Saturation

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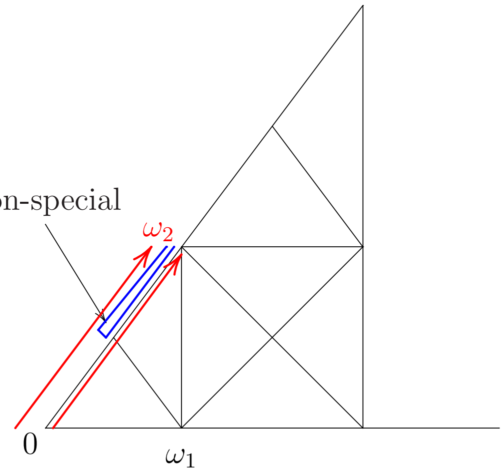
Recall the geometric interpretation of the saturation theorem for GL_n:
Suppose that $\lambda, \mu, \nu \in P_+^\vee$ are coweights such that $\lambda + \mu + \nu \in Q^\vee$ and there exists a triangle in the Euclidean building X for $G(\mathbb{Q}_p) = \mathrm{GL}_n(\mathbb{Q}_p)$ with side-lengths λ, μ, ν. Then

$$(V(\lambda) \otimes V(\mu) \otimes V(\nu))^{G^\vee(\mathbb{C})} \neq 0.$$

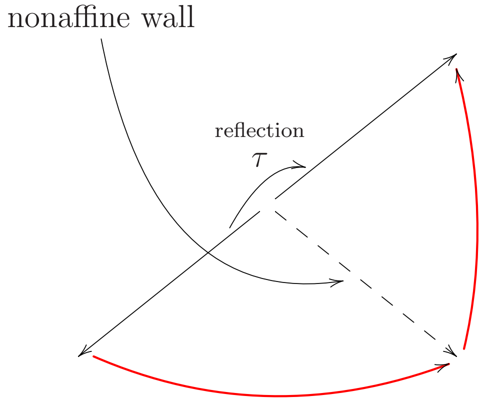
Here $G^\vee(\mathbb{C}) = \mathrm{GL}_n(\mathbb{C})$.

Below is an example of failure of the saturation theorem for $\mathrm{Sp}(4)$.
Take $\lambda = \mu = \nu = \varpi_2$, the longer fundamental coroot. We can form a triangle in the tree embedded in X, so that the side-lengths are λ, μ, ν. Project this triangle to the positive chamber $\Delta = P_+^\vee \otimes \mathbb{R}_+$. The result is a folded triangle with two geodesic side-lengths (drawn in red) and one positively folded path p (in blue). However the path p is not an LS path: the maximality condition fails. Indeed, by using a non-affine wall through the break-point $p(1/2)$ of p, one obtains a longer Bruhat chain from $p'(1/2-)$ to $p'(1/2+)$.

$B_2 \cong C_2$



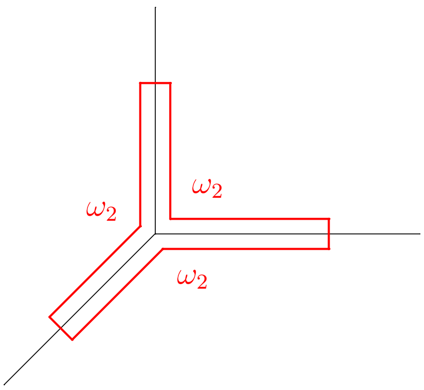
positively folded, but maximality condition fails, so not an LS path

