# LS Galleries and MV Cycles

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| $G$ group | $G^\vee$ |
|---|---|
| $\mathrm{SL}_3$ | $\mathrm{PSL}_3$ |
| $\mathrm{SO}_5$ | $\mathrm{Sp}_4$ |

Fix $A$, $P$ (the character lattice), $P^\vee$ (the co-character lattice), $(\alpha_i)$ (the simple roots), $(\alpha_i^\vee)$ (the co-roots)

Path model for $G^\vee$
Littelmann, 1994
$p : [0,1] \to P^\vee \otimes \mathbb{R}$
$\lambda \in (P^\vee)^+$ model of $V(\lambda)$
irreducible representation
of highest weight $\lambda$

$\subset$

Bruhat-Tits
building

$\supset$

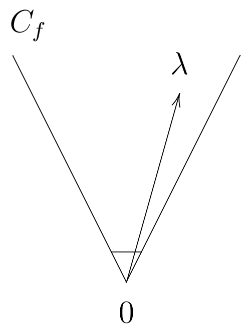
Mirković-Vilonen, 1996
Lusztig-Ginzburg: $IH^*(\mathrm{Gr}^{\overline{\lambda}}) \cong V(\lambda)$
$IH^*(\mathrm{Gr}^{\overline{\lambda}})$ intersection cohomology
$\lambda \in (P^\vee)^+$
$\bigoplus_{\lambda \in P^\vee} \# \ \mathrm{Irred}(Z_\mu^\lambda)$

**Remarks.**

1. Mirković-Vilonen found cycles in $IH^*(\mathrm{Gr}^{\overline{\lambda}})$ that give a canonical basis for $V(\lambda)$.
2. The first inclusion is combinatorial; the second inclusion is via geometry (inside the building).

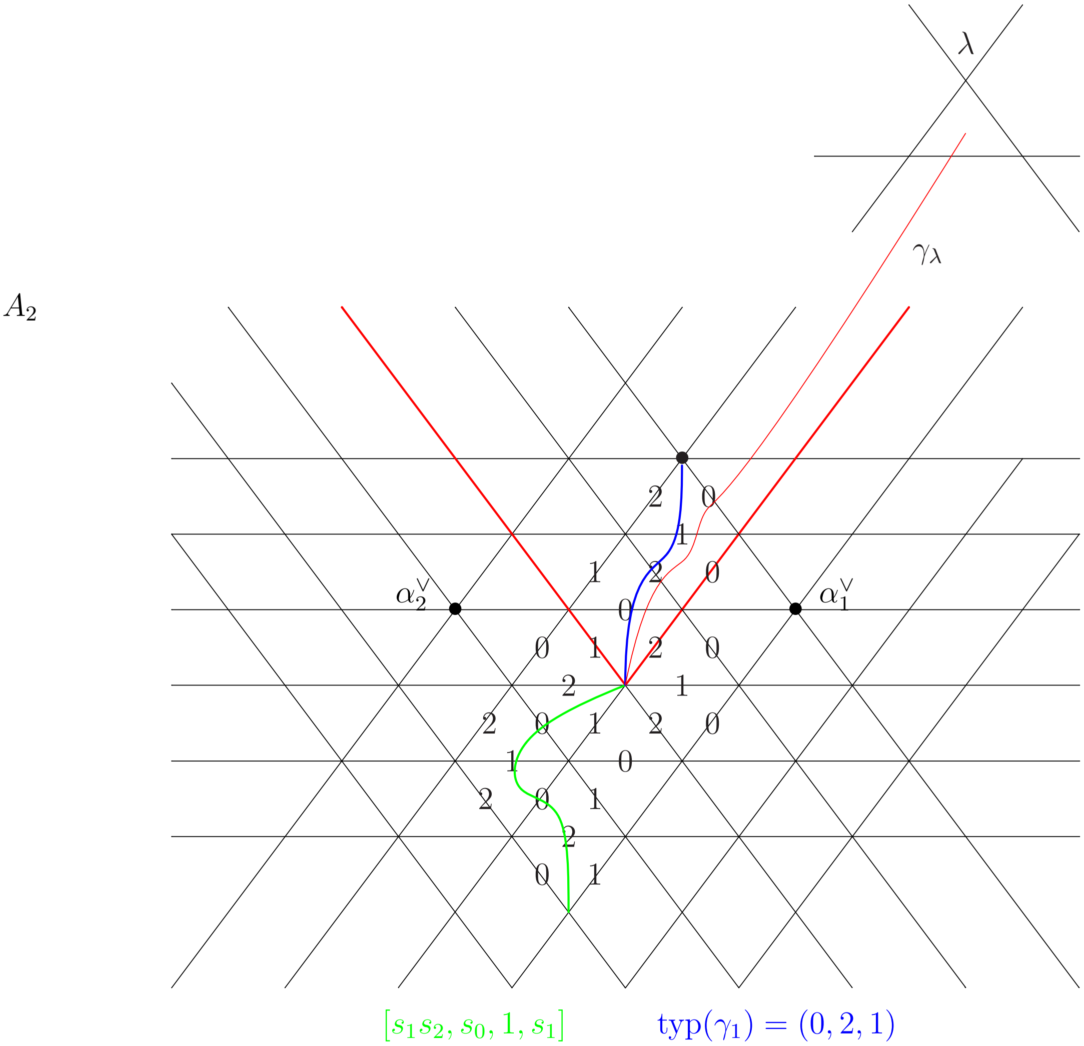
To compare the two models, we need to replace the path model by the gallery model.

