# OPEN PROBLEMS IN COMPACT MODULI SPACES AND BIRATIONAL GEOMETRY

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## 1. Foundational problems

1.1. **Relevant background.** If $\mathcal{F}$ is a coherent sheaf, denote by $\mathcal{F}^{[n]}$ the $n$th reflexive power of $\mathcal{F}$, that is, the double dual of the $n$th tensor power of $\mathcal{F}$. In most of the questions below, $\mathcal{F}$ will be a dualizing sheaf on a variety $X$ which is Gorenstein in codimension one. In this case, $\omega_X^{[n]}$ coincides with the pushforward of the dualizing sheaf of the smooth locus by the inclusion morphism.

For the definitions of semi-log canonical (slc) singularities and stable varieties, see [KSB88], which includes a classification of slc surface singularities), or [Ale96], which gives a definition that does not depend on resolution of singularities. Note that without some condition on the total space, the moduli problem of stable varieties is not separated. Dueling conditions to require on the total space appear in the first problem below. For an overview of the moduli space of stable varieties, see [vO04]. A family of stable surfaces will always be assumed to satisfy one of the conditions given in the problem below.

For higher dimensional moduli problems, we should admit varieties with canonical singularities, or at least rational Gorenstein singularities into the moduli problem of canonically polarized manifolds. These are not to be considered as being at the "boundary" of the moduli space. Consequently, *smoothable* will mean occuring in an admissible family whose general fiber is rational Gorenstein.

1.2. **Existence and scheme structure on moduli spaces.**

**Problem 1.1.** Let $X \to B$ be a family of surfaces. The condition "$\omega_{X/B}^{[m]}$ commutes with base change" means that for every base change $\phi : B' \to B$ and induced morphism $\phi_X : X' = X \times_B B' \to X$, $(\phi_X^* \omega_{X/B})^{[m]} \cong \phi_X^*(\omega_{X/B}^{[m]})$, where the superscript $[m]$ denotes the $m$th reflexive power, that is, the reflexive hull of the $m$th tensor power. *Kollár's condition* is the condition that $\omega_{X/B}^{[m]}$ commutes with base change for all $m > 0$.

Is the condition "$\omega_{X/B}$ is $\mathbf{Q}$-Cartier" equivalent to Kollár's condition? Note that this is true for smoothings over the spectrum of a discrete valuation ring [Hac], Prop. 10.14. The conditions are not equivalent in positive characteristic, as Kollár showed at the conference.

This problem is implicitly due to Kollár, who gave the stronger Kollár condition in the paper [Kol90] after using the weaker definition with Shepherd-Barron for the moduli functor in [KSB88].

**Problem 1.2.** The main purpose of the moduli space of stable surfaces is to serve as a compactification of the moduli space of canonically polarized surfaces. However, it is possible that some stable surfaces are not smoothable. Some parts of the construction of the moduli space have only been proved for smoothable varieties. Here are some problems:

(1) The moduli space $\overline{M^{\mathrm{sm}}_{K^2,\chi}}$ of smoothable stable surfaces does not have a natural scheme structure at the boundary. This is because infinitesimal information is lost by throwing away components of the moduli space parameterizing only surfaces with worse than rational double points. Can more sense be made of the notion of smoothable to get a good scheme structure?
(2) Probably we should keep all of the components and avoid the previous problem completely. In this case, however, the valuative criterion for properness has not been verified even in dimension two. The problem is: if $X \to \Delta'$ is a family of stable surfaces over a punctured disk satisfying Kollár's condition, can $X$ be completed, possibly after base change, to a family of stable surfaces over the disk, also satisfying Kollár's condition?

Alexeev alluded to these problems in one of his talks.

