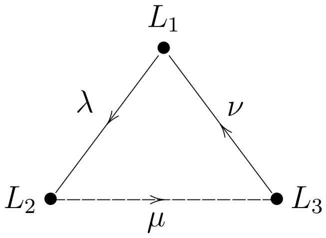
Mirkovic-Vilonen cycles and polytopes

Joel Kamnitzer (with J. Anderson)

Remark. All weights mentioned here are dominant.

For $\mu \in P^\vee$, let $t_\mu \in \mathrm{Gr} = G(K)/G(\mathcal{O})$.
Write $\mathrm{Gr}^\lambda = G(\mathcal{O})t_\lambda$.
If $L_1, L_2 \in \mathrm{Gr}$, then $\langle L_1, L_2\rangle = \mu \in P_+^\vee$.
If there is a $g \in G(K)$ such that $gL_1 = 1 = t_0$ and $gL_2 = t_\mu$, then the distance between L_1 and L_2 is μ.
Let

$$\begin{aligned}\mathrm{Gr}_{\lambda\mu\nu} &= \{(L_1, L_2, L_3) : L_1 = 1, \langle L_1, L_2\rangle = \lambda, \langle L_2, L_3\rangle = \mu, \langle L_3, L_1\rangle = \nu\} \\ &= \text{all triangles with side lengths } \lambda, \mu, \nu \text{ and special vertices.}\end{aligned}$$

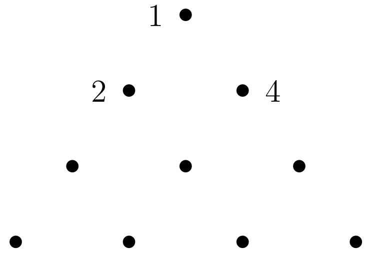


Theorem (Geometric Satake[1]). The number of components of $\mathrm{Gr}_{\lambda\mu\nu}$ of dimension $\langle \rho, \lambda + \mu + \nu\rangle$ is $\dim(V_\lambda \otimes V_\mu \otimes V_\nu)^{G^\vee}$.

Then

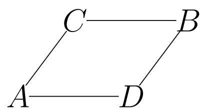
$$\begin{aligned}C_{\mathrm{Rep}} &= \{(\lambda, \mu, \nu) : \mathrm{Gr}_{\lambda\mu\nu} \text{ has a component of dimension } \langle \rho, \lambda + \mu + \nu\rangle\}, \\ C_{\mathrm{Hecke}} &= \{(\lambda, \mu, \nu) : \mathrm{Gr}_{\lambda\mu\nu} \neq \emptyset\}.\end{aligned}$$

Now $G = \mathrm{GL}_n = G^\vee$.
Hives



$$\Delta_n = \{(i, j, k) : i + j + k = n\}$$

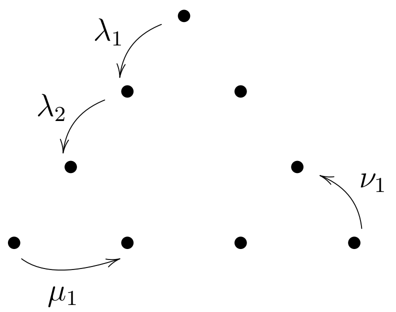
Fill with intgers; i.e., put a number at each vertex so that for all rhombi



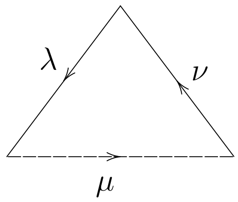
$C + D \geq A + B$ over $\mathbb{Z}$.

[1] See Haines' paper

A hive with boundary values λ, μ, ν:



$\xleftrightarrow{\text{bijection}}$ honeycombs



Theorem (Knutson-Tao, Woodward, J, Berenstein-Zelevinsky)**.** The number of hives with boundary values λ, μ, ν is $\dim(V_\lambda \otimes V_\mu \otimes V_\nu)^{\mathrm{GL}_n}$.

