1 Weyl characters

Your favourite group $G^\vee$ (probably $SL_3(\mathbb{C})$) corresponds to

$W = \{\text{chambers}\}$

and

$P^\vee = \{\text{dots}\}$

The irreducible $G^\vee$-modules $L(\lambda^\vee)$ are indexed by $\lambda^\vee \in (P^\vee)^+$ and

$$\text{char}(L(\lambda^\vee)) = \sum_{\mu^\vee \in P^\vee} \text{Card}(B(\lambda^\vee)_{\mu^\vee}) x^{\mu^\vee},$$

where

$B(\lambda^\vee)_{\mu^\vee} = \{\text{Littelmann paths of type } \lambda^\vee \text{ and end } \mu^\vee\}.$

If

$G = G(\mathbb{C}((t))), \quad K = G(\mathbb{C}[[t]]), \quad \text{and} \quad U^- = \left\{ \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \right\}. $

then $G/K$ is the loop Grassmanian and

$G = \bigsqcup_{\lambda^\vee \in (P^\vee)^+} K t_{\lambda^\vee} K \quad \text{and} \quad G = \bigsqcup_{\mu^\vee \in P^\vee} U^- t_{\mu^\vee} K.$
The MV cycles of type $\lambda^\vee$ and weight $\mu^\vee$ are the elements of
\[ MV(\lambda^\vee)_{\mu^\vee} = \{ \text{irreducible components of } Kt_{\lambda^\vee}K \cap U^{-t_{\mu^\vee}K} \}, \]
and
\[ \text{char}(L(\lambda^\vee)) = \sum_{\mu^\vee} \text{Card}(MV(\lambda^\vee)_{\mu^\vee}) x^{\mu^\vee}. \]

2 Hecke algebras

The spherical and affine Hecke algebras are
\[ \tilde{H}_{\text{sph}} = C(K\backslash G/K) \quad \text{and} \quad \tilde{H} = C(I\backslash G/I), \]
where
\[ G = G(\mathbb{C}(t)) \]
\[ \cup \quad \cup \]
\[ K = G(\mathbb{C}[[t]]) \quad \Phi \rightarrow \quad G(\mathbb{C}) \quad \text{where} \quad B = \left\{ \begin{pmatrix} \ast & \ast \\ 0 & \ast \end{pmatrix} \right\}. \]
\[ \cup \quad \cup \quad \cup \quad \cup \]
\[ I = \Phi^{-1}(B) \rightarrow B, \]
The Satake map is
\[ \mathbb{C}[X]^W = Z(\tilde{H}) \quad \sim \rightarrow \quad Z(\tilde{H})1_0 = 1_0 \tilde{H}1_0 = \tilde{H}_{\text{sph}} \]
\[ \quad f \quad \mapsto f1_0 \]
\[ P_{\lambda^\vee} \quad \leftarrow \quad 1_0 X^{\lambda^\vee}1_0 = \chi_{Kt_{\lambda^\vee}K} \quad \text{“obvious” basis} \]
and $P_{\lambda^\vee}$ are the Hall-Littlewood polynomials.
\[ P_{\lambda^\vee} = \sum_{\mu^\vee \in P_{\lambda^\vee}} \text{Card}_q(\mathcal{P}(\lambda^\vee)_{\mu^\vee}) x^{\mu^\vee}, \]
where
\[ \mathcal{P}(\lambda^\vee)_{\mu^\vee} = \{ \text{Hecke paths of type } \lambda^\vee \text{ and end } \mu^\vee \} \longleftrightarrow \{ \text{slices of } G/K \text{ in } Kt_{\lambda^\vee}K \cap U^{-t_{\mu^\vee}K} \} \]
and
\[ \text{Card}_q(\mathcal{P}(\lambda^\vee)_{\mu^\vee}) = \sum_{p \in \mathcal{P}(\lambda^\vee)_{\mu^\vee}} (\# \text{ of } \mathbb{F}_q \text{ points in slice } p). \]
After normalization,
\[ P_{\lambda^\vee} |_{q^{-1} = 0} = \text{char}(L(\lambda^\vee)). \]

3 Buildings

The group $B$ is a Borel subgroup of $G = G(\mathbb{C})$ and
\[ G/B = \text{flag variety} = \text{building}. \]
The cell decomposition of $G/B$ is
\[ G = \bigsqcup_{w \in W} BwB. \]
Idea: The points of $W$ are regions, or chambers.