**Problem 1.3** (Higher dimensions)**.** In [Kar00], Karu proves that the semistable minimal model program in dimension $n+1$ can be used (together with the weak semistable reduction of Abramovich and Karu) to construct the moduli space of smoothable stable $n$-folds. See the previous problem for problems with this approach. This is what needs to be done to avoid the MMP:

(1) Prove boundedness of slc $n$-folds with fixed $K^n$ (see Alexeev's problems below). The semistable MMP is not strong enough to do this for non-smoothable $n$-folds anyway.
(2) Prove that small $\mathbf{Q}$-Gorenstein deformations of slc singularities are slc. It is true that small $\mathbf{Q}$-Gorenstein deformations of slt singularities are slt. Again, the MMP in one dimension higher would verify this.
(3) Verify the valuative criterion for properness for non-smoothable $n$-folds. The MMP gives the valuative criterion in the smoothable case.

**Problem 1.4.** (Alexeev) Are the following classes of varieties bounded? Here bounded means that all members of the class can be put into a family over a base of finite type.

(1) Semi-log canonical varieties of general type with fixed $K^n$.
(2) $\epsilon$-log terminal Fano varieties.

### 1.3. **Positive characteristic issues.**

**Problem 1.5.** (Hassett) Prove the valuative criterion for properness for the moduli stack of stable surfaces in positive characteristic.

**Problem 1.6.** (Kollár) Consider the moduli space of canonically polarized surfaces over Spec $\mathbf{Z}$. Is the fiber over the prime $p$ the moduli space of surfaces over $\mathbf{F}_p$? A similar result for $M_g$ may be due to Oort. What about other moduli problems? $A_g$? K3 surfaces?

1.4. **Other moduli problems.** An article of Schumacher and Tsuji [ST04] asserts that a separated moduli space of smooth polarized manifolds is always quasi-projective. By Viehweg's work this was known for canonically polarized manifolds and for polarized manifolds $F$ with $\omega_F$ nef. In the case of polarized uniruled manifolds the separatedness of the moduli space is frequently hard to check. Kollár proposed a series of counterexamples by showing that every smooth toric variety is the moduli space for some class of polarized manifolds. There are many smooth, proper but nonprojective toric varieties.

**Problem 1.7.** Can some result between Viehweg's result and the statement of Schumacher and Tsuji be proved about quasi-projectivity? For instance, if the canonical class is assumed effective?

Kollár's class of examples comes from the observation that the blowups of points on a smooth variety $X$ are parameterized by $X$ modulo the automorphisms of $X$. Thus it could be true that every algebraic space which is a quotient of a quasi–projective variety by a proper group action is a moduli space for some class of polarized manifolds.

**Problem 1.8.** More generally, is every stack which is a quotient of a quasi–projective scheme by a group action a moduli stack for some class of polarized manifolds?

## 2. Moduli spaces of manifolds

The following problems are problems on the geometry of the "interior" of the moduli space, that is, on the moduli space of canonically polarized manifolds. They are motivated by the principle that even without local Torelli theorems, the geometry of moduli spaces should be similiar to the geometry of period domains.

**Problem 2.1.** (Viehweg). Let $M_H$ denote a moduli stack of canonically polarized manifolds. Can $\pi_1(M_H)$ be almost abelian (almost abelian means having an abelian subgroup of finite index)? More precisely, if $U$ is a nonsingular variety (not a point) mapping generically finitely to the moduli stack, can $\pi_1(U)$ be almost abelian. The conjecture is no. The result is known for $M_g$, and can be verified whenever some sort of local Torelli theorem holds.

A $U$ as above cannot be the complement in $\mathbf{P}^n$ of a normal crossings divisor with fewer than $n$ components. At present this is only known if the number of components of $H$ is strictly smaller than $n$ (see [VZ02]).

The following two questions are motivated by work of Viehweg and Zuo [VZ02] and [VZ05].

**Problem 2.2.** (Viehweg) Let $U$ be a smooth variety, étale over a moduli stack of polarized manifolds $M_H$. Let $Y$ be a log compactification of $U$ and $\Gamma = Y \backslash U$.

(1) Is $\Omega^1_Y(\log\ \Gamma)$ weakly positive with respect to $U$?
(2) Is $\omega_Y(\Gamma)$ ample with respect to $U$?

Both of these statements are known for period domains, and it is known that the second implies the first. These problems may be hard, but similar questions would be interesting and perhaps more tractable for configuration spaces.

