# Introduction to
# Buildings and Combinatorial Representation Theory
# American Institute of Mathematics (AIM)
# March 26, 2007

Arun Ram
Department of Mathematics
University of Wisconsin
Madison, WI 53706
ram@math.wisc.edu

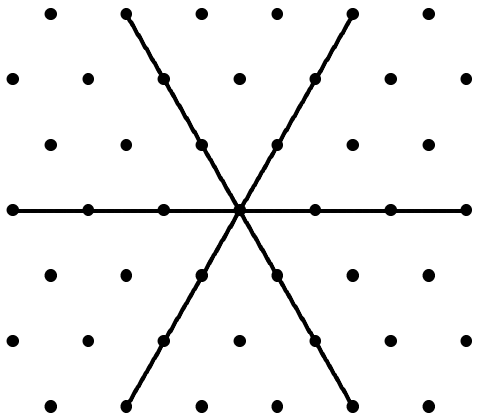
## 1 Weyl characters

Your favourite group $G^\vee$ (probably $SL_3(\mathbb{C})$) corresponds to

$W = \{\text{chambers}\}$

and

$P^\vee = \{\text{dots}\}$



The irreducible $G^\vee$-modules $L(\lambda^\vee)$ are indexed by $\lambda^\vee \in (P^\vee)^+$ and

$$\text{char}(L(\lambda^\vee)) = \sum_{\mu^\vee \in P^\vee} \text{Card}(B(\lambda^\vee)_{\mu^\vee} x^{\mu^\vee},$$

where

$$B(\lambda^\vee)_{\mu^\vee} = \{\text{Littelmann paths of type } \lambda^\vee \text{ and end } \mu^\vee\}.$$

If

$$G = G(\mathbb{C}((t))), \qquad K = G(\mathbb{C}[[t]]), \quad \text{and} \quad U^- = \left\{ \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix} \right\}.$$

then $G/K$ is the *loop Grassmanian* and

$$G = \bigsqcup_{\lambda^\vee \in (P^\vee)^+} K t_{\lambda^\vee} K \qquad \text{and} \qquad G = \bigsqcup_{\mu^\vee \in P^\vee} U^- t_{\mu^\vee} K.$$

The *MV cycles of type* $\lambda^\vee$ *and weight* $\mu^\vee$ are the elements of

$$MV(\lambda^\vee)_{\mu^\vee} = \{\text{irreducible components of } \overline{Kt_{\lambda^\vee}K \cap U^- t_{\mu^\vee}K}\ \},$$

and

$$\mathrm{char}(L(\lambda^\vee)) = \sum_{\mu^\vee} \mathrm{Card}(MV(\lambda^\vee)_{\mu^\vee}) x^{\mu^\vee}.$$

## 2 Hecke algebras

The *spherical* and *affine Hecke algebras* are

$$\tilde{H}_{\mathrm{sph}} = C(K\backslash G/K) \qquad \text{and} \qquad \tilde{H} = C(I\backslash G/I),$$

where

$$\begin{array}{ccccc} G & = & G(\mathbb{C}((t))) & & \\ \cup| & & \cup| & & \\ K & = & G(\mathbb{C}[[t]]) & \xrightarrow{\Phi} & G(\mathbb{C}) \\ \cup| & & \cup| & & \cup| \\ I & = & \Phi^{-1}(B) & \longrightarrow & B\,, \end{array} \qquad \text{where} \qquad B = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}.$$

The *Satake map* is

$$\begin{array}{rcl} \mathbb{C}[X]^W = Z(\tilde{H}) & \xrightarrow{\sim} & Z(\tilde{H})\mathbf{1}_0 = \mathbf{1}_0 \tilde{H} \mathbf{1}_0 = \tilde{H}_{\mathrm{sph}} \\ f & \longmapsto & f\mathbf{1}_0 \\ P_{\lambda^\vee} & \longleftarrow & \mathbf{1}_0 X^{\lambda^\vee} \mathbf{1}_0 = \chi_{Kt_{\lambda^\vee}K} \quad \text{“obvious” basis} \end{array}$$

and $P_{\lambda^\vee}$ are the *Hall-Littlewood polynomials*.

