Amoebas and tropical geometry

The American Institute of Mathematics

This is a hard–copy version of a web page available through http://www.aimath.org
Input on this material is welcomed and can be sent to `workshops@aimath.org`
Version: Wed Jan 14 21:58:23 2004

Table of Contents

Chapter A: Open problems

Tropical varieties are piecewise-linear objects in Euclidean space. The link between the classical complex geometry and the tropical geometry is provided by amoebas or logarithmic images of complex varieties. Tropical varieties appear as certain degenerations of amoebas. The workshop brought together researchers in such diverse areas as real algebraic geometry, mirror symmetry, control theory, mathematical physics, analysis of several variables, and dynamics. Amoebas, tropical geometry and the degeneration which relates them appeared in some form in all of these areas.

The collection was compiled by Thorsten Theobald.

A.1 Combinatorics of linear tropical varieties

We know that the Bergman complex of a variety (as defined in Chapter 9 of Sturmfels' book on "Solving systems of polynomial equations"), i.e. the tropical variety, is a polyhedral complex. For linear subspaces, the paper of Ardila and Klivans shows that this Bergman complex has a very nice combinatorial structure: a subdivision of it is the order complex of the lattice of flats of the associated matroid, a very well understood combinatorial object. (As a corollary we get the topology, etc.)

Question: Can we find a similar combinatorial description for other classes of Bergman complexes?

(contributed by Federico Ardila)

A.2 Monge-Ampère measure and mixed cells

Background: There exists a Bernstein theorem for tropical varieties (see Sturmfels' book on "Solving systems of polynomial equations"), there also exists a mixed Monge-Ampère measure whose value at any connected compact component K of the intersection of the considered amoebas is the number of solutions of the corresponding polynomial system in the pre-image (in the complex torus) by Log of K (see the paper "Amoebas, Monge-Ampère measures and triangulations of the Newton polytope" from M. Passare and H. Rullgård).

Questions: Is it true that this value coincides with the volume of the mixed cell corresponding to K (this volume participates in the Bernstein theorem for tropical varieties)? Is there a one-to-one correspondence with the solutions of our system in $\mathrm{Log}^{-1}(K)$ and the solutions of a binomial system corresponding to the mixed cell, and which sends real solutions to real solutions?

(contributed by Frederic Bihan)

A.3 Membership problems

Background: For every ideal $\mathfrak{a}$ in $R_d = \mathbb{Z}[x_1^{\pm 1}, \dots x_d^{\pm 1}]$ there is a related dynamical system generated by d commuting automorphisms of a compact abelian group via Pontryagin duality. For dynamics it is very important to determine when such systems have a finiteness condition called expansiveness. A theorem of Klaus Schmidt states in effect that when $\mathfrak{a}$ contains no nonzero integers, then the system is expansive if and only if the complex amoeba of $\mathfrak{a}$ does not contain the origin.

Question: Is there an algorithm to determine whether the complex amoeba of an ideal $\mathfrak{a}$ in R_d contains the origin?

(contributed by Manfred Einsiedler and Doug Lind)

A.4 Recognition problems

Background: Let k be an algebraically closed non-archimedean field, and $\mathfrak{p}$ be a prime ideal in $k[x_1^{\pm 1}, \dots x_d^{\pm 1}]$. We know that the non-archimedean amoeba of $\mathfrak{p}$ coincides with the Bieri-Groves set of the algebra $A = k[x_1^{\pm 1}, \dots x_d^{\pm 1}]/\mathfrak{p}$, and is thus a homogeneous polyhedral complex whose dimension is the Krull dimension of A, and which is rationally defined over the value group of k. It also has the geometric property of total concavity, a sort of harmonic condition of spreading for the complex.

Question: Given a homogeneous polyhedral complex that is rationally defined over a dense subgroup of the reals and is also totally concave, what further conditions are necessary in order for it to be the amoeba of a prime ideal in the ring of Laurent polynomials over an algebraically closed non-archimedean field?

(contributed by Manfred Einsiedler and Doug Lind)

A.5 Half-space behavior of amoebas

Background: Let $R_d = \mathbb{Z}[x_1^{\pm 1}, \dots x_d^{\pm 1}]$ and $\mathfrak{a}$ be an ideal in R_d with $\mathfrak{a} \cap \mathbb{Z} = \{0\}$. The *adelic amoeba* of $\mathfrak{a}$ is the union of its complex amoeba and its p-adic amoebas over all rational primes p. If $\mathfrak{a} = \langle f \rangle$ is principal, an argument from dynamics shows that every 1-dimensional ray from the origin must intersect the adelic amoeba of f. There should be a version of this for general ideals, and it is enough to state this for prime ideals.

Question: Let $\mathfrak{p}$ be a prime ideal in R_d, and r denote the Krull dimension of $R_d/\mathfrak{p}$. Then for every subspace of $\mathbb{R}^d$ with dimension $d-r+1$, does every half-space of the subspace intersect the adelic amoeba of $\mathfrak{p}$?