**Problem 2.3.** Assume that $M_H$ is a moduli stack for abelian varieties or for a class of manifolds for which local Torelli holds. Let $U$, $Y$, and $\Gamma$ be as in the previous problem. There is a variation of Hodge structures. Take an irreducible subvariation with Higgs bundle $E^{1,0} \oplus E^{0,1}$. Then one can show

$$\frac{\deg E^{1,0}}{\mathrm{rk} E^{1,0}} - \frac{\deg E^{0,1}}{\mathrm{rk} E^{0,1}} \leq \frac{\deg \Omega^1_Y(\log\ \Gamma)}{\dim Y}$$

where $\omega_Y(\Gamma)$ is nef and big, and degree is taken with respect to this sheaf. The problem is to determine when equality holds. A conjecture in this direction is that $U$ is a Shimura variety. This is true if $Y$ is a curve and for moduli spaces $A_g$ for small $g$.

**Problem 2.4.** (Viehweg) Does there exist a universal bound for the number of families of minimal manifolds with Hilbert polynomial $H$ over a curve of genus $q$ with $s$ singular fibers depending only on $H$, $q$, and $s$, and not any more features of the geometry of the fibers?

## 3. GIT

3.1. **Background.** Geometric invariant theory plays an essential role in the usual construction of $M_g$. GIT separates the points of a Hilbert scheme into three classes with respect to an action of a reductive group $G$: stable, semistable, and unstable. The quotient of the set of stable points by the group exists in a nice sense and is a moduli space of these GIT stable varieties. Note that GIT stability is not compatible with stability in the sense of the MMP. There are “MMP-stable” singularities which are asymptotically GIT-unstable. Also, some GIT semistable points are not MMP-stable. A good quotient of the semistable locus by the group $G$ exists, but is not a moduli space, since some orbits of the group action corresponding to nonisomorphic varieties have intersection closures.

3.2. **Problems.**

**Problem 3.1.** (from Kollár’s lecture) Fix a class of (pluri)canonically embedded manifolds in $\mathbf{P}^n$. There is a notion of GIT semistability for Hilbert points in the closure of the Hilbert points of the class of manifolds considered. Now embed the family in another $\mathbf{P}^m$ using a higher power of $K_X$ and consider the GIT semistability conditions in that projective space. Is there an integer $N$ such that the class of GIT semistable surfaces stabilizes for embeddings by $MK_X$ for $M > N$? This problem is fundamental, and the answer for curves is that $N = 5$.

**Problem 3.2.** Determine the GIT (semi)stable quintic surfaces, quartic threefolds, quintic threefolds. For hypersurfaces of a given dimension and degree, is there a bound on the exponents appearing in the diagonal 1-PS that need to be checked? All smaller cases have been checked by various authors.

**Problem 3.3.** (Morrison) For a variety $X$ and ample line bundle $L$, can we work harder on the combinatorics of the monomial subspaces of $H^0(X, L^m)$ to get easier estimates for the weights $r_{\lambda,X}$ of the Hilbert point of $X$ with respect to the filtration given by some 1-PS $\lambda$? For example

(1) Find more flexible filtrations of $H^0(X, L^m)$ with stages of the type occurring in geometric estimates (e.g. cusps are not semistable) and compute the leading term of the resulting estimates for $r_{\lambda,X}$.
(2) Find methods for producing estimates for $r_{\lambda,X}$ which use estimates for *non-nested* collections of subspaces of $H^0(X, L)$.

## 4. Examples and applications

**Problem 4.1.** Find applications of moduli of stable surfaces to surface theory. For example degenerations of curves are used to prove Brill-Noether type statements.

**Problem 4.2.** Understand the geometry of the moduli space of stable surfaces better. Here it is probably necessary to find some nice components and study their geometry. In general, after fixing even the differentiable structure of the underlying four-manifolds, the moduli space is disconnected, its connected components are reducible, and its irreducible components may be everywhere nonreduced. In addition, the boundary may not be a divisor, and the next best substitute is not a normal crossings divisor. Also, Vakil shows that the moduli space of canonically polarized surfaces is arbitrarily singular (in a precise sense).