$$P_{\lambda^\vee} = \sum_{\mu^\vee \in P^\vee} \mathrm{Card}_q(\mathcal{P}(\lambda^\vee)_{\mu^\vee}) x^{\mu^\vee},$$

where

$$\mathcal{P}(\lambda^\vee)_{\mu^\vee} = \{\text{Hecke paths of type } \lambda^\vee \text{ and end } \mu^\vee\} \longleftrightarrow \{\text{slices of } G/K \text{ in } Kt_{\lambda^\vee}K \cap U^- t_{\mu^\vee}K\}$$

and

$$\mathrm{Card}_q(\mathcal{P}(\lambda^\vee)_{\mu^\vee}) = \sum_{p \in \mathcal{P}(\lambda^\vee)_{\mu^\vee}} (\#\text{ of } \mathbb{F}_q \text{ points in slice } p).$$

After normalization,

$$P_{\lambda^\vee}\big|_{q^{-1}=0} = \mathrm{char}(L(\lambda^\vee)).$$

## 3 Buildings

The group $B$ is a Borel subgroup of $G = G(\mathbb{C})$ and

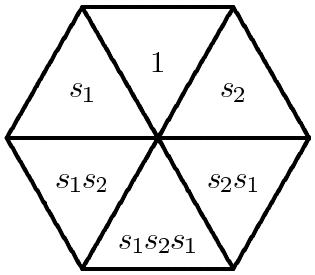
$$G/B = \text{flag variety} = \text{building}.$$

The cell decomposition of $G/B$ is

$$G = \bigsqcup_{w \in W} BwB.$$

Idea: The points of $W$ are regions, or chambers.

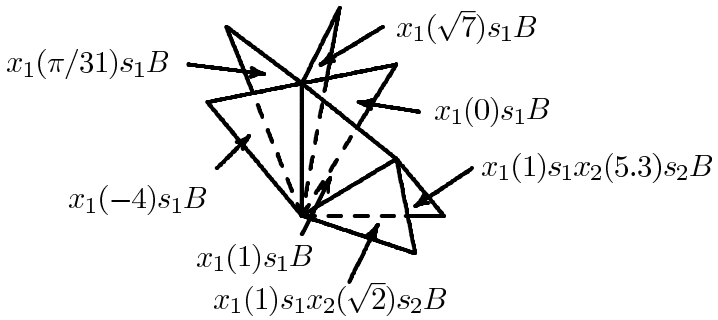
$$W = \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1, s_1 s_2 s_1 = s_2 s_1 s_2 \rangle$$




If $w = s_{i_1} \cdots s_{i_\ell}$ is a minimal length path to $w$ then

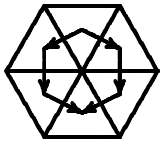
$$BwB = \{x_{i_1}(c_1)s_{i_1} \cdots x_{i_\ell}(c_\ell)s_{i_\ell}B \mid c_1, \ldots, c_\ell \in \mathbb{C}\}, \qquad \text{where} \quad x_i(c) = 1 + cE_{i,i+1},$$

with $E_{i,i+1}$ the matrix with a 1 in the $(i, i+1)$ entry and all other entries 0.
**IDEA**: The points of $G/B$ are regions, or chambers.



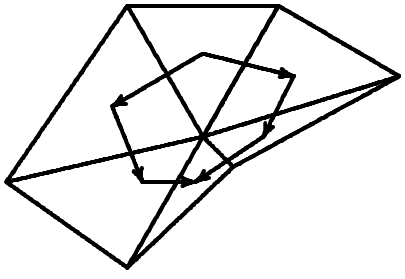

Just as the building of $W$, the *Coxeter complex*, has relations

$$s_1 s_2 s_1 = s_2 s_1 s_2$$



the building of $G/B$ also has relations

$$x_1(c_1)s_1x_2(c_2)s_2x_1(c_3)s_1 = x_2(c_3)s_2x_1(c_1c_3 - c_2)s_1x_2(c_3)s_2$$



An *apartment* is a subbuilding of $G/B$ that looks like $W$.

