite.

Theorem 3.8 (Equivariant Buchberger criterion)**.** *Let B be a subset of $K[X]$. Assume that for all $b_0, b_1 \in B$ there exists a complete set of S-polynomials, each of which has 0 as a G-remainder modulo B. Then B is a G-Gröbner basis of the ideal generated by GB.*

Proof. We may and will assume that all elements of B are monic. Let I denote the ideal generated by GB. If $\mathrm{lm}(GB)$ does not generate the ideal $\mathrm{lm}(I)$ in Mon then there exists a polynomial of the form

$$f = \sum_{g\in G, b\in B} f_{g,b}gb$$

with only finitely many of the $f_{g,b}$ non-zero, whose leading monomial is not in the ideal generated by $\mathrm{lm}(B)$. We may choose the expression above such that first, the *maximum* m of $\mathrm{lm}(f_{g,b}gb) = \mathrm{lm}(f_{g,b})\, g\, \mathrm{lm}(b)$ over all (g, b) for which $f_{g,b}$ is non-zero is *minimal* and second, the number of pairs (g, b) with $\mathrm{lm}(f_{g,b}gb) = m$ is also minimal. The maximum is then attained for at least two pairs $(g_0, b_0), (g_1, b_1)$, because otherwise m would be the leading monomial of f. Write $m_i := \mathrm{lm}(b_i)$ for $i = 0, 1$. We have

$$m = \mathrm{lm}(f_{g_0,b_0})g_0m_0 = \mathrm{lm}(f_{g_1,b_1})g_1m_1.$$

Now let H be a set of pairs $(h_0, h_1) \in G \times G$ giving rise to a complete set of S-polynomials for b_0 and b_1 that G-reduce to zero; such a set exists by assumption. Then we may write $g_0m_0 = g_2h_0m_0,\ g_1m_1 = g_2h_1m_1$ for some $(h_0, h_1) \in H$ and $g_2 \in G$. Let $\mathrm{lcm}(h_0m_0, h_1m_1) = t_0h_0m_0 = t_1h_1m_1$, so that

$$S := S(h_0b_0, h_1b_1) = t_0h_0b_0 - t_1h_1b_1;$$

where we have used that b_0 and b_1 are monic. We have

$$\mathrm{lm}(f_{g_0,b_0})g_2h_0m_0 = \mathrm{lm}(f_{g_1,b_1})g_2h_1m_1.$$

This implies that the left-hand side is a multiple of $\mathrm{lcm}(g_2h_0m_0, g_2h_1m_1)$, which equals $g_2\,\mathrm{lcm}(h_0m_0, h_1m_1)$. Hence $\mathrm{lm}(f_{g_0,b_0})$ is divisible by g_2t_0; set

$$A := \frac{\mathrm{lt}(f_{g_0,b_0})}{g_2t_0}.$$

Now 0 is a G-remainder of S modulo B, which implies that we can write S as a sum

$$\sum_{g\in G, b\in B} s_{g,b}gb$$

with only finitely many non-zero terms that moreover satisfy $\mathrm{lm}(s_{g,b}gb) \le \mathrm{lm}(S) < \mathrm{lcm}(h_0m_0, h_1m_1)$ for all g, b. Then we may rewrite f as

$$f = f - Ag_2(S - \sum_{g,b} s_{g,b}gb) = \sum_{g,b}(f_{g,b} + f'_{g,b} + f''_{g,b})gb$$

where

$$f'_{g,b} = \sum_{g' \in G, g_2 g' = g} A g_2 s_{g',b}$$

and

$$f''_{g,b} = \begin{cases} -\mathrm{lt}(f_{g_0,b_0}) & \text{if } (g,b) = (g_0,b_0), \\ \mathrm{lc}(f_{g_0,b_0})\mathrm{lm}(f_{g_1,b_1}) & \text{if } (g,b) = (g_1,b_1), \\ 0 & \text{otherwise.} \end{cases}$$

If $g_2 g' = g$ then for all b we have

$$\begin{aligned} \mathrm{lm}((A g_2 s_{g',b})(gb)) = \mathrm{lm}(A g_2 (s_{g',b} g' b)) &< \frac{\mathrm{lm}(f_{g_0,b_0})}{g_2 t_0} g_2 \,\mathrm{lcm}(h_0 m_0, h_1 m_1) \\ &= \mathrm{lm}(f_{g_0,b_0}) g_0 m_0 = m, \end{aligned}$$

so for all pairs (g,b) we have $\mathrm{lm}(f'_{g,b} gb) < m$. Moreover, $\mathrm{lm}((f_{g_0,b_0} + f''_{g_0,b_0}) g_0 b_0)$ is strictly smaller than m. Finally, $\mathrm{lm}(f''_{g_1,b_1} g_1 b_1) = m$. We conclude that either $\max_{g,b} \mathrm{lm}((f_{g,b} + f'_{g,b} + f''_{g,b}) gb)$ is strictly smaller than m, or else the number of pairs (g,b) for which it equals m is smaller than the number of pairs (g,b) for which $\mathrm{lm}(f_{g,b} gb)$ equals m. This contradicts the minimality of the expression chosen above. □

In addition to our set-up so far—a monomial order on monomials in the variables in X and an action of a monoid G on X preserving the strict order—we make the following finiteness assumption:

(*) $\forall b_0, b_1 \in K[X]$ the set $Gb_0 \times Gb_1$ is the union of a finite number of G-orbits.

This ensures that a finite, complete set of equivariant S-polynomials exists for any pair b_0, b_1. We then have the following theoretical algorithm. We do not claim that it terminates, but if it does, then it returns a finite equivariant Gröbner basis by Theorem 3.8.

Algorithm 3.9 (Equivariant Buchberger algorithm)**.**

Input: a finite subset B of $K[X]$.

Output (assuming termination): a finite equivariant Gröbner basis of the ideal generated by GB.

Procedure: (1) $P := B \times B$;

- (2) while $P \neq \emptyset$ do
  - (a) choose $(b_0, b_1) \in P$ and set $P := P \setminus \{(b_0, b_1)\}$;
  - (b) let $\mathcal{S}$ be a finite complete set of equivariant S-polynomials for (b_0, b_1);
  - (c) for all $f \in \mathcal{S}$ compute a G-remainder r of f modulo B; if $r \neq 0$ then set $B := B \cup \{r\}$ and $P := P \cup (B \times r)$;
- (3) return B.