4.1. **Intersection theory.**

**Problem 4.3.** (Hassett) Does the moduli space of stable surfaces have a virtual fundamental class?

**Problem 4.4.** Find natural loci in moduli spaces of surfaces which would be interesting to intersect with each other.

**Problem 4.5.** (Alexeev) Is there a Gromov-Witten theory or quantum cohomology theory arising from stable pairs, perhaps from stable maps $f : (\mathbf{P}^2, D = D_1 + \cdots + D_n) \to X$, where $D$ is a union of lines?

4.2. **Explicit examples.**

**Problem 4.6.** Can one work out a geometric compactification of polarized K3 surfaces or Calabi-Yau manifolds in general using the framework of Alexeev/Kollár/Shepherd-Barron? For example, let $(X, H)$ be a pair where $X$ is a K3 surface and $H$ is a very ample divisor with $H^2 = 4$. Can the stable pairs occurring as degenerations of these objects be classified? Compare the resulting singularities with the work of Shah on GIT-semistable quartic surfaces. Also compare with Martin Olsson's work on moduli of log K3 surfaces.

**Problem 4.7.** Can one similarly give explicit examples of moduli spaces of surfaces of general type? For example, it would be interesting to understand the stable degenerations of

(1) octic double planes,
(2) quintic hypersurfaces,
(3) bidouble covers, and other abelian covers, following Catanese, Manetti, Pardini, etc.

**Problem 4.8.** Study the geometry of Hacking's moduli space of plane curves. Are there applications to questions about families of smooth plane curves? Similarly for configuration spaces or moduli spaces of marked del Pezzo surfaces.

## 5. Moduli of curves

5.1. **Birational geometry.** Some references are: [FG03], [GKM02], [FP]. This is by no means a complete list; see the references in these papers for more details.

**Problem 5.1.** Determine the cone of curves of $\overline{M_{0,n}}$.

**Problem 5.2.** Determine the Kodaira dimension of $\overline{M_g}$ when $g = 15$ or $17 \leq g \leq 22$. For $g > 22$ $\overline{M_g}$ is of general type, and in the other known cases, the Kodaira dimension is negative.

5.2. **Other questions.**

**Problem 5.3.** Does the coarse moduli space $M_g$ contain a $\mathbf{P}^1$? An $\mathbf{A}^1$? It is known that the moduli stack in this case is algebraically hyperbolic, but the same is not likely to be true for the moduli space.

**Problem 5.4.** (Farkas) For $g$ large, what is the minimal genus of a curve passing through a general point of $M_g$? This can be asked for the moduli space or the moduli stack. For low $g$, the answer for the moduli stack is zero.

One could ask this question for moduli spaces of canonically polarized manifolds, although I think the point of making $g$ large is to ensure that $M_g$ is not uniruled. There is nothing even approximating such a condition to ensure that $M_H$ is not unirational.

## 6. Moduli of abelian varieties

**Problem 6.1.** (Grushevsky)
1. Construct divisors on $\overline{A_g}$ of small slope.
2. Find explicit complete subvarieties of $A_g$. There is a bound on the codimension of such a variety in [KS03].

**Problem 6.2.** (Shepherd-Barron) Questions on $A_g$ and the first Voronoi compactification $A_g^F$.
1. Run the MMP on $A_g^F$ when $g \leq 10$; in particular, flip its extremal ray. If $g \geq 12$, $A_g^F$ is the canonical model of $A_g$, and when $g = 11$ it is a minimal model.
2. There is a family of natural inclusions $A_g^F \to A_{g+1}^F$ parameterized by the $j$-line. Does $H^*(A_g^F, \mathbf{C})$ stabilize? To what?
3. Let $M$ be the bundle of weight 1 modular forms and $D$ the boundary divisor. Compute the intersection numbers $M^a.D^b$ and the plurigenera of $A_g^F$.
4. Is the total coordinate ring of $A_g^F$ finitely generated?
5. Is the divisor $12M - D$ semi-ample for $g \geq 12$?
6. Find the effective cone of $A_g^F$. This is the well-known question of finding the possible slopes of Siegel cusp forms.

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