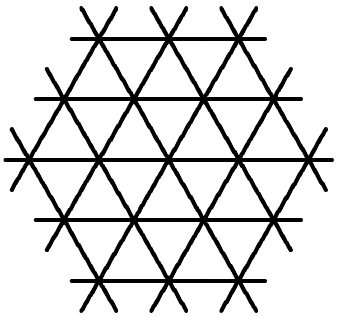
The Borel subgroup of $G = G(\mathbb{C}((t)))$ is $I$ and

$G/I$ is the *affine flag variety*

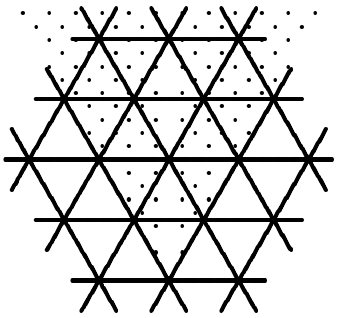
with

$$G = \bigsqcup_{w \in \tilde{W}} IwI, \qquad \text{where} \quad \tilde{W} = W \ltimes P^\vee$$

is the *affine Weyl group*



The *affine building* $G/I$ has *sectors*



since $G = \bigsqcup_{v \in \tilde{W}} U^- vI.$

## 4 MV polytopes

Let

$$T = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \qquad \text{and} \quad \text{let } V \text{ be a } T\text{-module}$$

with $T$-invariant inner product $\langle , \rangle$ (such that $\langle v, v \rangle = 0 \Leftrightarrow v = 0$). Let

$$\mathfrak{h} = Lie(T) \qquad \text{and} \qquad \mathbb{P}V = \{[v] \mid v \in V, v \neq 0\},$$

where $[v] = \text{span}\{v\}$. The *moment map* on $\mathbb{P}V$ is

$$\begin{array}{rccc} \mu\colon & \mathbb{P}V & \to & \mathfrak{h}^* \\ & [v] & \longmapsto & \mu_v \end{array} \qquad \text{where} \qquad \mu_v(h) = \frac{\langle hv, v \rangle}{\langle v, v \rangle}.$$

Now let $V = L(\gamma)$ be a simple $G$-module ($G = G(\mathbb{C})$) with highest weight vector $v^+$. Then

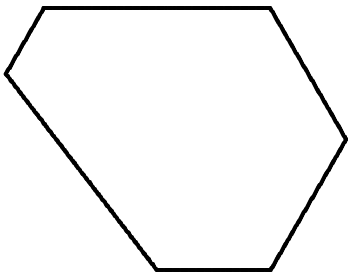
$$B[v^+] = [v^+] \qquad \text{and} \qquad G[v^+] \subseteq \mathbb{P}V$$

is the image of $G/B$ in $\mathbb{P}V$. The *moment map* on $G/B$ (associated to $\gamma$) is

$$\begin{array}{rccccc} \mu\colon & G/B & \to & \mathbb{P}V & \to & \mathfrak{h}^* \\ & gB & \mapsto & g[v^+] & \mapsto & \mu_{gv^+} \end{array}$$

Joel(Kamnitzer)'s favourite case is $G/K$ with $\gamma = \omega_0$ (the fundamental weight corresponding to the added node on the extended Dynkin diagram) and

$$\mu(\text{MV cycle of type } \lambda^\vee \text{ and weight } \mu^\vee) = (\textit{MV polytope of type } \lambda^\vee \textit{ and weight } \mu^\vee)$$



## 5 Tropicalization

Let $G = G(\mathbb{C}((t)))$.

$$\mathbb{C}((t)) = \{a_\ell t^\ell + a_{\ell+1}t^{\ell+1} + \cdots \mid \ell \in \mathbb{Z}, a_i \in \mathbb{C}\}.$$

Points of $G/I$ are

$$gI, \qquad \text{where} \quad g = (g_{ij}), \quad g_{ij} \in \mathbb{C}((t)).$$

The *valuation* on $\mathbb{C}((t))$

$$v(a_\ell t^\ell + a_{\ell+1}t^{\ell+1} + \cdots) = \ell,$$

is like log

$$v(f_1 f_2) = v(f_1) + v(f_2) \qquad \text{and} \qquad v(f_1 + f_2) = \min(v(f_1), v(f_2)).$$

Then $v(gI)$ is a *tropical point* of $v(G/I)$, the *tropical flag variety*. An *amoeba*, or *tropical subvariety*, is the image, under $v$, of a subvariety of $G/I$.