$$W = \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1, s_1s_2s_1 = s_2s_1s_2 \rangle$$

If $w = s_{i_1} \cdots s_{i_\ell}$ is a minimal length path to $w$ then

$$BwB = \{ x_{i_1}(c_1)s_{i_1} \cdots x_{i_\ell}(c_\ell)s_{i_\ell}B \mid c_1, \ldots, c_\ell \in \mathbb{C} \},$$

where $x_i(c) = 1 + cE_i$, with $E_{i,i+1}$ the matrix with a 1 in the $(i, i+1)$ entry and all other entries 0.

**IDEA:** The points of $G/B$ are regions, or chambers.

Just as the building of $W$, the Coxeter complex, has relations

$$s_1s_2s_1 = s_2s_1s_2$$

the building of $G/B$ also has relations

$$x_1(c_1)s_1x_2(c_2)s_2x_1(c_3)s_1 = x_2(c_3)s_2x_1(c_1c_3 - c_2)s_1x_2(c_3)s_2$$

An apartment is a subbuilding of $G/B$ that looks like $W$.

The Borel subgroup of $G = G(\mathbb{C}(t))$ is $I$ and

$G/I$ is the affine flag variety.
with
\[ G = \bigsqcup_{w \in \tilde{W}} IwI, \quad \text{where} \quad \tilde{W} = W \ltimes P^\vee \]

is the affine Weyl group

The affine building $G/I$ has sectors

since $G = \bigsqcup_{v \in \tilde{W}} U^{-v}I$.

4 MV polytopes

Let
\[ T = \begin{cases} \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \end{cases} \]

and let $V$ be a $T$-module

with $T$-invariant inner product $(\langle \cdot, \cdot \rangle$ (such that $\langle v, v \rangle = 0 \iff v = 0$). Let
\[ \mathfrak{h} = \text{Lie}(T) \quad \text{and} \quad \mathbb{P}V = \{ [v] \mid v \in V, v \neq 0 \}, \]

where $[v] = \text{span}\{v\}$. The moment map on $\mathbb{P}V$ is
\[ \mu: \mathbb{P}V \to \mathfrak{h}^* \quad \text{where} \quad \mu_v(h) = \frac{\langle hv, v \rangle}{\langle v, v \rangle}. \]

Now let $V = L(\gamma)$ be a simple $G$-module ($G = G(\mathbb{C})$) with highest weight vector $v^+$. Then
\[ B[v^+] = [v^+] \quad \text{and} \quad G[v^+] \subseteq \mathbb{P}V \]

is the image of $G/B$ in $\mathbb{P}V$. The moment map on $G/B$ (associated to $\gamma$) is
\[ \mu: G/B \to \mathbb{P}V \to \mathfrak{h}^* \quad \text{where} \quad gB \mapsto g[v^+] \mapsto \mu_{gv^+}. \]

Joel(Kamnitzer)'s favourite case is $G/K$ with $\gamma = \omega_0$ (the fundamental weight corresponding to the added node on the extended Dynkin diagram) and
\[ \mu(\text{MV cycle of type } \lambda^\vee \text{ and weight } \mu^\vee) = (\text{MV polytope of type } \lambda^\vee \text{ and weight } \mu^\vee) \]
5 Tropicalization

Let $G = G(\mathbb{C}((t)))$. 
\[ \mathbb{C}((t)) = \{ a_\ell t^\ell + a_{\ell+1} t^{\ell+1} + \cdots \mid \ell \in \mathbb{Z}, a_i \in \mathbb{C} \}. \]

Points of $G/I$ are 
\[ gI, \quad \text{where} \quad g = (g_{ij}), \quad g_{ij} \in \mathbb{C}((t)). \]

The valuation on $\mathbb{C}((t))$ 
\[ v(a_\ell t^\ell + a_{\ell+1} t^{\ell+1} + \cdots) = \ell, \]

is like log 
\[ v(f_1 f_2) = v(f_1) + v(f_2) \quad \text{and} \quad v(f_1 + f_2) = \min(v(f_1), v(f_2)). \]

Then $v(gI)$ is a tropical point of $v(G/I)$, the tropical flag variety. An amoeba, or tropical subvariety, is the image, under $v$, of a subvariety of $G/I$. 

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