If (i, j, k) is such that $i + j + k = n$, consider

$$\wedge^i \mathbb{C}^n \otimes \wedge^j \mathbb{C}^n \otimes \wedge^k \mathbb{C}^n \supset \det .$$

Let ϵ_{ijk} be a basis vector for $\det \subset \wedge^i \otimes \wedge^j \otimes \wedge^k$ (ϵ_{ijk} is well-defined up to scalar).
Write $\mathrm{Gr} = G(\mathcal{O})\backslash G(K)$.
For (i, j, k) such that $i + j + k = n$, define $H_{ijk} : \mathrm{Gr}_{\lambda\mu\nu} \to \mathbb{Z}$ by

$$H_{ijk}([g_i], [g_j], [g_k]) = \mathrm{val}(g_1 \otimes g_2 \otimes g_3 \cdot \epsilon_{ijk}) \in \wedge^i K^n \otimes \wedge^j K^n \otimes \wedge^k K^n,$$

where $\mathrm{val} : K \to \mathbb{Z}$ is given by $t^i + \cdots \mapsto i$. If $\{e_1, \ldots, e_n\}$ is a basis for V and $v \in V \otimes K$ with $v = \sum a_i e_i$, where $a_i \in K$, then $\mathrm{val}(v) = \min_i \mathrm{val}(a_i)$.

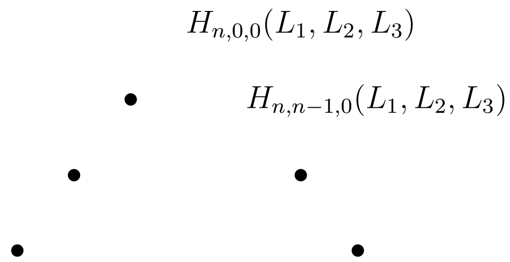
Remarks.

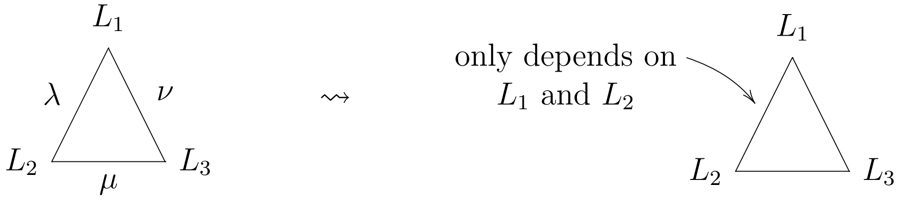
1. If $g \in \mathrm{GL}_V(\mathcal{O})$, then $\mathrm{val}(v) = \mathrm{val}(gv)$ for all $v \in V$, so the function H_{ijk} is independent of choice of representative.

2. $V \otimes K$ comes with a filtration $V \otimes K \supset V \otimes t^i\mathcal{O}$.

3. If V is a representation of G, then $V \otimes K$ is a representation of $G(K)$.

$H : \mathrm{Gr}_{\lambda\mu\nu} \to \{\text{fillings of } \Delta_n\}$
It's easy to show that the boundary values are always λ, μ, ν.

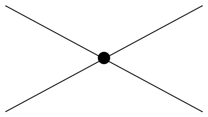




Theorem.

1. The function H is constant on an open part of each component of $\mathrm{Gr}_{\lambda\mu\nu}$.

2. For each component, this constant value is a hive.



   This gives a bijection between the top-dimensional components of $\mathrm{Gr}_{\lambda\mu\nu}$ and the hives with boundary λ, μ, ν.

General G:
Fix $\lambda, \mu, \nu \in P_+^\vee$.
Note that $\nu^\vee = -w_0\nu$.
Let $S^\mu = t_\mu \cdot U(K)$ and $T^\nu = t_\nu \cdot U_-(K)$.
Write kpf for the Kostant partition function.
Recall that

$$\mathrm{Gr}_{\lambda\mu\nu} = \{(L_1, L_2, L_3) : L_1 = 1, \langle L_1, L_2\rangle = \lambda, \langle L_2, L_3\rangle = \mu, \langle L_3, L_1\rangle = \nu\}.$$

Then by using $L_3 \rightsquigarrow t_{\nu^\vee}$,

$$\begin{aligned}&\#\text{ (top-dimensional) components of } \mathrm{Gr}_{\lambda\mu\nu}\\ &= \#\text{ (top-dimensional) components of } \{(1, L_2, t_{\nu^\vee}) : L_2 \in \mathrm{Gr}^\lambda, L_2^{-1}t_{\nu^\vee} \in \mathrm{Gr}^\mu\}.\end{aligned}$$

