Tropical Linear Algebra

Worksheet for the Evening Session
led by Bernd Sturmfels at AIM Palo Alto
on Thursday, October 23, 7:00pm

Tropical arithmetic requires some practise, even for grown-ups. I wish to invite the participants of the *Amoebas and Tropical Geometry* workshop to try the following elementary exercises before (or while) traveling to Palo Alto. The results of your efforts will be the starting point for our discussions in the evening of October 23. All computations are to be carried out over the *tropical semiring* $(\mathbb{R}, \oplus, \otimes)$, where $x \oplus y = \min(x, y)$ and $x \otimes y = x + y$.

1. Compute the *matrix products* $A \otimes B$ and $B \otimes A$ for

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 5 & 6 \\ 3 & 4 \end{pmatrix}.$$

2. Fix positive numbers d_{12}, d_{13}, d_{23} and consider the symmetric matrix

$$D = \begin{pmatrix} 0 & d_{12} & d_{13} \\ d_{12} & 0 & d_{23} \\ d_{13} & d_{23} & 0 \end{pmatrix}$$

Show that $D^2 = D$ if and only if the three triangle inequalities hold.

3. Determine the images of the two linear maps $\mathbb{R}^3 \to \mathbb{R}^3$ which are given respectively by the matrices

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4. Compute the *determinant* of the matrix

$$\begin{pmatrix} 2 & 3 & 5 \\ 7 & 11 & 13 \\ 17 & 19 & 23 \end{pmatrix}.$$

5. Which square submatrices of the following 3×4-matrix are *singular*?

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix}.$$

6. Consider four distinct points A, B, C and D in the *projective plane*. Suppose that the triple A, B, C is collinear and the triple A, B, D is collinear. Does this imply that the triple A, C, D is collinear as well ?

7. Determine the *kernel* of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

8. Regard the *image* of this matrix as a subset of *projective* 3-*space*. Find the smallest *linear subspace* of projective 3-space which contains it.

9. Compute the *rank* of each of the matrices in Exercises (1), (2), (3), (6).

10. Determine the *rank* of the 7×7-matrix

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$