Note the order in which B and P are updated: one needs to add (r, r) to P, as well.

4. A G-Gröbner basis for the 2-factor model

Our main theorem will follow from the following result. Let $X = \{y_{ij} \mid i, j \in \mathbb{N}, i \geq j\}$ be a set of variables representing the entries of a symmetric matrix. We consider the lexicographic monomial order on Mon in which the diagonal variables

$l(p)$	3	4	5	6	7	8	9
$\#p \in B$	1	6	11	10	8	5	1
degrees	3^1	3^6	$3^{10}5^1$	$3^5 5^5$	5^8	5^5	5^1
$\#p \in B \cap K[y_{ij} \mid i > j]$			1	5	8	5	1
degrees			5^1	$3^5 5^5$	5^8	5^5	5^1

TABLE 1. Largest indices and degrees of the Inc($\mathbb{N}$)-Gröbner basis of $I_{\mathbb{N}}(K)$; multiplicities written as exponents.

y_{ii} are larger than all variables y_{ij} with $i > j$, and apart from that $y_{ij} \geq y_{i'j'}$ if and only if $i > i'$ or $i = i'$ and $j \geq j'$. So for instance we have

$$y_{2,2} > y_{1,1} > y_{5,2} > y_{4,3}.$$

Note that this monomial order is compatible with the action of the monoid Inc($\mathbb{N}$) of all increasing maps $\mathbb{N} \to \mathbb{N}$. For any polynomial $p \in K[X]$ let $l(p)$ denote the *largest index of* p, i.e., the largest index appearing in any of the variables in any of the monomials of p.

Theorem 4.1. *For any field K, let $I_{\mathbb{N}}(K)$ be the ideal in $K[X]$ generated by all 3×3-minors of the matrix y (recall that we identify y_{ji} for $j < i$ with y_{ij}). Relative to the monomial order $\leq$ the ideal $I_{\mathbb{N}}(K)$ has an* Inc($\mathbb{N}$)*-Gröbner basis B consisting of* 42 *polynomials. The intersection $B \cap K[y_{ij} \mid i > j]$ is an* Inc($\mathbb{N}$)*-Gröbner basis of $I_{\mathbb{N}}(K) \cap K[y_{ij} \mid i > j]$ consisting of* 20 *polynomials. The largest indices and the degrees of the elements in these bases are summarised in Table 4.1.*

Remark 4.2. The polynomial with largest index 5 in the Inc($\mathbb{N}$)-Gröbner basis $B \cap K[y_{ij} \mid i > j]$ is the pentad P. The five degree-3 polynomials with largest index 6 in that Gröbner basis form the Sym($\mathbb{N}$)-orbit of the off-diagonal minor M. All 14 remaining polynomials are already in the Inc($\mathbb{N}$)-stable ideal generated by these polynomials; this latter statement also follows from the result in [7] that at least up to $n = 9$ the ideal of the two-factor model is generated by pentads and off-diagonal minors.

Remark 4.3. A Gröbner basis of the ideal of the two-factor model $F_{2,n}$ relative to *circular term orders* was already found in [10]. The proof involves general techniques for determining the ideal of secant varieties, especially of toric varieties; see also [9]. The Gröbner basis found there, however, does not stabilise as n grows—and indeed, circular term orders are not compatible with the action of Inc($\mathbb{N}$). It would be interesting to find a direct translation between Sullivant's Gröbner basis and ours.

Theorem 4.1 implies our Main Theorem.

Proof of the Main Theorem. It is well known that the $(k+1) \times (k+1)$-minors of the symmetric matrix $(y_{ij})_{i,j=1,\ldots,n}$ generate the ideal of all polynomials vanishing on all rank-k matrices (for a recent combinatorial proof of this fact, see [9, Example 4.12]; in characteristic 0 this fact is known as the Second Fundamental Theorem for the orthogonal group). Hence the ideal $I_n(K)$ is the intersection of the ideal J_n generated by the 3×3-minors of $(y_{ij})_{i,j=1,\ldots,n}$ with the ring $K[y_{ij} \mid i > j]$. Theorem 4.1 implies that one obtains a Gröbner basis of J_n, relative to the restriction of the monomial order on $K[y_{ij} \mid i, j \in \mathbb{N}, i \geq j]$ to $K[y_{ij} \mid 1 \leq j < i \leq n]$ by applying all increasing maps $\{1,\ldots,l(p)\} \to \{1,\ldots,n\}$ to all $p \in B \cap K[y_{ij} \mid i > j]$ with $l(p) \le n$. Such an increasing map can be extended to an element of $\mathrm{Sym}(n)$, and Remark 4.2 concludes the proof. □

We conclude with some remarks on the computation that proved Theorem 4.1. First we need to verify Condition (*).

Lemma 4.4. *For all $b_0, b_1 \in K[y_{ij} \mid i,j \in \mathbb{N}, i \ge j]$ the set $(\mathrm{Inc}(\mathbb{N})b_0) \times (\mathrm{Inc}(\mathbb{N})b_1)$ is the union of a finite number of $\mathrm{Inc}(\mathbb{N})$-orbits.*

Proof. Consider all pairs (S_0, S_1) of sets $S_0, S_1 \subseteq \mathbb{N}$ with $|S_i| = l(b_i)$ for which $S_0 \cup S_1$ is an interval of the form $\{1,\ldots,k\}$ for some k, which is then at most $l(b_0) + l(b_1)$. Note that there are only finitely many such pairs (S_0, S_1). For each such pair let (π_0, π_1) be a pair of elements of $\mathrm{Inc}(\mathbb{N})$ such that π_i maps $\{1,\ldots,l(b_i)\}$ onto S_i; it is irrelevant how π acts on the rest of $\mathbb{N}$. Then we have

$$\mathrm{Inc}(\mathbb{N})b_0 \times \mathrm{Inc}(\mathbb{N})b_1 = \bigcup_{(S_0,S_1)} \mathrm{Inc}(\mathbb{N})(\pi_0 b_0, \pi_1 b_1),$$