Since $L_2^{-1}t_{\nu^\vee} \in \mathrm{Gr}^\mu$ if and only if $L_2 \in t_{\nu^\vee}\mathrm{Gr}^{-\mu}$,

stable weight multiplicities	$\geq$	weight multiplicities	$\geq$	tensor product multiplicities
$\mathrm{kpf}(\lambda + \mu - \nu^\vee)$		$\dim\ V_\lambda(\nu^\vee - \mu)$		$\dim(V_\lambda \otimes V_\mu \otimes V_\nu)^{G^\vee}$
= # (top-dimensional) components of $T^{\lambda+\mu-\nu^\vee} \cap S^0$		= # (top-dimensional) components of $\mathrm{Gr}^\lambda \cap S^{\nu^\vee-\mu}$		= # (top-dimensional) components of $\mathrm{Gr}_{\lambda\mu\nu}$
= # (top-dimensional) components of $T^\lambda \cap t_{\nu^\vee}S^{-\mu}$	$\supset$	= # (top-dimensional) components of $\mathrm{Gr}^\lambda \cap t_{\nu^\vee}S^{-\mu}$	$\supset$	= # (top-dimensional) components of $\mathrm{Gr}^\lambda \cap t_{\nu^\vee}\mathrm{Gr}^{-\mu}$

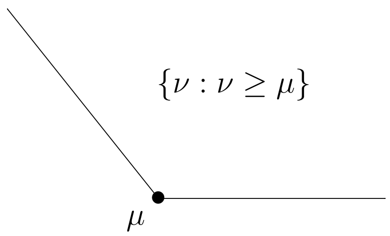
Remarks.

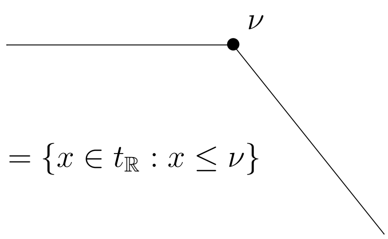
1. The containments are in terms of the top-dimensional components.

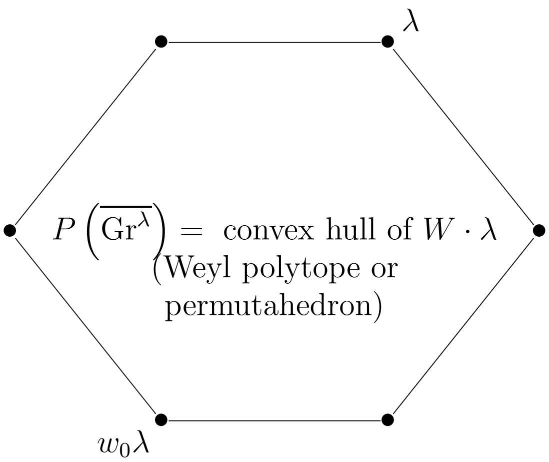
2. The second containment is obtained by taking the closure: it's not true that $\mathrm{Gr}^{-\lambda} \subset S^{-\lambda}$, but $\overline{\mathrm{Gr}^{-\lambda}} \subset \overline{S^{-\lambda}}$. To see that $\mathrm{Gr}^{-\lambda} \subset S^{-\lambda}$, but $\overline{\mathrm{Gr}^{-\lambda}} \subset \overline{S^{-\lambda}}$, consider $\mathbb{C}^\times \overset{\rho}{\subset} T \subset G$. For $s \in \mathbb{C}^\times$ and $L \in \mathrm{Gr}$, $s \cdot L = t_\nu$ for some $\nu \in P^\vee$ and
$$\lim_{\substack{s\to 0\\ s\in\mathbb{C}^\times}} s \cdot L = \text{ retraction of } L.$$
Then
$$S^\nu = \left\{L : \lim_{s\to 0} s \cdot L = t_\nu\right\}$$
and
$$\overline{S^\nu} = \left\{L : \lim_{s\to 0} s \cdot L = t_\mu \text{ for some } \mu \geq \nu\right\}.$$
On the other hand, if $L \in \overline{\mathrm{Gr}^{-\lambda}}$, then $s \cdot L = t_\mu$ and
$$\lim_{s\to 0} s \cdot L \in \overline{\mathrm{Gr}^{-\lambda}}.$$
Thus, $\mu \geq -\lambda$.