(contributed by Manfred Einsiedler and Doug Lind)

A.6 Higher order connectedness of amoebas

Background: Let k be an algebraically closed non-archimedean field, and $\mathfrak{p}$ be a prime ideal in $R = k[x_1^{\pm 1}, \dots x_d^{\pm 1}]$. An argument in a forthcoming paper by Einsiedler, Lind, and Kapranov shows that the non-archimedean amoeba of $\mathfrak{p}$ is a connected set in $\mathbb{R}^d$. We also know that it is a homogeneous polyhedral complex of dimension r, where r is the Krull dimension of $R/\mathfrak{p}$. But examples show that the amoeba may always have a higher type of connectivity as well.

Question: Let $\mathfrak{p}$ be a prime ideal in R, and r be the Krull dimension of $R/\mathfrak{p}$. Form the finite graph whose vertices are the r-dimensional faces of the amoeba of $\mathfrak{p}$, and for which two vertices are joined if they share an $(r-1)$-face of the amoeba. Then is this graph always connected?

(contributed by Manfred Einsiedler, Doug Lind, and Rekha Thomas)

A.7 What does the Riemann-Roch theorem say in the tropical world?

Background: One way to approach this would be to build up the machinery of line bundles (or maybe coherent sheaves?). A different, more immediately geometric approach might be called the "Brill-Noether" approach. This requires only two ingredients:

A. "plane curve with ordinary nodes" and
B. If A, B and C are curves with a common point of intersection, what is "$A \cap B - A \cap C$", the "residual intersection of C in $A \cap B$". If $\mathcal{O}_x$ is the local ring of x on A, and f_B, f_C are the images in $\mathcal{O}_x$ of the equations of B and C, then in the classical case the parts of the intersections $A \cap B$ and $A \cap C$ supported at x are represented by the ideals (f_B) and (f_C) in $\mathcal{O}_x$, and the residual is represented by the ideal

$$(f_B : f_C) := \{g \in \mathcal{O}_x \,|\, g f_C \in (f_B)\}\,.$$

The early work on Riemann-Roch treated only the case where A is smooth at x. Then $A \cap B$ at x is represented just by a multiplicity, and residuation is just subtraction. When A is arbitrary, things still work because $\mathcal{O}_x$ is a Gorenstein ring for any smooth curve, no matter how singular.

Question: How do these notions play out for tropical plane curves?

(contributed by David Eisenbud)

A.8 Tropical Calabi-Yau manifolds and tropical line bundles

An affine manifold is a real manifold with coordinate charts whose transition maps are in $\mathrm{Aff}(\mathbb{R}^n)$.

We will call a *tropical Calabi-Yau manifold* a real manifold B with a dense open subset $B_0 \subseteq B$ which has an affine structure with transition maps in $\mathbb{R}^n \rtimes \mathrm{GL}_n(\mathbb{Z})$, and such that $B \setminus B_0 =: \Delta$ is a locally finite union of locally closed submanifolds of B.

It makes sense to call B_0 a tropical variety. Certainly B_0 locally looks like tropical affine space, and maps in $\mathbb{R}^n \rtimes \mathrm{GL}_n(\mathbb{Z})$ look like maps defined by tropical monomials, so this seems natural. One can additionally talk about the sheaf of piecewise linear functions on B_0 with integral slope, or the sheaf of continuous functions on B which restrict to piecewise linear functions on B_0 with integral slope. This should play the role of the structure sheaf.

Question (Sturmfels): Is it natural to call B a tropical Calabi-Yau variety? In other words, do these singularities make sense in the tropical context? This is related to Zharkov's question of cutting tentacles.

Let $\mathrm{Aff}(B, \mathbb{R})$ denote the sheaf of functions on B which are continuous and restrict to affine linear functions with integral slope on B_0. We define a *tropical line bundle* to be an element of $H^1(B, \mathrm{Aff}(B, \mathbb{R}))$. Representing an element by a Čech 1-cocycle

(α_{ij}) for an open cover $\{U_i\}$, a section of this tropical line bundle is a collection of tropical functions s_i on U_i such that $s_i - s_j = \alpha_{ij}$. (Here this is ordinary subtraction).

We saw how sections of tropical line bundles over tori are tropical theta functions.

Question (Eisenbud, see also the question on Riemann-Roch[33]): What is tropical Riemann-Roch?

[33]page 4, *What does the Riemann-Roch theorem say in the tropical world?*

The above discussion should go over to tropical varieties in general, if we have the right definitions. The same question applies.

Questions: What is the notion of an ample line bundle? Is it interesting to study embeddings into tropical projective space?

Exercise: Consider a tropical plane cubic, say

$$-6x^3 - 4x^2y - 3xy^2 - 6y^3 - 4y^2z - 3yz^2 - 0xyz - 3x^2z - 1xz^2 - 3z^3.$$

Draw a picture of this curve. Cut off the infinite rays, to get a polygon. The affine length of each edge is defined as follows. For the vertices of an edge, v and w, write $v - w = ld$, where d is a primitive integral vector and l is a real number. Then the affine length is $|l|$. Check that the sum of the affine lengths of the edges is 13. Show this polygon can be obtained as an embedding $\mathbb{R}/13\mathbb{Z} \to T\mathbb{P}^2$, using three tropical sections of a tropical line bundle of degree *five*.