where the union is over all pairs (S_0, S_1) as above. □

Computational proof of Theorem 4.1. The 42 polynomials of B were constructed by computing a Gröbner basis for $I_9(\mathbb{Q})$ with `Singular` and retaining only those polynomials p for which the set of indices occurring in their variables form an interval of the form $\{1,\ldots,k\}$ with $k \le 9$. All elements of B are monic and have integral coefficients (in fact, equal to ± 1 except for the 3×3-minor with largest index 3, which has a coefficient 2). By the equivariant Buchberger criterion and the proof of Lemma 4.4, we need only $\mathrm{Inc}(\mathbb{N})$-reduce modulo B all S-polynomials of pairs $(\pi_0 b_0, \pi_1 b_1)$ with $b_0, b_1 \in B$ and $\pi_i : \{1,\ldots,l(b_i)\} \to \mathbb{N}$ increasing and such that $\mathrm{im}\,\pi_0 \cup \mathrm{im}\,\pi_1 = \{1,\ldots,k\}$ for some k. For instance, for $b_0 = b_1 = b$ equal to the polynomial in B with largest index 9, we having to $\mathrm{Inc}(\mathbb{N})$-reduce $S(\pi_0 b, \pi_1 b)$ modulo B for all increasing maps $\pi_0, \pi_1 : \{1,\ldots,9\} \to \{1,\ldots,18\}$ whose image union is an interval $\{1,\ldots,k\}$. However, if $k = 17$ or $k = 18$, then $\pi_0 b$ and $\pi_1 b$ turn out to have leading monomials with gcd 1, so these cases can be skipped. This reduces the theorem to a finite computation involving polynomials with largest indices up to 16, which we have implemented directly in `C`. Finally, to deduce the result for all base fields—and to speed up the computation—we used the following trick. Since $\mathrm{Inc}(\mathbb{N})B \cap K[y_{ij} \mid 1 \le j \le i \le n]$ is a subset of the ideal of 3×3-minors, it is a Gröbner basis if and only if the ideal generated by $\mathrm{lm}(B)$ has the same Hilbert series as the ideal generated by 3×3-minors. Since this Hilbert series is known and does not depend on the field [3], we may do all our computations over one field and conclude that it holds over all fields. We have verified the equivariant Buchberger criterion over $\mathbb{F}_2$, which made the computation slightly faster than working over $\mathbb{Q}$. □

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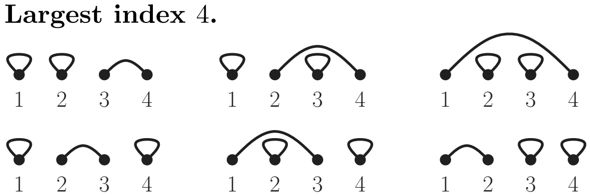
Appendix: the basis B

Below is the complete equivariant Gröbner basis of Theorem 4.1. To distinguish the diagonal entries y_{ii} from the off-diagonal entries, we have denoted them a_i. We precede the polynomials by graphs representing their leading monomials; here the variable y_{ij} is depicted as an undirected edge between i and j. For larger indices, the edges have been given different shades; this is only to make the pictures more readable. Ideally, one would hope to prove Theorem 4.1 by hand by giving a bijection between the standard monomials relative to B and the known standard monomials relative to the Gröbner basis of [3], but we have not yet found such a bijection so far.

Largest index 3.

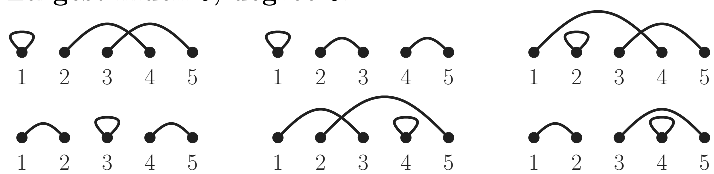
$$a_3 * a_2 * a_1 - a_3 * y_{21}^2 - a_2 * y_{31}^2 - a_1 * y_{32}^2 + 2 * y_{32} * y_{31} * y_{21}$$

Largest index 4.



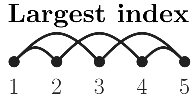
$$a_2 * a_1 * y_{43} - a_2 * y_{41} * y_{31} - a_1 * y_{42} * y_{32} - y_{43} * y_{21}^2 + y_{42} * y_{31} * y_{21} + y_{41} * y_{32} * y_{21}$$
$$a_3 * a_1 * y_{42} - a_3 * y_{41} * y_{21} - a_1 * y_{43} * y_{32} + y_{43} * y_{31} * y_{21} - y_{42} * y_{31}^2 + y_{41} * y_{32} * y_{31}$$
$$a_3 * a_2 * y_{41} - a_3 * y_{42} * y_{21} - a_2 * y_{43} * y_{31} + y_{43} * y_{32} * y_{21} + y_{42} * y_{32} * y_{31} - y_{41} * y_{32}^2$$
$$a_4 * a_1 * y_{32} - a_4 * y_{31} * y_{21} - a_1 * y_{43} * y_{42} + y_{43} * y_{41} * y_{21} + y_{42} * y_{41} * y_{31} - y_{41}^2 * y_{32}$$
$$a_4 * a_2 * y_{31} - a_4 * y_{32} * y_{21} - a_2 * y_{43} * y_{41} + y_{43} * y_{42} * y_{21} - y_{42}^2 * y_{31} + y_{42} * y_{41} * y_{32}$$
$$a_4 * a_3 * y_{21} - a_4 * y_{32} * y_{31} - a_3 * y_{42} * y_{41} - y_{43}^2 * y_{21} + y_{43} * y_{42} * y_{31} + y_{43} * y_{41} * y_{32}$$

Largest index 5, degree 3.