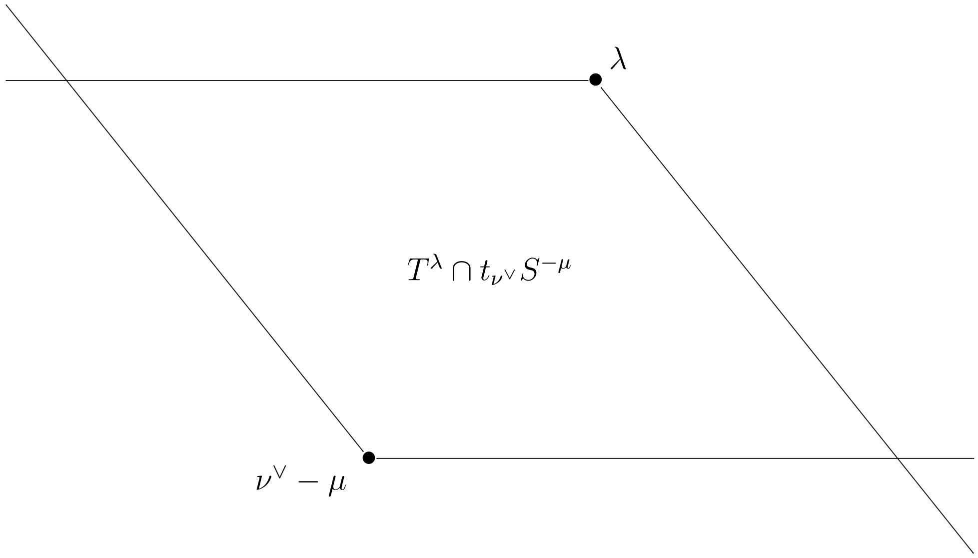
It follows that if we want to describe the top-dimensional components, we can just describe the top-dimensional components in $T^\lambda \cap t_{\nu^\vee} S^{-\mu}$ and then specialize.
Note that $V_\lambda(\mu) = \mathrm{kpf}(\lambda - \mu)$ when $\lambda - \mu \ll \lambda, \mu$.
If $S \subset \mathrm{Gr}$ is a T-invariant closed subset, then the polytope of S, denoted $P(S)$, is
$$\begin{aligned} P(S) &= \text{ convex hull of } \{\mu : t_\mu \in S\} \subseteq t_{\mathbb{R}} := \mathbb{R} \otimes P^\vee \\ &= \text{image of the moment map} \end{aligned}$$
and
$$P\left(\overline{S^\mu}\right) = \{x \in t_{\mathbb{R}} : x \geq \mu\},$$
where $x \geq \mu$ means that $x - \mu$ is a positive linear combination of positive coroots.