Question: This seems a bit strange, doesn't it?

Observation: If one uses a line bundle of degree 3 to try to map to $T\mathbb{P}^2$, certain line segments in the circle will be contracted! Does this mean that the line bundle of degree 3 isn't very ample?

Given such a B, we can form two manifolds of twice the dimension, both torus bundles over B_0. Let $\Lambda \subseteq \mathcal{T}_{B_0}$ be a family of lattices in the tangent bundle generated locally by $\partial/\partial y_1, \ldots, \partial/\partial y_n$ where $y_1, \ldots, y_n$ are local affine coordinates on B_0. Because of the $\mathrm{GL}_n(\mathbb{Z})$ restriction on transition functions, this is well-defined. Let $X(B_0) = \mathcal{T}_{B_0}/\Lambda$. This carries a complex structure which interchanges horizontal and vertical directions in the tangent bundle. Similarly, let $\check{\Lambda} \subseteq \mathcal{T}^*_{B_0}$ be the dual family of lattices generated by $dy_1, \ldots, dy_n$. Then we set $\check{X}(B_0) = \mathcal{T}^*_{B_0}/\check{\Lambda}$. This is canonically a symplectic manifold.

One particularly important question relevant for the Strominger-Yau-Zaslow conjecture is the following. We would like to find *classical* sections of tropical line bundles (i.e. smooth functions (U_i, s_i) with $s_i - s_j = \alpha_{ij}$) satisfying the Monge-Ampère equation

$$\det(\partial^2 s_i/\partial y_j \partial y_k) = \mathit{constant}.$$

If one does this, then pulling back the functions s_i to $X(B_0)$ will give Kähler potentials for Ricci-flat metrics.

Question (Gross, Siebert, Zharkov): Is there a tropical Monge-Ampère equation? Fixing the line bundle $\mathcal{L}$, can one find a sequence of sections $s_i \in \Gamma(B, \mathcal{L}^n)$ (hopefully satisfying this tropical form) such that $\hbar s_n$ converges to a classical solution of the equation. (Here $\hbar = 1/n$). Write a computer program to produce numerical solutions in this way, and draw a picture of a genuine Ricci-flat metric!

Remark: There was some discussion of phases for complex patching during the conference. This can be interpreted as the B-field. See Gross' book with Joyce and Huybrechts for details, but the basic idea is that one can twist the standard complex structure on $X(B_0)$ with an element of $H^1(B_0, \Lambda \otimes \mathbb{R}/\Lambda)$. Under mirror symmetry, this element corresponds to what physicists call the B-field.

References: Thinking about Calabi-Yau manifolds in a tropical sort of way first arose in Kontsevich's and Soibelman's paper from 2000. For examples of tropical Calabi-Yau manifolds, see Gross' book with Joyce and Huybrechts, and the preprints of Haase and Zharkov. For more types of tropical varieties of this flavor, see Symington's work. For a general construction of tropical Calabi-Yau manifolds arising from degenerations of genuine Calabi-Yau manifolds, see Gross' recent paper with Siebert. This latter paper includes quite a bit on tropical Calabi-Yau manifolds (section 1) and gives applications to mirror symmetry.

(contributed by Mark Gross)

A.9 Real tropical varieties

Real tropical hypersurfaces are directly related to T-hypersurfaces (piecewise-linear hypersurfaces arising in the combinatorial patchworking). Many restrictions on the topology of T-hypersurfaces are known. It would be interesting to look at these restrictions from the point of view of tropical geometry and to study the topology of real tropical varieties. For example, the following question arises.

Question: What can be said about Betti numbers of a real tropical variety?

(contributed by Ilia Itenberg)

A.10 The tropical Grassmannian

The tropical Grassmannian, studied by Speyer and Sturmfels, turns out, at least in the cases they study, $(2, n)$ and $(3, 6)$ to have strong combinatorial connections with the Kapranov's Chow quotient $G(r, n)//(\mathbb{C}^*)^n$.

Question (Eugene Tevelev and Sean Keel): Try to understand the precise relationship.

(contributed by Sean Keel)

A.11 Real Gromov-Witten invariants and tropical geometry

Background: G. Tian and S. Kwon recently defined a real Gromov-Witten invariant on each chamber in the real Chow cycles' parameter space when the target space is $\mathbb{CP}^2$. That is a real enumerative invariant, counting the number of intersection points of pull back of real Chow cycles in the real part of the Kontsevich's moduli space of stable maps from genus 0 curves. To use Mikhalkin's work on counting plane rational nodal curves, we showed that the classical nodal Severi variety is embedded as a Zariski open dense subset in the Kontsevich's moduli space.

Question: It will be interesting to develop techniques to calculate real Gromov-Witten invariants by using tropical geometry.