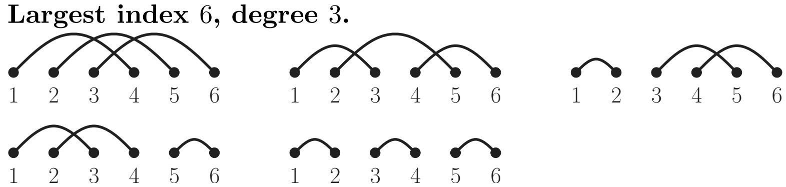
$$a_1 * y_{53} * y_{42} - a_1 * y_{52} * y_{43} - y_{53} * y_{41} * y_{21} + y_{52} * y_{41} * y_{31} + y_{51} * y_{43} * y_{21} - y_{51} * y_{42} * y_{31}$$
$$a_1 * y_{54} * y_{32} - a_1 * y_{52} * y_{43} - y_{54} * y_{31} * y_{21} + y_{52} * y_{41} * y_{31} + y_{51} * y_{43} * y_{21} - y_{51} * y_{41} * y_{32}$$
$$a_2 * y_{53} * y_{41} - a_2 * y_{51} * y_{43} - y_{53} * y_{42} * y_{21} + y_{52} * y_{43} * y_{21} - y_{52} * y_{41} * y_{32} + y_{51} * y_{42} * y_{32}$$
$$a_2 * y_{54} * y_{31} - a_2 * y_{51} * y_{43} - y_{54} * y_{32} * y_{21} + y_{52} * y_{43} * y_{21} - y_{52} * y_{42} * y_{31} + y_{51} * y_{42} * y_{32}$$
$$a_3 * y_{52} * y_{41} - a_3 * y_{51} * y_{42} + y_{53} * y_{42} * y_{31} - y_{53} * y_{41} * y_{32} - y_{52} * y_{43} * y_{31} + y_{51} * y_{43} * y_{32}$$
$$a_3 * y_{54} * y_{21} - a_3 * y_{51} * y_{42} - y_{54} * y_{32} * y_{31} - y_{53} * y_{43} * y_{21} + y_{53} * y_{42} * y_{31} + y_{51} * y_{43} * y_{32}$$
$$a_4 * y_{52} * y_{31} - a_4 * y_{51} * y_{32} - y_{54} * y_{42} * y_{31} + y_{54} * y_{41} * y_{32} - y_{52} * y_{43} * y_{41} + y_{51} * y_{43} * y_{42}$$
$$a_4 * y_{53} * y_{21} - a_4 * y_{51} * y_{32} - y_{54} * y_{43} * y_{21} + y_{54} * y_{41} * y_{32} - y_{53} * y_{42} * y_{41} + y_{51} * y_{43} * y_{42}$$
$$a_5 * y_{42} * y_{31} - a_5 * y_{41} * y_{32} - y_{54} * y_{52} * y_{31} + y_{54} * y_{51} * y_{32} + y_{53} * y_{52} * y_{41} - y_{53} * y_{51} * y_{42}$$
$$a_5 * y_{43} * y_{21} - a_5 * y_{41} * y_{32} - y_{54} * y_{53} * y_{21} + y_{54} * y_{51} * y_{32} + y_{53} * y_{52} * y_{41} - y_{52} * y_{51} * y_{43}$$

Largest index 5, degree 5.



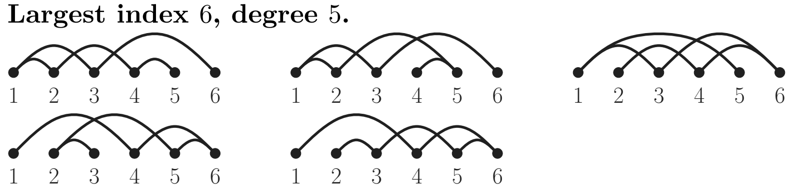
$$\begin{aligned} & y_{54} * y_{53} * y_{42} * y_{31} * y_{21} - y_{54} * y_{53} * y_{41} * y_{32} * y_{21} - y_{54} * y_{52} * y_{43} * y_{31} * y_{21} \\ & + y_{54} * y_{52} * y_{41} * y_{32} * y_{31} + y_{54} * y_{51} * y_{43} * y_{32} * y_{21} - y_{54} * y_{51} * y_{42} * y_{32} * y_{31} \\ & + y_{53} * y_{52} * y_{43} * y_{41} * y_{21} - y_{53} * y_{52} * y_{42} * y_{41} * y_{31} - y_{53} * y_{51} * y_{43} * y_{42} * y_{21} \\ & + y_{53} * y_{51} * y_{42} * y_{41} * y_{32} + y_{52} * y_{51} * y_{43} * y_{42} * y_{31} - y_{52} * y_{51} * y_{43} * y_{41} * y_{32} \end{aligned}$$

Largest index 6, degree 3.



$$y_{63} * y_{52} * y_{41} - y_{63} * y_{51} * y_{42} - y_{62} * y_{53} * y_{41} + y_{62} * y_{51} * y_{43} + y_{61} * y_{53} * y_{42} - y_{61} * y_{52} * y_{43}$$
$$y_{64} * y_{52} * y_{31} - y_{64} * y_{51} * y_{32} - y_{62} * y_{54} * y_{31} + y_{62} * y_{51} * y_{43} + y_{61} * y_{54} * y_{32} - y_{61} * y_{52} * y_{43}$$
$$y_{64} * y_{53} * y_{21} - y_{64} * y_{51} * y_{32} - y_{63} * y_{54} * y_{21} + y_{63} * y_{51} * y_{42} + y_{61} * y_{54} * y_{32} - y_{61} * y_{53} * y_{42}$$
$$y_{65} * y_{42} * y_{31} - y_{65} * y_{41} * y_{32} - y_{62} * y_{54} * y_{31} + y_{62} * y_{53} * y_{41} + y_{61} * y_{54} * y_{32} - y_{61} * y_{53} * y_{42}$$
$$\begin{aligned} & y_{65} * y_{43} * y_{21} - y_{65} * y_{41} * y_{32} - y_{63} * y_{54} * y_{21} + y_{63} * y_{51} * y_{42} + y_{62} * y_{53} * y_{41} - y_{62} * y_{51} * y_{43} \\ & + y_{61} * y_{54} * y_{32} - y_{61} * y_{53} * y_{42} \end{aligned}$$

Largest index 6, degree 5.



$$y_{63} * y_{54} * y_{42} * y_{31} * y_{21} - y_{63} * y_{54} * y_{41} * y_{32} * y_{21} - y_{63} * y_{51} * y_{42}^2 * y_{31}$$
$$+ y_{63} * y_{51} * y_{42} * y_{41} * y_{32} - y_{62} * y_{54} * y_{43} * y_{31} * y_{21} + y_{62} * y_{54} * y_{41} * y_{32} * y_{31}$$
$$+ y_{62} * y_{53} * y_{43} * y_{41} * y_{21} - y_{62} * y_{53} * y_{42} * y_{41} * y_{31} + y_{62} * y_{51} * y_{43} * y_{42} * y_{31}$$
$$- y_{62} * y_{51} * y_{43} * y_{41} * y_{32} + y_{61} * y_{54} * y_{43} * y_{32} * y_{21} - y_{61} * y_{54} * y_{42} * y_{32} * y_{31}$$
$$- y_{61} * y_{53} * y_{43} * y_{42} * y_{21} + y_{61} * y_{53} * y_{42}^2 * y_{31}$$