<u>Combinatorics</u>
Let $\lambda \in (P^\vee)^+$ be regular.
($\lambda$ is a dominant co-weight, regular)




The apartment is $\mathbb{A} = P^\vee \otimes \mathbb{R}$.

The affine Weyl group is $W^a = W \ltimes Q^\vee$, where $Q^\vee$ is the coroot lattice.
An apartment is a Coxeter complex.
labelling of it
Fix a minimal alcove gallery $\gamma_\lambda$ from 0 to $\lambda$.



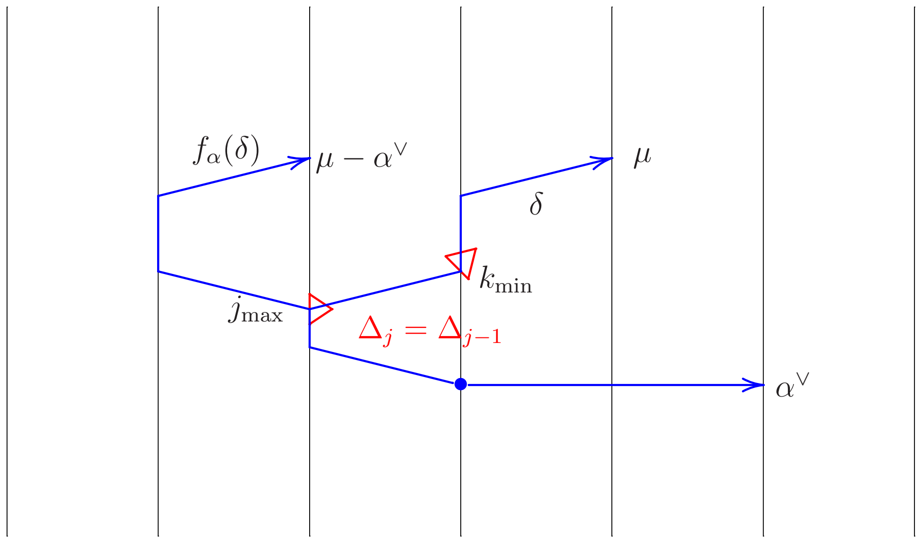

If $\gamma_\lambda \leftrightarrow w_\lambda \in W^a$ has minimal decomposition $w_\lambda = s_{i_1} s_{i_2} \cdots s_{i_r}$, then $\mathrm{typ}(\gamma_\lambda) = (i_1, i_2, \ldots, i_r)$ (this depends on the walls the minimal alcove gallery crosses).
Let $\Gamma(\gamma_\lambda)$ be the set of all galleries in $\mathbb{A}$ starting at 0 and having type $\mathrm{typ}(\gamma_\lambda)$.
Then we have a bijection