(contributed by Seongchun Kwon)

A.12 Idempotent geometry

Questions:

1. Is it possible to construct a version of algebraic geometry over a class of algebraically closed idempotent semifields (not only tropical semifields)?

Remark: A simple criterion for an idempotent semifield to be algebraically closed is proved in the paper of G. Shpiz "Solving algebraic equations in idempotent semifields", Uspekhi Mat. Nauk, v.55, #5 (2000), p.185-186 (in Russian; there is an English translation in Russian Mathematical Surveys, 2000). There are many examples of algebraically closed idempotent semifields. For example, some standard linear function spaces and all the Banach lattices generate algebraically closed idempotent semifields; see, e.g., the paper of G.L. Litvinov, V.P. Maslov, and G.B. Shpiz "Idempotent functional analysis: an algebraic approach", Math. Notes, v.69, #5 (2001), p.758-797.

2. Is it possible to define a notion of an abstract algebraic (not only affine or projective) variety over tropical and idempotent semifields?
3. Is it possible to define idempotent/tropical versions of such concepts as regular functions and regular maps to get a natural category of idempotent/tropical "affine" algebraic varieties? Is it possible to construct a natural correspondence between this category and a category of idempotent semirings of functions in the spirit of the traditional algebraic geometry?
4. It would be useful to define tropical/idempotent versions of such notions as algebraic equations and ideals of affine algebraic varieties in such a way that points and subvarieties correspond to analogs of ideals.
5. It would be useful to describe tropical/idempotent versions of such notions as prime ideals and irreducible varieties. How to investigate the corresponding decomposition into irreducible components?
6. It would be nice to construct dequantization procedures for a natural correspondence between traditional algebraic varieties and tropical varieties? Is it possible to construct something like a functor ("almost functor") between the corresponding categories?

(contributed by G.L. Litvinov, in cooperation with G.B. Shpiz)

A.13 Moduli space of holomorphic polygons

Background: In the paper with Fukaya "Zero loop open strings in the cotangent bundle and Morse homotopy", Asian J. Math. 1 (1997), 96 - 180, we proved that

"The moduli space of holomorphic polygons with boundary lying on k-tuples of Lagrangian graphs of k-Morse functions is diffeomorphic to that of graph flows of the Morse functions in the adiabatic limit or (in the large complex structure limit). The projections, near the limit, of the holomorphic polygons on the base of the cotangent bundle resembles amoeba-type shapes and it shrinks to the graphs of Morse flows in the limit."

In the paper, we dealt with the case of discs, i.e., open Riemann surfaces of genus zero.

Problem: Study the similar degeneration problem for the higher genus case.

(contributed by Yong-Geun Oh)

A.14 Solidness of amoebas of maximally sparse polynomials

Let $f(z) = \sum_{\alpha \in A} a_\alpha z^\alpha$, with A a finite subset of the integer lattice $\mathbf{Z}^n$, be a complex Laurent polynomial. Its amoeba is the subset of $\mathbf{R}^n$ obtained as the image of $\{f(z) = 0\}$ under the mapping $(z_1, \ldots, z_n) \mapsto (\log|z_1|, \ldots, \log|z_n|)$. The amoeba is said to be *solid* if the number of connected components of its complement is minimal, that is, equal to the number of vertices of the Newton polytope Δ_f of f. Solid amoebas are particularly well adapted to tropical geometry. The polynomial f is said to be *maximally sparse* if the support of summation A is minimal, that is, equal to the set of vertices of Δ_f. When $n = 1$ a maximally sparse polynomial is a binomial.

Question: Does every maximally sparse polynomial have a solid amoeba?

The conjecture is mainly based on empirical data (=computer pictures). I did prove with Hans Rullgård that if the number of vertices is less than or equal to $n+2$, then the tropical spine is contained in the amoeba. (So it would seem very plausible that the number of complement components is minimal for maximally sparse polynomials with at most $n+2$ terms.)

(contributed by Mikael Passare)

A.15 Topology of amoebas of linear spaces

Consider a d-dimensional linear subspace V of $\mathbb{C}^n$, and let M be the intersection of V with $(\mathbb{C}^*)^n$. Then M is the complement of a collection H of n hyperplanes in V, and (virtually) any arrangement of hyperplanes arises in this way. It is a classical problem to study the topology of M in terms of the combinatorics (for example the matroid) of H.

Questions:

A. What are the fibers of the map $\text{Log} : M \to A$, where A is the amoeba of V?
B. What conditions on H will guarantee that this map is a homeomorphism?
C. What can we say in general about the topology of the amoeba of a linear space?
D. How does this relate to Federico Ardila's characterization of the tropicalization of V in terms of the matroid of H?

When $d = 1$, the answer to (1) is easy. In this case, H is a collection of points on a complex line. If there exist three points that do not lie on a common real line, then Log is injective. If all n points lie on a real line, then the fibers of Log are the orbits of the $\mathbb{Z}_2$ action given by reflection over this line.

Higher dimensional examples of hyperplane arrangements such that Log is injective can be constructed by taking a product of d copies of three generic points on a complex line, and then adding arbitrarily many more hyperplanes to this collection of 3d-hyperplanes in $V = \mathbb{C}^d$. But there should be many examples that are simpler than these.