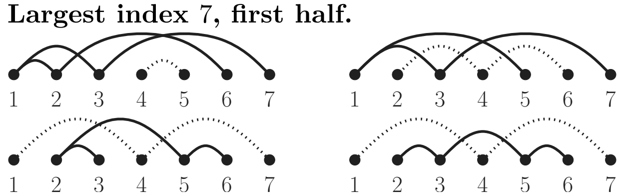
$$y_{63} * y_{54} * y_{52} * y_{31} * y_{21} - y_{63} * y_{54} * y_{51} * y_{32} * y_{21} - y_{63} * y_{52} * y_{51} * y_{42} * y_{31}$$
$$+ y_{63} * y_{51}^2 * y_{42} * y_{32} - y_{62} * y_{54} * y_{53} * y_{31} * y_{21} + y_{62} * y_{54} * y_{51} * y_{32} * y_{31}$$
$$+ y_{62} * y_{53} * y_{51} * y_{43} * y_{21} - y_{62} * y_{51}^2 * y_{43} * y_{32} + y_{61} * y_{54} * y_{53} * y_{32} * y_{21}$$
$$- y_{61} * y_{54} * y_{52} * y_{32} * y_{31} - y_{61} * y_{53} * y_{52} * y_{43} * y_{21} + y_{61} * y_{53} * y_{52} * y_{42} * y_{31}$$
$$- y_{61} * y_{53} * y_{51} * y_{42} * y_{32} + y_{61} * y_{52} * y_{51} * y_{43} * y_{32}$$

$$y_{64} * y_{63} * y_{51} * y_{42} * y_{31} - y_{64} * y_{63} * y_{51} * y_{41} * y_{32} + y_{64} * y_{62} * y_{53} * y_{41} * y_{31}$$
$$- y_{64} * y_{62} * y_{51} * y_{43} * y_{31} - y_{64} * y_{61} * y_{53} * y_{42} * y_{31} + y_{64} * y_{61} * y_{51} * y_{43} * y_{32}$$
$$- y_{63} * y_{62} * y_{54} * y_{41} * y_{31} + y_{63} * y_{62} * y_{51} * y_{43} * y_{41} + y_{63} * y_{61} * y_{54} * y_{41} * y_{32}$$
$$- y_{63} * y_{61} * y_{51} * y_{43} * y_{42} + y_{62} * y_{61} * y_{54} * y_{43} * y_{31} - y_{62} * y_{61} * y_{53} * y_{43} * y_{41}$$
$$- y_{61}^2 * y_{54} * y_{43} * y_{32} + y_{61}^2 * y_{53} * y_{43} * y_{42}$$

$$y_{65} * y_{64} * y_{52} * y_{41} * y_{32} - y_{65} * y_{64} * y_{51} * y_{42} * y_{32} - y_{65} * y_{62} * y_{54} * y_{41} * y_{32}$$
$$+ y_{65} * y_{62} * y_{51} * y_{43} * y_{42} + y_{65} * y_{61} * y_{54} * y_{42} * y_{32} - y_{65} * y_{61} * y_{52} * y_{43} * y_{42}$$
$$+ y_{64} * y_{62} * y_{54} * y_{51} * y_{32} - y_{64} * y_{62} * y_{53} * y_{52} * y_{41} - y_{64} * y_{61} * y_{54} * y_{52} * y_{32}$$
$$+ y_{64} * y_{61} * y_{53} * y_{52} * y_{42} + y_{62}^2 * y_{54} * y_{53} * y_{41} - y_{62}^2 * y_{54} * y_{51} * y_{43}$$
$$- y_{62} * y_{61} * y_{54} * y_{53} * y_{42} + y_{62} * y_{61} * y_{54} * y_{52} * y_{43}$$

$$y_{65} * y_{64} * y_{53} * y_{41} * y_{32} - y_{65} * y_{64} * y_{51} * y_{43} * y_{32} - y_{65} * y_{63} * y_{54} * y_{41} * y_{32}$$
$$+ y_{65} * y_{63} * y_{51} * y_{43} * y_{42} + y_{65} * y_{61} * y_{54} * y_{43} * y_{32} - y_{65} * y_{61} * y_{53} * y_{43} * y_{42}$$
$$+ y_{64} * y_{63} * y_{54} * y_{51} * y_{32} - y_{64} * y_{63} * y_{53} * y_{51} * y_{42} - y_{64} * y_{62} * y_{53}^2 * y_{41}$$
$$+ y_{64} * y_{62} * y_{53} * y_{51} * y_{43} - y_{64} * y_{61} * y_{54} * y_{53} * y_{32} + y_{64} * y_{61} * y_{53}^2 * y_{42}$$
$$+ y_{63} * y_{62} * y_{54} * y_{53} * y_{41} - y_{63} * y_{62} * y_{54} * y_{51} * y_{43}$$

Largest index 7, first half.



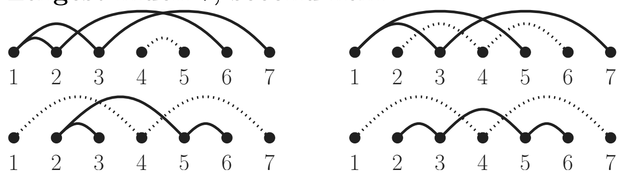
$$y_{73} * y_{62} * y_{54} * y_{31} * y_{21} - y_{73} * y_{61} * y_{54} * y_{32} * y_{21} - y_{73} * y_{61} * y_{52} * y_{42} * y_{31}$$
$$+ y_{73} * y_{61} * y_{51} * y_{42} * y_{32} - y_{72} * y_{63} * y_{54} * y_{31} * y_{21} + y_{72} * y_{61} * y_{54} * y_{32} * y_{31}$$
$$+ y_{72} * y_{61} * y_{53} * y_{43} * y_{21} - y_{72} * y_{61} * y_{51} * y_{43} * y_{32} + y_{71} * y_{63} * y_{54} * y_{32} * y_{21}$$
$$+ y_{71} * y_{63} * y_{52} * y_{42} * y_{31} - y_{71} * y_{63} * y_{51} * y_{42} * y_{32} - y_{71} * y_{62} * y_{54} * y_{32} * y_{31}$$
$$- y_{71} * y_{62} * y_{53} * y_{43} * y_{21} + y_{71} * y_{62} * y_{51} * y_{43} * y_{32}$$