$$\begin{aligned} W \times \{1, s_{i_1}\} \times \cdots \times \{1, s_{i_r}\} &\xrightarrow{\simeq} \Gamma(\gamma_\lambda) \\ [\delta_0, \delta_1, \ldots, \delta_r] &\longmapsto (0 \subset \overline{\Delta}_0 \supset \Delta'_1 \subset \overline{\Delta}_1 \supset \cdots \Delta'_r \subset \overline{\Delta}_r \supset \mu) = \delta, \end{aligned}$$

where $\overline{\Delta}_i$ denotes an alcove (more precisely, $\overline{\Delta}_i = \delta_0 \delta_1 \cdots \delta_i(\Delta_f)$, where $\Delta_f$ is the fundamental alcove), $\Delta'_j$ is a face of codimension 1.
Note that $\mathrm{type}(\mu) = \mathrm{type}(\lambda)$ and $\pi(\delta) = \mu$ (end of the gallery).

Root operators

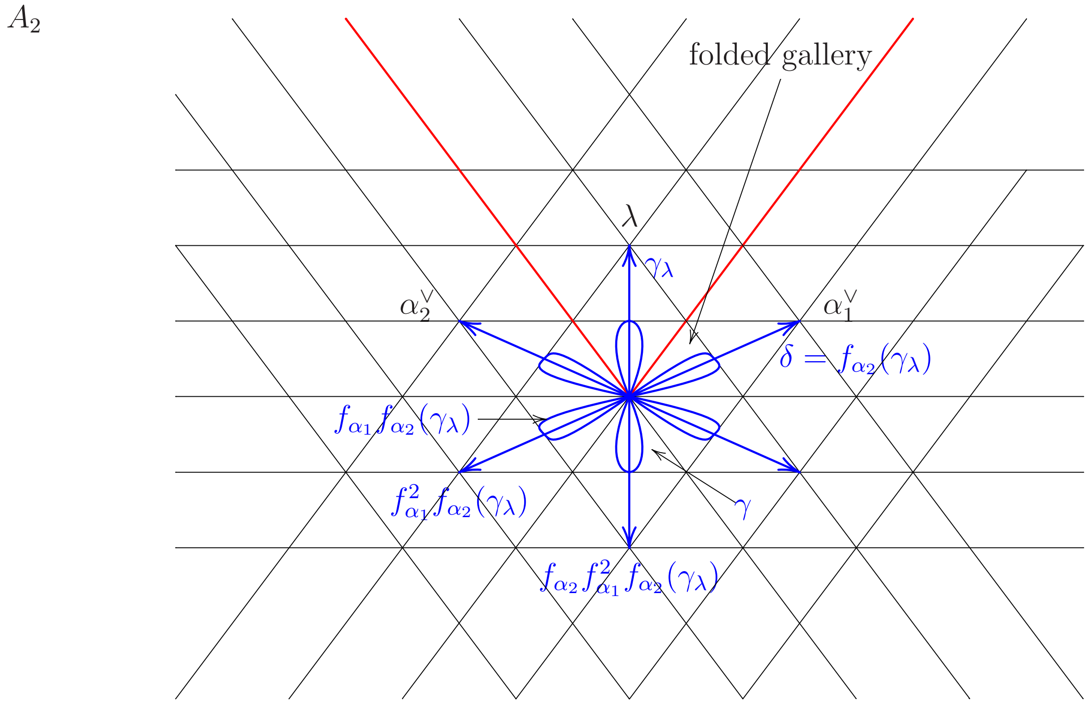
Let $\alpha$ be a simple root, $f_\alpha$, $e_\alpha$.




Let $H_{\alpha,m}$ with $m$ minimum such that there exists $j$ so that $\Delta'_j$ is in $H_{\alpha,m}$. Let $j_{\max}$ be the maximum of the indices $j$ such that $\Delta'_j$ is in $H_{\alpha,m}$. Let $k_{\min}$ be the minimum of the indices $k$ such that $\Delta'_k$ is in $H_{\alpha,m+1}$. Then $f_\alpha(\delta)$ is obtained from $\delta$ by reflecting the part of $\delta$ between $j_{\max}$ and $k_{\min}$ and translating the part after $k_{\min}$.
Define $e_\alpha$ in the same way. Then

$$\pi(f_\alpha\delta) = \mu - \alpha^\vee, \qquad \pi(e_\alpha\delta) = \mu + \alpha^\vee, \qquad f_\alpha(e_\alpha\delta) = e_\alpha(f_\alpha\delta) = \delta.$$

In the following example, $\lambda = \alpha_1^\vee + \alpha_2^\vee$ and we draw all the galleries in $\Gamma(\gamma_\lambda)$.