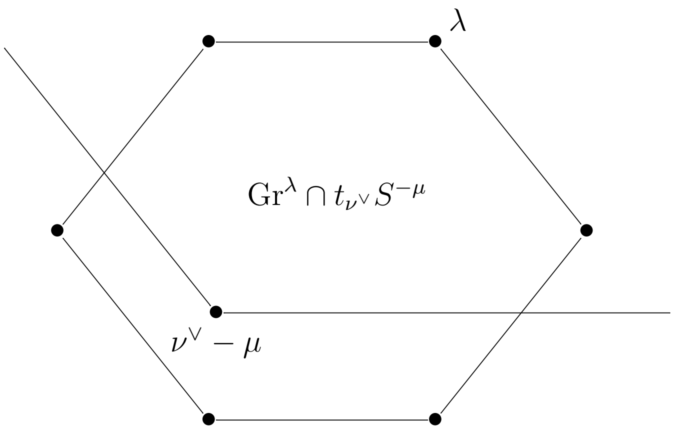
Example. A_2

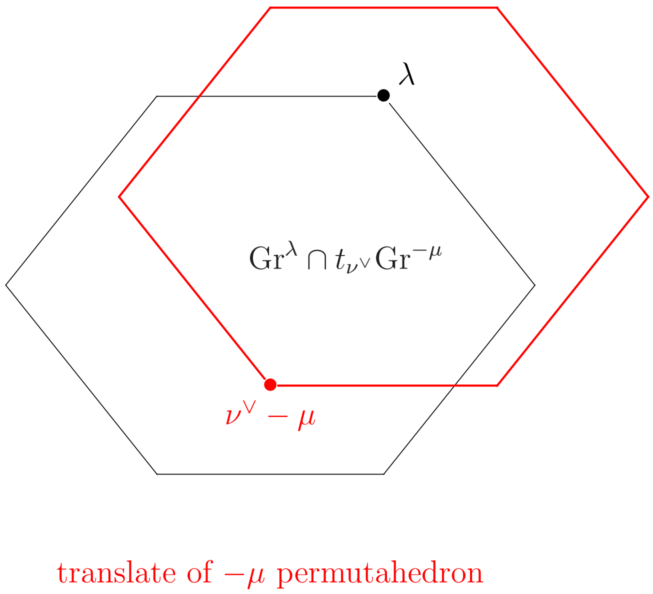







λ
$T^\lambda \cap t_{\nu^\vee} S^{-\mu}$
$\nu^\vee - \mu$


λ
$\mathrm{Gr}^\lambda \cap t_{\nu^\vee} S^{-\mu}$
$\nu^\vee - \mu$


λ
$\mathrm{Gr}^\lambda \cap t_{\nu^\vee} \mathrm{Gr}^{-\mu}$
$\nu^\vee - \mu$
translate of $-\mu$ permutahedron

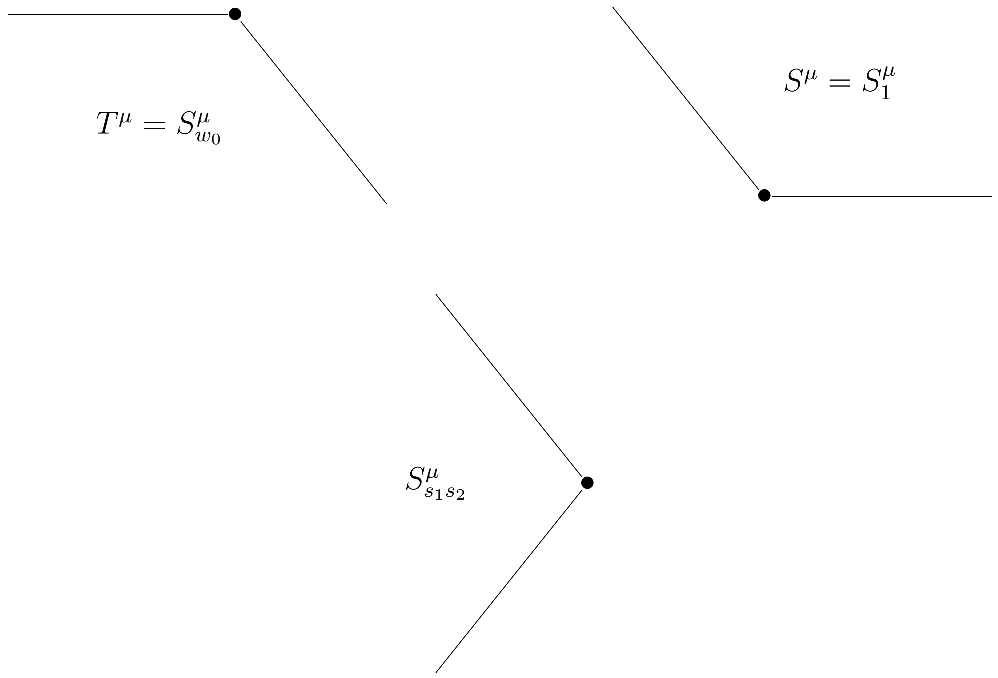
Theorem.

$$\left\{\begin{array}{c}\text{top-dimensional}\\ \text{components of}\\ T^\lambda \cap S^\mu\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{polytopes with}\\ \text{highest vertex } \lambda,\\ \text{lowest vertex } \mu\end{array}\right\}$$
$$X \longmapsto P(X)$$

This map is injective. Moreover, $X \subset \overline{\mathrm{Gr}^\lambda}$ if and only if $P(X) \subset P\left(\overline{\mathrm{Gr}^\lambda}\right)$.