$$y_{73} * y_{64} * y_{51} * y_{42} * y_{31} - y_{73} * y_{64} * y_{51} * y_{41} * y_{32} - y_{73} * y_{61} * y_{54} * y_{42} * y_{31}$$
$$+ y_{73} * y_{61} * y_{54} * y_{41} * y_{32} + y_{72} * y_{64} * y_{53} * y_{41} * y_{31} - y_{72} * y_{64} * y_{51} * y_{43} * y_{31}$$
$$- y_{72} * y_{63} * y_{54} * y_{41} * y_{31} + y_{72} * y_{63} * y_{51} * y_{43} * y_{41} + y_{72} * y_{61} * y_{54} * y_{43} * y_{31}$$
$$- y_{72} * y_{61} * y_{53} * y_{43} * y_{41} - y_{71} * y_{64} * y_{53} * y_{42} * y_{31} + y_{71} * y_{64} * y_{51} * y_{43} * y_{32}$$
$$+ y_{71} * y_{63} * y_{54} * y_{42} * y_{31} - y_{71} * y_{63} * y_{51} * y_{43} * y_{42} - y_{71} * y_{61} * y_{54} * y_{43} * y_{32}$$
$$+ y_{71} * y_{61} * y_{53} * y_{43} * y_{42}$$

$$y_{74} * y_{65} * y_{52} * y_{41} * y_{32} - y_{74} * y_{65} * y_{51} * y_{42} * y_{32} - y_{74} * y_{62} * y_{53} * y_{52} * y_{41}$$
$$+ y_{74} * y_{61} * y_{53} * y_{52} * y_{42} - y_{72} * y_{65} * y_{54} * y_{41} * y_{32} + y_{72} * y_{65} * y_{51} * y_{43} * y_{42}$$
$$+ y_{72} * y_{64} * y_{54} * y_{51} * y_{32} + y_{72} * y_{62} * y_{54} * y_{53} * y_{41} - y_{72} * y_{62} * y_{54} * y_{51} * y_{43}$$
$$- y_{72} * y_{61} * y_{54} * y_{53} * y_{42} + y_{71} * y_{65} * y_{54} * y_{42} * y_{32} - y_{71} * y_{65} * y_{52} * y_{43} * y_{42}$$
$$- y_{71} * y_{64} * y_{54} * y_{52} * y_{32} + y_{71} * y_{62} * y_{54} * y_{52} * y_{43}$$

$$y_{74} * y_{65} * y_{53} * y_{41} * y_{32} - y_{74} * y_{65} * y_{51} * y_{43} * y_{32} - y_{74} * y_{62} * y_{53}^2 * y_{41}$$
$$+ y_{74} * y_{61} * y_{53} * y_{52} * y_{43} - y_{73} * y_{65} * y_{54} * y_{41} * y_{32} + y_{73} * y_{65} * y_{51} * y_{43} * y_{42}$$
$$+ y_{73} * y_{64} * y_{54} * y_{51} * y_{32} - y_{73} * y_{64} * y_{53} * y_{51} * y_{42} + y_{73} * y_{61} * y_{54} * y_{53} * y_{42}$$
$$- y_{73} * y_{61} * y_{54} * y_{52} * y_{43} + y_{72} * y_{64} * y_{53} * y_{51} * y_{43} + y_{72} * y_{63} * y_{54} * y_{53} * y_{41}$$
$$- y_{72} * y_{63} * y_{54} * y_{51} * y_{43} - y_{72} * y_{61} * y_{54} * y_{53} * y_{43} + y_{71} * y_{65} * y_{54} * y_{43} * y_{32}$$
$$- y_{71} * y_{65} * y_{53} * y_{43} * y_{42} - y_{71} * y_{64} * y_{54} * y_{53} * y_{32} + y_{71} * y_{64} * y_{53}^2 * y_{42}$$
$$- y_{71} * y_{64} * y_{53} * y_{52} * y_{43} - y_{71} * y_{63} * y_{54} * y_{53} * y_{42} + y_{71} * y_{63} * y_{54} * y_{52} * y_{43}$$
$$+ y_{71} * y_{62} * y_{54} * y_{53} * y_{43}$$

Largest index 7, second half.



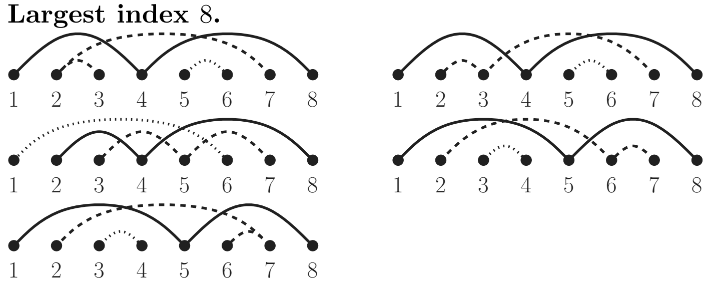
$$y_{73} * y_{62} * y_{54} * y_{31} * y_{21} - y_{73} * y_{61} * y_{54} * y_{32} * y_{21} - y_{73} * y_{61} * y_{52} * y_{42} * y_{31}$$
$$+ y_{73} * y_{61} * y_{51} * y_{42} * y_{32} - y_{72} * y_{63} * y_{54} * y_{31} * y_{21} + y_{72} * y_{61} * y_{54} * y_{32} * y_{31}$$
$$+ y_{72} * y_{61} * y_{53} * y_{43} * y_{21} - y_{72} * y_{61} * y_{51} * y_{43} * y_{32} + y_{71} * y_{63} * y_{54} * y_{32} * y_{21}$$
$$+ y_{71} * y_{63} * y_{52} * y_{42} * y_{31} - y_{71} * y_{63} * y_{51} * y_{42} * y_{32} - y_{71} * y_{62} * y_{54} * y_{32} * y_{31}$$
$$- y_{71} * y_{62} * y_{53} * y_{43} * y_{21} + y_{71} * y_{62} * y_{51} * y_{43} * y_{32}$$