**Remarks.** See the definition of $\dim(\gamma)$ later.

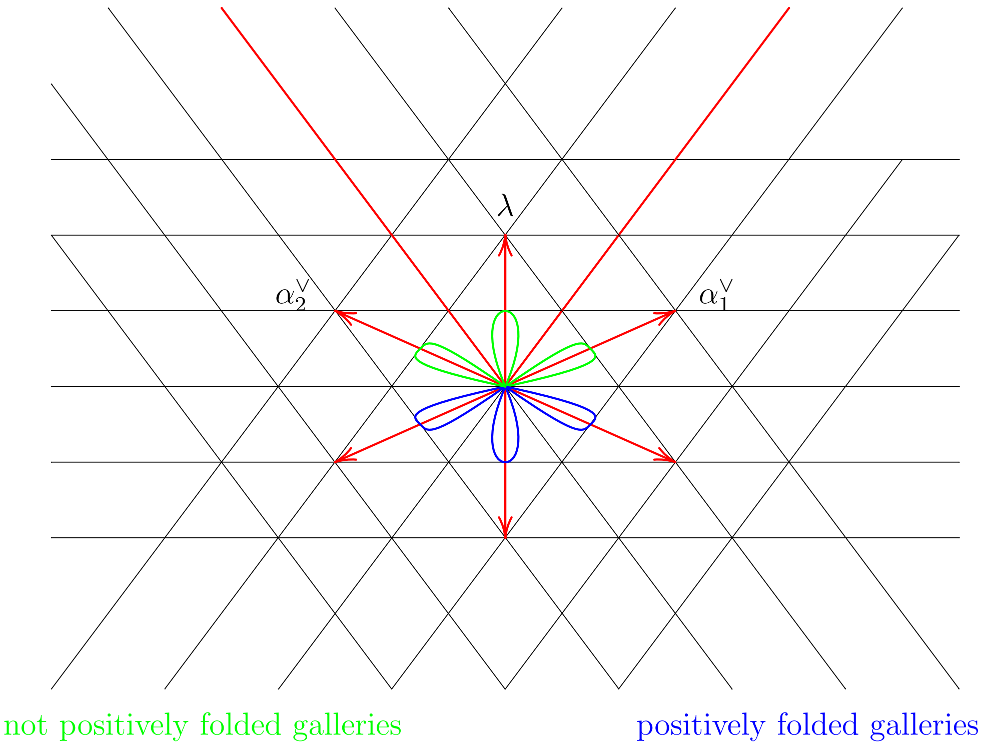
1. $\dim(\gamma) = 1$

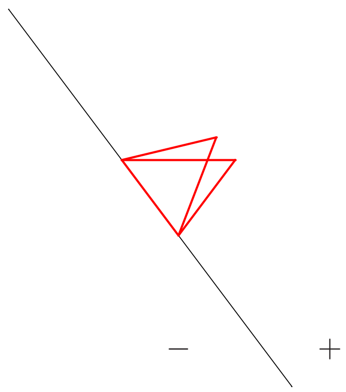
2. $\dim(\delta) = 3$

3. $\dim(f_{\alpha_1} f_{\alpha_2}(\gamma_\lambda)) = 2$

The operators preserve $\Gamma(\gamma_\lambda)$, $f_\alpha, e_\alpha : \Gamma(\gamma_\lambda) \to \Gamma(\gamma_\lambda)$.
Let $\Gamma^+(\gamma_\lambda) = \{\gamma \in \Gamma(\gamma_\lambda) \text{ positively folded}\}$.
A gallery $\delta$ is positively folded if $\delta_j = 1 \Rightarrow \overline{\Delta_{j-1}} = \overline{\Delta_j}$ is on the positive side of the wall at $j$.