(contributed by Nicholas Proudfoot)

A.16 Nullstellensatz for amoebas

Let $f(x_1, \ldots, x_n)$ be a Laurent polynomial, and write $f(x_1, \ldots, x_n) = \sum_{n=1}^{l} m_i(x)$, where $m_i(x)$ are the monomial terms of x. Given a point $a \in \mathbb{R}^n$, let $f\{a\}$ denote the list of positive reals $[|m_1(\text{Log}^{-1}(a))|, \ldots, |m_l(\text{Log}^{-1}(a))|]$. Note this is well defined, even though Log is not injective.

We say that a list of positive numbers satisfies the polygon condition if it is possible to make a polygon with those side lengths, i.e. no number is greater than the sum of all the others.

Theorem 1. Let I be an ideal, and $A(I)$ its amoeba. Then $a \in A(I)$ if and only if $f\{a\}$ satisfies the polygon condition for all $f \in I$.

Let $P(f) = \{a \in \mathbb{R}^n \ : \ f\{a\} \text{ satisfies the polygon condition}\}$. Think of this as an approximation to the amoeba of a hypersurface.

Theorem 2. Let $A(f)$ be the amoeba of a hypersurface. Let

$$f_m(x_1, \ldots, x_n) = \text{the product of } f(u_1 x_1, \ldots, u_n x_n)$$

over all u_i such that $u_i^m = 1$. The family $P(f_m)$ converges uniformly (in the Euclidean norm) to $A(f)$.

Questions:

A. Is there a version of theorem 2 (an explicit family approximating the amoeba) in the higher codimension case?
B. An analogous statement to theorem 1 is known for non-archimedean Amoebas. Is theorem 1 true in an even more general context?
C. The convergence of the family in theorem 2 is of order $O(\log m/m)$, at least in worst case situations. How fast does this family converge for a randomly chosen f? If the approximation is within $(a \log m + b)/m$ of the actual amoeba, what are a and b, in typical examples?
D. What open problems can this be used to solve?

(contributed by Kevin Purbhoo)

A.17 Tropical Calabi-Yau structures

An example of a tropical Calabi-Yau is the base of a Lagrangian fibered K3 surface. This is a sphere with, generically, an affine structure $\mathcal{A}$ on the complement of 24 points where the singularity at each point has a structure specified by two features:

A. The monodromy in the affine structure $\mathcal{A}$ along a simple loop around a singular point is conjugate to
$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
and
B. there is an **injective** map $\Phi : (U - R, \mathcal{A}) \to (\mathbb{R}^2, \mathcal{A}_0)$ where U is a neighborhood of the singularity and R is a ray based at the singular point. (Here the map Φ is assumed to be a local isomorphism of the affine structures $\mathcal{A}$ and $\mathcal{A}_0$.) The injectivity follows from an argument involving three-dimensional contact geometry.

A natural question is what closed surfaces admit such a singular affine structure, and how many singular points there can be on such a surface. In fact, the possibilities are: a torus or Klein bottle with no singular points, a sphere with 24 singular points, or an $\mathbb{R}P^2$ with 12 singular points. Each one can be realized as the base of a (singular) Lagrangian fibration. The singular fibers in each are diffeomorphic to the singular fibers in a genus one Lefschetz fibration, i.e. they are spheres with one positive self-intersection.

Question: What can one say about the geometry or topology of the set of tropical Calabi-Yau structures on S^2?

Remark: If one is willing to give up the second condition on the singular points, retaining only the monodromy constraint, then one can construct affine structures on S^2 with $12k$ singularities for any $k \geq 2$.

Motivated by the moment map images of Kähler toric varieties, one can consider tropical manifolds that are not necessarily Calabi-Yau. Such a manifold would be built out of strata that are tropical Calabi-Yau manifolds with boundary that satisfy appropriate compatibility conditions. A simple example would be a cylinder equipped with an affine structure such that the boundary of the cylinder is an affine submanifold.

Question: Zharkov asked whether one can perform tropical Gromov-Witten calculations on a Calabi-Yau. Continuing on this line of thought, can one make such calculations on manifolds that have Lagrangian fibrations over these more general tropical manifolds? In particular, on $S^2 \times T^2$ fibering over the cylinder?

(contributed by Margaret Symington)

A.18 Contour of an amoeba

For $f \in \mathbb{C}[x_1, x_2]$, let $\mathcal{C}_f \subset \mathbb{R}^2$ denote the contour of the amoeba of f, i.e., the locus of the critical points of the Gauss map. The singular points V on $\mathcal{C}_f$ naturally divides $\mathcal{C}_f$ into several arcs E, and thus (V, E) defines a planar graph.

Question: What combinatorial properties does the graph (V, E) have? Which graphs can be realized by some function f?

Background: Some examples of the contour can be found e.g., in T. Theobald, Computing amoebas, Exp. Math. 11:513-526, 2002, or in M. Passare and A. Tsikh, Amoebas: their spines and their contours, Preprint, 2003. Since tracing the contour can be used to (numerically) compute the boundary of the amoeba,understanding the combinatorial properties of the contour helps to compute the boundary of the amoeba.