$$y_{73} * y_{64} * y_{51} * y_{42} * y_{31} - y_{73} * y_{64} * y_{51} * y_{41} * y_{32} - y_{73} * y_{61} * y_{54} * y_{42} * y_{31}$$
$$+ y_{73} * y_{61} * y_{54} * y_{41} * y_{32} + y_{72} * y_{64} * y_{53} * y_{41} * y_{31} - y_{72} * y_{64} * y_{51} * y_{43} * y_{31}$$
$$- y_{72} * y_{63} * y_{54} * y_{41} * y_{31} + y_{72} * y_{63} * y_{51} * y_{43} * y_{41} + y_{72} * y_{61} * y_{54} * y_{43} * y_{31}$$
$$- y_{72} * y_{61} * y_{53} * y_{43} * y_{41} - y_{71} * y_{64} * y_{53} * y_{42} * y_{31} + y_{71} * y_{64} * y_{51} * y_{43} * y_{32}$$
$$+ y_{71} * y_{63} * y_{54} * y_{42} * y_{31} - y_{71} * y_{63} * y_{51} * y_{43} * y_{42} - y_{71} * y_{61} * y_{54} * y_{43} * y_{32}$$
$$+ y_{71} * y_{61} * y_{53} * y_{43} * y_{42}$$

$$y_{74} * y_{65} * y_{52} * y_{41} * y_{32} - y_{74} * y_{65} * y_{51} * y_{42} * y_{32} - y_{74} * y_{62} * y_{53} * y_{52} * y_{41}$$
$$+ y_{74} * y_{61} * y_{53} * y_{52} * y_{42} - y_{72} * y_{65} * y_{54} * y_{41} * y_{32} + y_{72} * y_{65} * y_{51} * y_{43} * y_{42}$$
$$+ y_{72} * y_{64} * y_{54} * y_{51} * y_{32} + y_{72} * y_{62} * y_{54} * y_{53} * y_{41} - y_{72} * y_{62} * y_{54} * y_{51} * y_{43}$$
$$- y_{72} * y_{61} * y_{54} * y_{53} * y_{42} + y_{71} * y_{65} * y_{54} * y_{42} * y_{32} - y_{71} * y_{65} * y_{52} * y_{43} * y_{42}$$
$$- y_{71} * y_{64} * y_{54} * y_{52} * y_{32} + y_{71} * y_{62} * y_{54} * y_{52} * y_{43}$$

$$y_{74} * y_{65} * y_{53} * y_{41} * y_{32} - y_{74} * y_{65} * y_{51} * y_{43} * y_{32} - y_{74} * y_{62} * y_{53}^2 * y_{41}$$
$$+ y_{74} * y_{61} * y_{53} * y_{52} * y_{43} - y_{73} * y_{65} * y_{54} * y_{41} * y_{32} + y_{73} * y_{65} * y_{51} * y_{43} * y_{42}$$
$$+ y_{73} * y_{64} * y_{54} * y_{51} * y_{32} - y_{73} * y_{64} * y_{53} * y_{51} * y_{42} + y_{73} * y_{61} * y_{54} * y_{53} * y_{42}$$
$$- y_{73} * y_{61} * y_{54} * y_{52} * y_{43} + y_{72} * y_{64} * y_{53} * y_{51} * y_{43} + y_{72} * y_{63} * y_{54} * y_{53} * y_{41}$$
$$- y_{72} * y_{63} * y_{54} * y_{51} * y_{43} - y_{72} * y_{61} * y_{54} * y_{53} * y_{43} + y_{71} * y_{65} * y_{54} * y_{43} * y_{32}$$
$$- y_{71} * y_{65} * y_{53} * y_{43} * y_{42} - y_{71} * y_{64} * y_{54} * y_{53} * y_{32} + y_{71} * y_{64} * y_{53}^2 * y_{42}$$
$$- y_{71} * y_{64} * y_{53} * y_{52} * y_{43} - y_{71} * y_{63} * y_{54} * y_{53} * y_{42} + y_{71} * y_{63} * y_{54} * y_{52} * y_{43}$$
$$+ y_{71} * y_{62} * y_{54} * y_{53} * y_{43}$$

Largest index 8.



$$y_{84} * y_{72} * y_{65} * y_{41} * y_{32} - y_{84} * y_{72} * y_{62} * y_{53} * y_{41} - y_{84} * y_{71} * y_{65} * y_{42} * y_{32}$$
$$+ y_{84} * y_{71} * y_{62} * y_{53} * y_{42} - y_{82} * y_{74} * y_{65} * y_{41} * y_{32} + y_{82} * y_{74} * y_{62} * y_{53} * y_{41}$$
$$+ y_{82} * y_{71} * y_{65} * y_{43} * y_{42} + y_{82} * y_{71} * y_{64} * y_{54} * y_{32} - y_{82} * y_{71} * y_{64} * y_{53} * y_{42}$$
$$- y_{82} * y_{71} * y_{62} * y_{54} * y_{43} + y_{81} * y_{74} * y_{65} * y_{42} * y_{32} - y_{81} * y_{74} * y_{62} * y_{53} * y_{42}$$
$$- y_{81} * y_{72} * y_{65} * y_{43} * y_{42} - y_{81} * y_{72} * y_{64} * y_{54} * y_{32} + y_{81} * y_{72} * y_{64} * y_{53} * y_{42}$$
$$+ y_{81} * y_{72} * y_{62} * y_{54} * y_{43}$$