Remark. The $P(X)$ in the last theorem are *MV polytopes*.

Let
$$S^\mu = U(K)\cdot t^\mu, \qquad T^\mu = U_-(K)\cdot t^\mu, \qquad S^\mu_w = (wUw^{-1})(K)\cdot t^\mu.$$

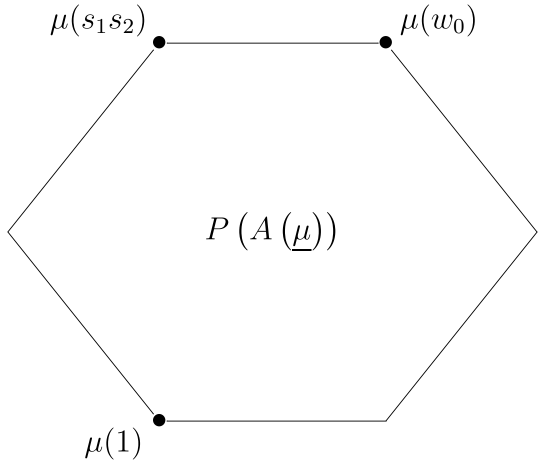


Then
$$P\left(S^\mu_w\right) = \{x \in t_{\mathbb{R}} : w^{-1}(x-\mu) \geq 0\}.$$

If $\underline{\mu} : W \to P^\vee$, then
$$A\left(\underline{\mu}\right) = \bigcap_{w \in W} S_w^{\underline{\mu}(w)}.$$

Lemma. If $A\left(\underline{\mu}\right) \neq \emptyset$, then
$$P\left(A\left(\underline{\mu}\right)\right) = \{x \in t_{\mathbb{R}} : w^{-1}(x-\mu) \geq 0 \text{ for all } w\} = \bigcap_{w \in W} \text{cones}.$$

Remark. The set $\{x \in t_{\mathbb{R}} : w^{-1}(x-\mu) \geq 0 \text{ for all } w\}$ is the convex hull of $\underline{\mu}(w)$.

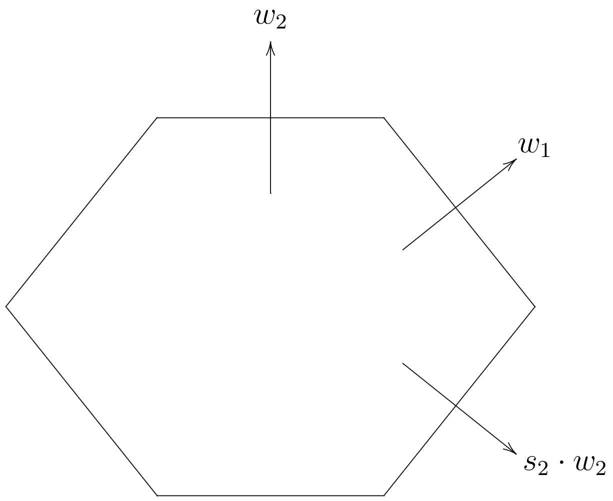


For some good choices of $\underline{\mu}$, $\overline{A\left(\underline{\mu}\right)}$ is a top-dimensional component of $S_1^{\mu(1)} \cap S_{w_0}^{\mu(w_0)}$, and all the top-dimensional components arise in this way.

A pseudo Weyl-polytope/generalized permutahedron is

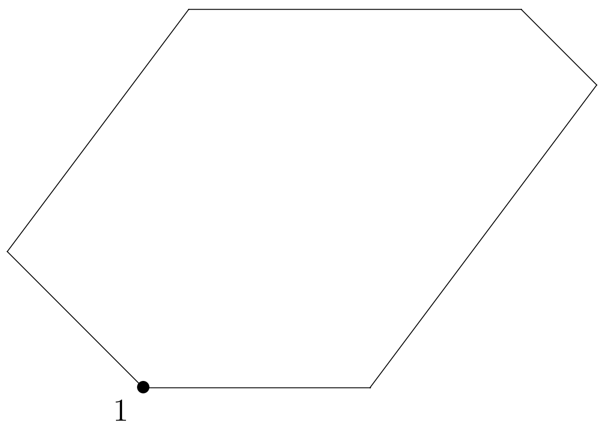
$$P(M.) = \{x \in t_{\mathbb{R}} : \langle x, \gamma \rangle \leq M_\gamma \text{ for all } \gamma \in \Gamma\}$$