$A_2$



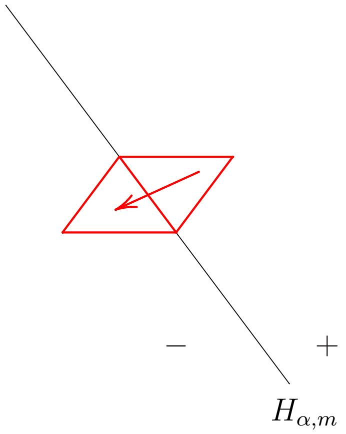

$$\overline{\Delta}_{j-1} = \overline{\Delta}_j \text{ in } \delta \rightsquigarrow \delta = [\delta_0, \delta_1, \ldots, \delta_{j-1}, 1, \delta_{j+1}, \ldots, \delta_r]$$

The operators preserve the set $\Gamma^+(\gamma_\lambda)$.

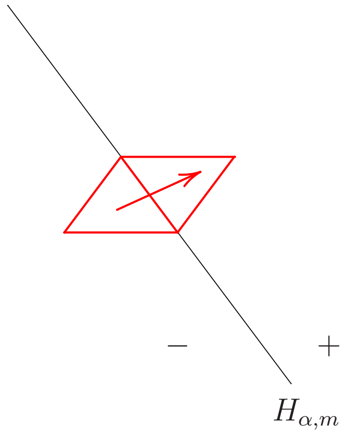
<u>Define LS galleries</u>

Let $\delta \in \Gamma(\gamma_\lambda)$.

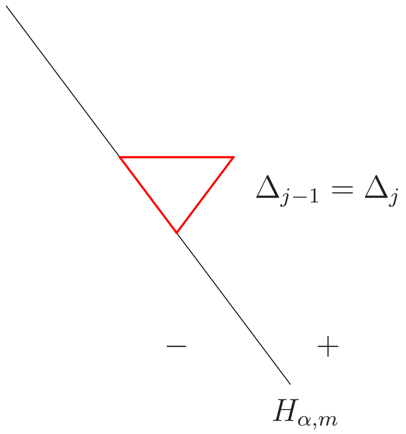
Then $\dim(\delta) = \#\{H_{\alpha,m} \text{ left positively by } \delta\}$, where $H_{\alpha,m}$ denotes an affine hyperplane.



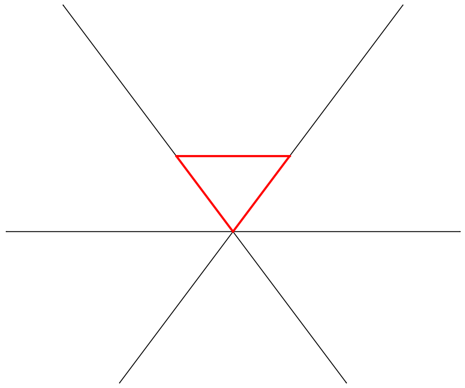

not counted in $\dim(\delta)$




counted in $\dim(\delta)$




counted in $\dim(\delta)$



Peculiar rule for the starting alcove:
one looks at several walls going through 0

**Proposition.** If $\delta \in \Gamma^+(\gamma_\lambda)$ and $\pi(\delta) = \mu$, then $\dim(\delta) \leq \rho(\lambda + \mu)$, where $\rho = (\sum_{\alpha>0} \alpha)/2$ is half the sum of the positive roots.

**Definition.** A gallery $\delta \in \Gamma^+(\gamma_\lambda)$ is LS if $\dim(\delta) = \rho(\lambda + \mu)$.

The operators preserve $\Gamma^+_{\text{LS}}(\gamma_\lambda)$.

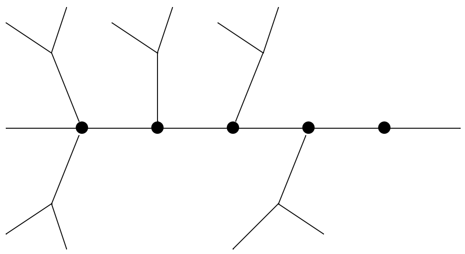
**Theorem.**

1. character formula:
$$\text{Char} V(\lambda) = \sum_\nu \sharp\Gamma^+_{\text{LS}}(\gamma_\lambda, \nu) e^\nu,$$
where $\Gamma^+_{\text{LS}}(\gamma_\lambda, \nu)$ is the set of all LS galleries ending in $\nu$.
2. The graph built on $\Gamma^+_{\text{LS}}(\gamma_\lambda)$ with the root operators is isomorphic to the crystal graph of $V(\lambda)$ of $G^\vee$.