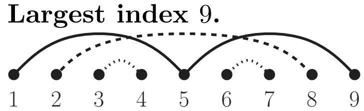
$$y_{84} * y_{73} * y_{65} * y_{41} * y_{32} - y_{84} * y_{72} * y_{63} * y_{53} * y_{41} - y_{84} * y_{71} * y_{65} * y_{43} * y_{32}$$
$$+ y_{84} * y_{71} * y_{62} * y_{53} * y_{43} - y_{83} * y_{74} * y_{65} * y_{41} * y_{32} + y_{83} * y_{71} * y_{65} * y_{43} * y_{42}$$
$$+ y_{83} * y_{71} * y_{64} * y_{54} * y_{32} - y_{83} * y_{71} * y_{62} * y_{54} * y_{43} + y_{82} * y_{74} * y_{63} * y_{53} * y_{41}$$
$$- y_{82} * y_{71} * y_{64} * y_{53} * y_{43} + y_{81} * y_{74} * y_{65} * y_{43} * y_{32} - y_{81} * y_{74} * y_{62} * y_{53} * y_{43}$$
$$- y_{81} * y_{73} * y_{65} * y_{43} * y_{42} - y_{81} * y_{73} * y_{64} * y_{54} * y_{32} + y_{81} * y_{73} * y_{62} * y_{54} * y_{43}$$
$$+ y_{81} * y_{72} * y_{64} * y_{53} * y_{43}$$

$$y_{84} * y_{75} * y_{61} * y_{53} * y_{42} - y_{84} * y_{75} * y_{61} * y_{52} * y_{43} - y_{84} * y_{71} * y_{65} * y_{53} * y_{42}$$
$$+ y_{84} * y_{71} * y_{65} * y_{52} * y_{43} + y_{83} * y_{75} * y_{64} * y_{51} * y_{42} - y_{83} * y_{75} * y_{61} * y_{54} * y_{42}$$
$$- y_{83} * y_{74} * y_{65} * y_{51} * y_{42} + y_{83} * y_{74} * y_{61} * y_{54} * y_{52} + y_{83} * y_{71} * y_{65} * y_{54} * y_{42}$$
$$- y_{83} * y_{71} * y_{64} * y_{54} * y_{52} - y_{82} * y_{75} * y_{64} * y_{51} * y_{43} + y_{82} * y_{75} * y_{61} * y_{54} * y_{43}$$
$$+ y_{82} * y_{74} * y_{65} * y_{51} * y_{43} - y_{82} * y_{74} * y_{61} * y_{54} * y_{53} - y_{82} * y_{71} * y_{65} * y_{54} * y_{43}$$
$$+ y_{82} * y_{71} * y_{64} * y_{54} * y_{53} - y_{81} * y_{75} * y_{64} * y_{53} * y_{42} + y_{81} * y_{75} * y_{64} * y_{52} * y_{43}$$
$$+ y_{81} * y_{74} * y_{65} * y_{53} * y_{42} - y_{81} * y_{74} * y_{65} * y_{52} * y_{43}$$

$$y_{85} * y_{76} * y_{62} * y_{51} * y_{43} - y_{85} * y_{76} * y_{61} * y_{52} * y_{43}$$
$$- y_{85} * y_{72} * y_{64} * y_{63} * y_{51} + y_{85} * y_{71} * y_{64} * y_{63} * y_{52} - y_{82} * y_{76} * y_{65} * y_{51} * y_{43}$$
$$+ y_{82} * y_{76} * y_{61} * y_{54} * y_{53} + y_{82} * y_{75} * y_{65} * y_{61} * y_{43} + y_{82} * y_{73} * y_{65} * y_{64} * y_{51}$$
$$- y_{82} * y_{73} * y_{65} * y_{61} * y_{54} - y_{82} * y_{71} * y_{65} * y_{64} * y_{53} + y_{81} * y_{76} * y_{65} * y_{52} * y_{43}$$
$$- y_{81} * y_{76} * y_{62} * y_{54} * y_{53} - y_{81} * y_{75} * y_{65} * y_{62} * y_{43} - y_{81} * y_{73} * y_{65} * y_{64} * y_{52}$$
$$+ y_{81} * y_{73} * y_{65} * y_{62} * y_{54} + y_{81} * y_{72} * y_{65} * y_{64} * y_{53}$$

$$y_{85} * y_{76} * y_{72} * y_{51} * y_{43} - y_{85} * y_{76} * y_{71} * y_{52} * y_{43} - y_{85} * y_{73} * y_{72} * y_{64} * y_{51}$$
$$+ y_{85} * y_{73} * y_{71} * y_{64} * y_{52} - y_{82} * y_{76} * y_{75} * y_{51} * y_{43} + y_{82} * y_{76} * y_{71} * y_{54} * y_{53}$$
$$+ y_{82} * y_{75} * y_{73} * y_{64} * y_{51} + y_{82} * y_{75} * y_{71} * y_{65} * y_{43} - y_{82} * y_{75} * y_{71} * y_{64} * y_{53}$$
$$- y_{82} * y_{73} * y_{71} * y_{65} * y_{54} + y_{81} * y_{76} * y_{75} * y_{52} * y_{43} - y_{81} * y_{76} * y_{72} * y_{54} * y_{53}$$
$$- y_{81} * y_{75} * y_{73} * y_{64} * y_{52} - y_{81} * y_{75} * y_{72} * y_{65} * y_{43} + y_{81} * y_{75} * y_{72} * y_{64} * y_{53}$$
$$+ y_{81} * y_{73} * y_{72} * y_{65} * y_{54}$$

Largest index 9.



$$
\begin{aligned}
& y_{95} * y_{82} * y_{76} * y_{51} * y_{43} - y_{95} * y_{82} * y_{73} * y_{64} * y_{51} - y_{95} * y_{81} * y_{76} * y_{52} * y_{43} \\
& + y_{95} * y_{81} * y_{73} * y_{64} * y_{52} - y_{92} * y_{85} * y_{76} * y_{51} * y_{43} + y_{92} * y_{85} * y_{73} * y_{64} * y_{51} \\
& + y_{92} * y_{81} * y_{76} * y_{54} * y_{53} + y_{92} * y_{81} * y_{75} * y_{65} * y_{43} - y_{92} * y_{81} * y_{75} * y_{64} * y_{53} \\
& - y_{92} * y_{81} * y_{73} * y_{65} * y_{54} + y_{91} * y_{85} * y_{76} * y_{52} * y_{43} - y_{91} * y_{85} * y_{73} * y_{64} * y_{52} \\
& - y_{91} * y_{82} * y_{76} * y_{54} * y_{53} - y_{91} * y_{82} * y_{75} * y_{65} * y_{43} + y_{91} * y_{82} * y_{75} * y_{64} * y_{53} \\
& + y_{91} * y_{82} * y_{73} * y_{65} * y_{54}
\end{aligned}
$$

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