Question: How does this generalize to higher dimension?

(contributed by Thorsten Theobald)

A.19 Tropical bases

Let $I \subset \mathbb{C}[x_1, \ldots, x_n]$ be an ideal. The problem is to characterize/compute subsets J of I which suffice to define the tropical variety $\mathcal{T}(I)$, i.e. $\mathcal{T}(I) = \cap_{j \in J} \mathcal{T}(j)$.

Theorem. The 3×3-minors of an $n \times n$-matrix of indeterminates (which are not a not a universal Gröbner basis) suffice to define the tropical variety of that ideal.

Question (Sturmfels): Do the 4×4-minors of a 5×5-matrix of indeterminates (which are far from a universal Gröbner basis) suffice to define the tropical variety?

Question: Find a characterization of a (smaller) sufficient set (which should be easier/better to compute).

(contributed by Rekha Thomas)

A.20 Real enumerative invariants

In the lecture I gave at the AIM workshop on Amoebas and tropical geometry, I defined some enumerative invariants of real algebraic convex 3-manifolds. For example, through a generic configuration of $2d$ real points in the complex projective space, there passes only finitely many irreducible real rational curves of degree d. Their real parts provide a collection of embedded knots in $\mathbb{RP}^3$. Equip this real projective space with a spin structure. Then it is possible to define a spinor orientation on these knots. Indeed, considering a real subholomorphic line bundle of maximal degree in the normal bundle of the curves, one first defines a framing on these knots. From this framing, one can then build a loop in the $SO_3(\mathbb{R})$-principal bundle of orthonormal frames of $\mathbb{RP}^3$. Then, the spinor orientation of the real curve is the obstruction to lift this loop as a loop of the Spin_3-principal bundle given by the spin structure. Now the algebraic number of real curves, counted with respect to their spinor orientation, turns out to be independent of the choice of the configuration of points and this is my invariant.

Question: Is it possible to compute this invariant with the help of tropical algebraic geometry?

(contributed by Jean-Yves Welschinger)

A.21 Positive tropical varieties and cluster algebras

Question: What is the connection between the tropicalization of the totally positive part of a variety, and the cluster algebra structure of the variety?

Background: In joint work with David Speyer, we have described the tropicalization of the totally positive part of the Grassmannian $G(k, n)$. When $k = 2$, we get a fan which is closely related to the type A associahedron. For $G(3, 6)$ and $G(3, 7)$, we get fans which are related to the type $D4$ and type $E6$ associahedra. Our results seem to be related to the results of Joshua Scott, who showed that the cluster algebra structure of the Grassmannians $G(2, n)$, $G(3, 6)$, and $G(3, 7)$ are of types A, $D4$, and $E6$, respectively.

(contributed by Lauren Williams)

A.22 Statistical algebraic geometry

Background: In statistical algebraic geometry, we put a Gaussian probability measure on the space of polynomials of degree N in m real or complex variables. For simplicity, we think mainly of the $U(m + 1)$-invariant Gaussian measure in the complex case and the $O(m + 1)$ invariant measure in the real case. We then consider probabilities and expected values for interesting random variables. The real and complex cases are quite different, since deterministic problems in the complex case can become random in the real case.

Questions for real algebraic plane curves:

- Consider the ensemble of plane algebraic curves of degree N. Let the random variable be: the number of components of the curve. What is the most probable number of connected components? What is the expected number?
- Consider random spherical harmonics of degree N. Let the random variable be the number of nodal domains (i.e. components of the complement of the zero set of the harmonic). What is the most probable number of nodal domains? The expected number?

Random real fewnomials. We fix a number f. In dimension m, we select m real fewnomials of degree N at random, each with at most f monomials. We pick the spectrum of each fewnomial at random (f lattice points in $\mathbb{Z}_+^m \cap N\Sigma$). We then pick the coefficients of these fewnomials at random from the $O(m + 1)$ ensemble. The problem is:

Question: What is the expected number of real zeros of a random fewnomial system of degree N with f monomials in each fewnomial?

The current bound, due to Khovanski, is

$$\#\text{real zeros} \quad \leq 2^m 2^{f(f-1)/2} (m + 1)^f.$$

It is believed to be an enormous over-estimate.

Zeros of random real fewnomials with fixed Newton polytope. We now pick m random fewnomials $p_1, \dots, p_m$ with prescribed Newton polytopes $\Delta_1, \dots, \Delta_m$ and fixed fewnomial number f. How does the number of simultaneous zeros behave as the polytopes are dilated, $\Delta_j \to N\Delta_j$. I.e. we increase the degrees, but keep the fewnomial number f fixed and keep the spectra in the dilates of the polytopes.

Zeros of random real Kac fewnomials. We ask the same questions but define random real fewnomial as $\sum_\alpha c_\alpha x^\alpha$ where c_α are normal. That is, we do not use projective space to define norms of monomials. [The number of real zeros then goes way down.]