**Remark.** If $\lambda$ is singular, there is still a well-defined alcove containing $\lambda$ in $C_f$ (the fundamental chamber) realizing the minimum distance to $\Delta_f$ and one can adapt the preceeding to get the same result (with P. Baumann).

<u>Geometry</u>
Let $G \supset B \supset T$ (be a borus as Jantzen is saying).
Then $B \supset U \supset T$, $U^- \subset B^-$, and $B^- \cap B = T$.
We have $\mathscr{B}^{\text{sph}}$ (spherical building), $\mathscr{B} = \mathscr{B}(G, K)$ (affine building), and $\mathscr{B}^\infty = \mathscr{B}(G(K), B(K))$ (spherical building at infinity).
Note that $\mathscr{B}(G, K) = G(K) \times \mathbb{A}/\sim$ for some equivalence relation $\sim$, and

$$\mathscr{B}(G, K) \cong \{\text{parahoric subgroups } P \supset I = ev_0^{-1}(B_{\mathbb{C}})\}.$$



Affine building of type $A_1$

<u>Affine Grassmanian</u>

$$\text{Gr} = G(K)/G(\mathcal{O}) = \coprod_{\lambda \in (P^\vee)^+} G(\mathcal{O}) t_\lambda G(\mathcal{O})/G(\mathcal{O}) = \coprod_{\mu \in P^\vee} U^\mp(K) t_\mu G(\mathcal{O})/G(\mathcal{O})$$

Mirković-Vilonen cycles: The intersections $U^-(K) \cdot t_\mu \cap G(\mathcal{O}) \cdot t_\lambda$ are of pure dimension $\rho(\lambda + \mu)$ (Mirković-Vilonen).

An MV cycle $Z_\mu^\lambda$ is the closure of an irreducible component in $U^-(K)\cdot t_\mu \cap G(\mathcal{O})\cdot t_\lambda$ in $\overline{\mathrm{Gr}^\lambda} = \overline{G(\mathcal{O})\cdot t_\lambda}$.

Suppose $\lambda$ is regular (again, one can manage the singular case as well), and let $\hat{\sum}(\gamma_\lambda)$ be the set of all galleries in the affine building $\mathscr{B}$ starting at 0 of type $\mathrm{typ}(\gamma_\lambda)$.
Then

$$\begin{tikzcd} \hat{\sum}(\gamma_\lambda) \arrow[r, "\pi"] & \overline{\mathrm{Gr}^\lambda} = \overline{G(\mathcal{O})\cdot t_\lambda} \\ G(\mathcal{O})\cdot\gamma_\lambda \arrow[u, hook] \arrow[r, "\simeq"] & G(\mathcal{O})\cdot t_\lambda \arrow[u, hook] \end{tikzcd}$$

Here the upper arrow is a Bott-Samelson resolution of singularities.

**Remarks.**

1. $G(\mathcal{O})\cdot\gamma_\lambda$ is the set of all minimal galleries.
2. $\hat{\sum}(\gamma_\lambda) = G(\mathcal{O}) \times_I P_{i_1} \times_I \cdots \times_I P_{i_r}/I$, where $P_{i_1} = I\{1, s_{i_1}\}I$.
3. $\pi([g_0, g_1, \ldots, g_r]) = g_0 g_1 \cdots g_r t_{\lambda_{\mathrm{fund}}}$, where $\lambda_{\mathrm{fund}} \in \Delta_f$ such that $\mathrm{typ}(\lambda_{\mathrm{fund}}) = \mathrm{typ}(\lambda)$

**Theorem.**
$$U^-(K)\cdot t_\mu \cap G(\mathcal{O})\cdot t_\lambda \cong \coprod_{\substack{\delta\in\Gamma^+(\gamma_\lambda),\\ \pi(\delta)=\mu}} C(\delta)\cap G(\mathcal{O})\cdot\gamma_\lambda,$$

where $C(\delta)$ is a cell of dimension $\dim(\delta)$ and therefore, each LS gallery gives an open of a unique MV cycle.