for some collection $\{M_\gamma\}_{\gamma \in \Gamma}$, where the $M_\gamma \in \mathbb{Z}$.
The walls of the permutahedron are perpendicular to $\Gamma = \cup W \cdot w_i$.



Then

$$P\left(\overline{\mathrm{Gr}^\lambda}\right) = \text{ convex hull of } W \cdot \lambda = \{x \in t_{\mathbb{R}} : \langle x, w \cdot w_i \rangle \leq \langle \lambda, w_i \rangle\}.$$



If $P(M.)$ is a permutahedron, then $\underline{\mu} : W \to P^\vee$ (the vertices).
For each $\gamma \in \Gamma$,

$$\begin{aligned} D_\gamma : \mathrm{Gr} &\longrightarrow \mathbb{Z} \\ [g] &\longmapsto \mathrm{val}(g \cdot v_\gamma) = \min_\delta \ \mathrm{val}\ \Delta_{\gamma\delta}(g), \end{aligned}$$

where the expression to the right of the equal sign is in Berenstein's notation.
If $\gamma = w \cdot \omega_i$, then $v_\gamma \in V_{\omega_i}$ is of weight γ.
If $G = \mathrm{SL}_n$, $\gamma = (0, 0, 1, 0, \dots, 0)$, and $\omega_1 = (1, 0, \dots, 0)$ (note that the corresponding representation is the standard representation), then $D_\gamma(g)$ is the minimum valuation of the entries in the third column of g.

Lemma. If $\underline{\mu}$ are the vertices of a generalized permutahedron $P(M.)$, then

$$A\left(\underline{\mu}\right) = \{L \in \mathrm{Gr} : D_\gamma(L) = M_\gamma\} =: A(M.).$$

and $P\left(\overline{A\left(\underline{\mu}\right)}\right) = P(M.)$ if it's non-empty.

Remark. Showing that $A\left(\underline{\mu}\right) = A(M.)$ is easy.

We use the fact that $\left\langle \underline{\mu}(w), w \cdot \omega_i \right\rangle = M_{w \cdot \omega_i}$.

Let $G = \mathrm{SL}_3$, and let $g \in \mathrm{SL}_3(K)$.
Then "most of the time" (since we are talking in terms of representations)

$$D_1(g) = \mathrm{val}\ \Delta_{1,1}(g), \qquad D_2(g) = \mathrm{val}\ \Delta_{1,2}(g), \qquad \Delta_{12}(g) = \mathrm{val}\ \Delta_{13,12}(g),$$

and we have the Plücker relations

$$\Delta_{1,1}\Delta_{12,23} + \Delta_{3,3}\Delta_{12,12} = \Delta_{2,2}\Delta_{12,13}.$$

Remark. The Pl ucker relations have been generalized to other root systems by Berenstein-Fomin-Zelevinsky.

We get

$$\mathrm{val}(ab) = \mathrm{val}(a) + \mathrm{val}(b), \qquad \mathrm{val}(a + b) = \min\{\mathrm{val}(a), \mathrm{val}(b)\},$$

where the equality in the expression involving $\mathrm{val}(a + b)$ holds most of the time.
A collection $\{M_\gamma\}_{\gamma \in \Gamma}$ is called a BZ datum if they satisfy the tropical Plücker relations, examples of which are

$$\min\{D_1(L) + D_{23}(L), \underline{\qquad}\} = \underline{\qquad}\,, \qquad \min\{M_1 + M_{23}, M_3 + M_{12}\} = M_2 + M_{13}.$$

Theorem. If $\{M_\gamma\}$ is a BZ datum, then $A(M.)$ is an MV cycle, and these are all the MV cycles. Hence, all the MV polytopes have the form $P(M.)$.