Current result: Shiffman and I currently have an exact formula for the expected number of real zeros of random fewnomial ensembles, but we have not yet found its asymptotics.

Proposition. The density $K_f^N(x)$ of real zeros of random f-fewnomial systems of degree N is given by:

$$K_f^N(x) = \tfrac{1}{\pi^{knm}} \tfrac{Q}{|\mathcal{S}(N,f)|} \textstyle\sum_{S \in \mathcal{S}(N,f)} \frac{\sqrt{\det \nabla_x \nabla_y \log \Pi_{N|S}(x,y)|_{x=y}}}{[\sqrt{\Pi_{N|S}(x,x)}]^m},$$

where $Q := \int_{\mathbb{R}^m} |\xi| \exp\left(-\langle \xi, \xi \rangle\right) d\xi$, and where $\Pi_{N|S}$ is the Szegö kernel for the spectrum S

$$\Pi_{N|S}(x,y) = \sum_{\beta \in S} \binom{N}{\beta} x^\beta y^\beta.$$

Here, $\mathcal{S}(N,f)$ is the set of possible spectra.

Critical points of holomorphic sections. Critical points of holomorphic functions have a long history (Picard-Lefschetz, Milnor-Orlik, Arnold, etc.). Critical points of holomorphic sections have arisen recently in string theory, where they are 'supersymmetric vacua'. Mike Douglas has posed the problem of counting them, finding how they are distributed, and many other statistic problems relevant in string/M theory.

Critical points of a holomorphic section $s \in H^0(M,L)$ of a holomorphic line bundle depend on a choice of Hermitian metric h or connection ∇, which we usually pick to be the metric connection. The equation reads $\nabla s(z) = 0$ and hence the number of critical points depends on the connection or metric.

(contributed by Steve Zelditch)

A.23 Compact tropical varieties, Monge-Ampère equation, Calabi conjecture and curve counting

Background: We adapt Gross's definition of tropical Calabi-Yau manifolds as well as notations (see his contribution on tropical Calabi-Yau manifolds and tropical line bundles).

Compact tropical varieties. The natural question is how to make sense of compact tropical manifolds, not necessarily Calabi-Yau. There has to be a procedure of deleting pseudo-pods and leaving as much of affine structure as possible. My guess is that this will require a choice of polarization (tropical Kähler class). But the affine structure should not depend on this choice and has to be of purely algebro-geometric nature.

Question: How to modify naturally the valuation map for compact tropical varieties?

Tropical Monge-Ampère equation. Let me also add to Gross's question on Monge-Ampère equation. The beauty and importance of the real (and complex) MA equation is that given a boundary conditions it has a unique solution. This is crucial in proving various Calabi conjectures.

In the differential-geometric picture a metric is determined locally in an affine chart $U \subset B_0$ by the graph of the differential of a potential $dK \subset \mathbb{R}^n \times (\mathbb{R}^n)^*$ (we can consistently identify tangent spaces at different points in U with $\mathbb{R}^n$ and cotangent spaces – with $(\mathbb{R}^n)^*$). In the tropical world the C^∞ graphs should be replaced by piece-wise linear ones, which will define distribution-like metrics (or measures) on B. The Monge-Ampère condition – the equality of euclidean measures on $\mathbb{R}^n$ and $(\mathbb{R}^n)^*$ provided by the graph dK – has to be understood in this distributional sense as well.

Question: (Tropical Calabi conjecture) Is there a way to define a tropical Monge-Ampère operator whose solution gives a (unique?) Monge-Ampère measure on B in a given polarization class in $H^1(B, \mathrm{Aff}(B, \mathbb{R}))$?

The uniqueness seems to be false in an obvious assumption that the bendings of the potentials are regulated by the integral lattice. On a torus this corresponds to several possible Voronoi cell decompositions in dimension higher than one.

Curve counting. The symplectic area of a straight line interval is easily seen to be proportional to the scalar product of the primitive vector along the interval and the distance vector between the end points in the *dual* structure. The behavior of a curve near the singular locus is very restrictive. Namely, it can end on a singular point only coming from a unique (eigen) direction.

Question: Given this can we perform a tropical Gromov-Witten calculation on a Calabi-Yau?

As was shown by Mikhalkin the tropical invariants coincide with the genuine ones for curves in surfaces. In higher dimensions, however, there are tropical curves which are not limits of true holomorphic curves.

Question: Is there a simple recipe deciding which tropical curves are the limits of classical ones?

(contributed by Ilia Zharkov)

Chapter B: Snapshot of the pre-open problem session

Relevant aspects:

Amoebas

- maximally sparse polynomials
- amoebas for fewnomials
- spine
- for general varieties (non-hypersurfaces): what are spine, solid, maximally sparse?
- topological structure of amoebas; convexity
- for specific classes of varieties?
- discriminants and amoebas

Tropical geometry

- tropical linear algebra
- line bundles and vector bundles
- variations of tropical varieties

Amoebas vs. tropical geometry

- What is gained or lost in the transition?

Chapter C: Snapshot of the open problem session

The workshop included a moderated, open-problem discussion session.

C.1 Relevant lines of research

- Basic definitions
- Computational issues
- Amoebas of higher codimension
- Families of examples
- Applications of abstract data types in tropical and idempotent calculus
- Recognition problems
- Applications to complex algebraic geometry
- Applications to real algebraic geometry
- Applications to dynamical systems
- Applications to differential equations
- Applications to optimization and control theory
- Applications to representation theory
- Applications to number theory
- Applications to statistical mechanics
- Tropical representation theory

C.2 Basic definitions

A. What is an abstract tropical variety (without embedding, as ringed spaces, rigid analytical space, non-archimedean field given by a covering of charts) ?
What is a good family of local models? Gluing maps.
B. What is an abstract idempotent variety?

Detailed discussion (moderated and contributed by Margaret Symington; see also the figure of the white board of that session):

In the subsequent discussion of the basic definition of tropical varieties, the workshop participants expressed a desire to lay out a couple of definitions, sorting out the names of different objects arising in the tropical realm. Here is what was proposed (in the lower right hand section of the white board):

Given an ideal I in $K^* \subset K = \overline{\mathbb{C}(t)}$ (the Puiseux series), consider two maps,

$$\text{val} : K^* \to \mathbb{R}^*$$

and

$$(\text{val},\ \text{phase}) : K^* \to \mathbb{C}^*$$

where the valuation map "val" takes the value of the smallest exponent in a Puiseux series and "phase" takes the argument of the coefficient of the term with the smallest exponent. Then

- a *tropical variety* is the image of val,
- a *complex tropical variety* is the image of (val, phase), and
- a *real tropical variety* is the subset of a complex tropical variety on which the phase is real-valued.

In the p-adic setting (lower center of the white board) one has analogous objects of interest: the image of the valuation map in $\mathbb{Q}$ and the image of the valuation and the phase in $\mathbb{Q} \times \overline{F_p^*}$ where $\overline{F_p^*}$ is the closure of the set of multiplicative generators of the algebraic closure of F_p.

More generally (in characteristic zero), let I be an ideal in the Laurent polynomial ring $K[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ and let $V(I)$ be its affine variety $V(I) \subset (K^*)^n$. Then the corresponding tropical and complex tropical varieties are the images of val and (val, phase) in $(\mathbb{Q}^*)^n$ and $(\mathbb{C}^*)^n$. The fiber of the projection from such a complex tropical variety to the corresponding tropical variety is a torus, while the fiber of the projection from the real tropical variety is a subset of $\mathbb{Z}_2^n$.

Now view an ideal I in $K[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ as a family of ideals I_t in $\mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$. As $t \to 0$ the complex algebraic varieties $V(I_t)$ converge in the Hausdorff limit to the complex tropical variety for I. In general the complex tropical variety is not homeomorphic to $V(I_t)$ for $t \neq 0$. For curves, if the complex tropical variety is smooth, then it is homeomorphic to $V(I_t)$ for small t.

Question: When is the complex tropical variety homeomorphic or homotopic to the algebraic variety $V(I_t)$ for small t?

Question: Can one define a Hodge theory for complex tropical varieties that is consistent with the limit of the classical Hodge theory?

Moreover, the following questions and needs were stated:

A. Patchworking theorem: Is there a refinement of tropical geometry that will give information about a single (classical) hypersurface?
B. Need to define maps, line bundles, vector bundles

C.3 Computational issues

A. How to compute examples of amoebas and tropical varieties?
B. Special features to compute:
    - homology groups of the complement;
    - is there a Hermitean form whose signature is equal to the number of complement components?

C. Which classical varieties are expansive (i.e., is 0 is in the amoeba?) ? E.g.,
    - Grassmannians, generalized flag manifolds, spherical varieties, determinantal varieties, Schubert varieties;
    - Calabi-Yaus.

D. Develop the theory of smooth curves.
E. What is the complexity of deciding the vanishing of a resultant?
F. When does $\mathrm{Log}(X)$ lie in a lower-dimensional subspace?
G. How hard is it to compute (i.e., what is the complexity of computing) the distance of a point to the discriminantal variety?

C.4 Recognition problems

A. How to recognize whether a a pure d-complex is a tropical variety?
B. Characterize the image of linear subspaces w.r.t. under the phase map ("phlat"). i.e., characterize $\pi(L)$ where $L \subset \mathbb{C}^*$ is a linear subspace, and π is the phase map $\pi : \mathbb{C}^* \to T^n$ (T^n: n-dimensional torus).

C.5 Applications

A. From number theory: Consider one polynomial in one variable with integer coefficients, written as a straight line program of complexity τ. How many roots are there which are congruent to 1 modulo p, as a function of τ ?
There exist estimates using amoeba theory, by work of Maurice Rojas.
B. Prove Calabi conjecture for tropical Calabi-Yaus: in each metric class there is a unique metric which solves the tropical Monge-Ampère measure.
In the real case this corresponds to $\det(\text{Hessian}) = \text{const}$.