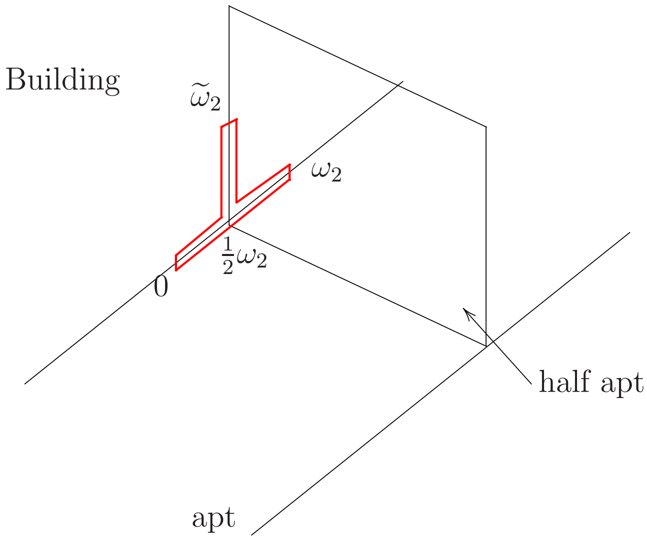


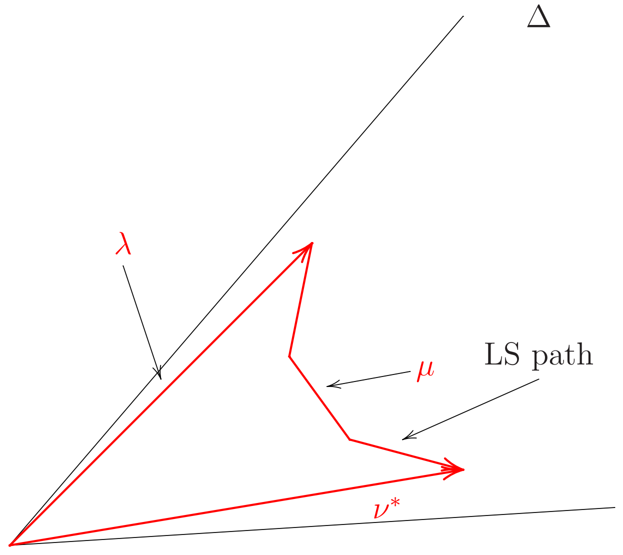
longer Bruhat chain

highest weights
$(\lambda, \mu, \nu) = (\omega_2, \omega_2, \omega_2)$

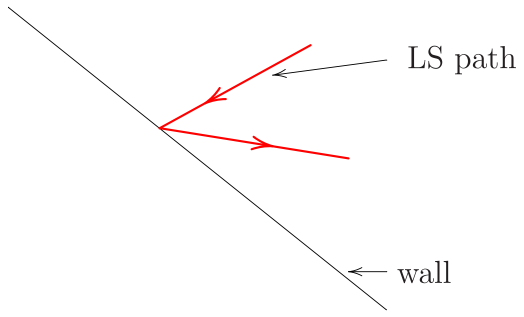
$c_{\lambda\mu\nu} = 0$







$c_{\lambda\mu\nu} \neq 0$ $\qquad\qquad$ $\nu^* = -w_0(\nu) = \nu$

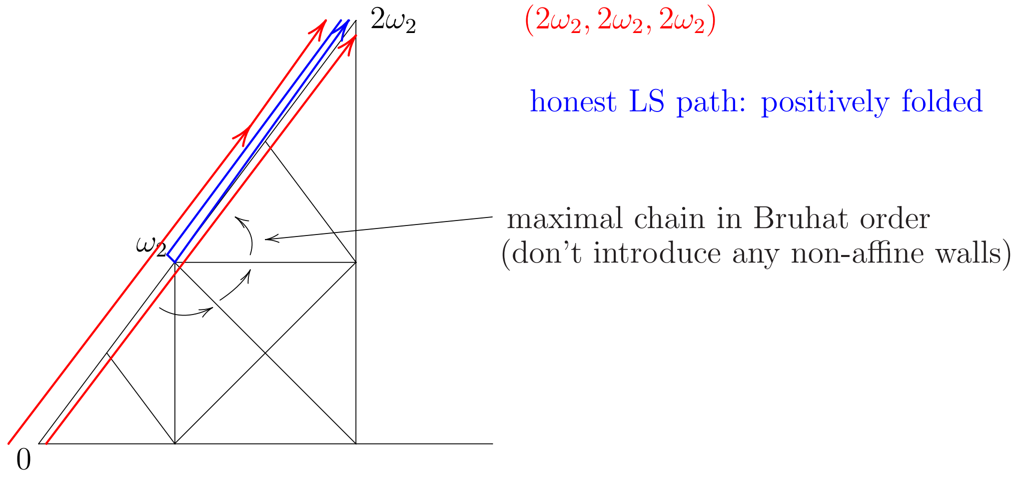


One observes (by inspection) that the path p is the only positively folded path (in Δ) of length μ connecting ϖ_2 to itself. Thus,

$$c_{\lambda\mu\nu} = 0.$$

However, by scaling the side-lengths by 2 one obtains an honest LS path (in Δ) of length $2\varpi_2$ connecting $2\varpi_2$ to itself. The corresponding folded triangle is obtained by taking the original one and scaling it by 2. Therefore

$$c_{2\lambda,2\mu,2\nu} \neq 0.$$



ω_2 vertex special $\qquad (2\lambda, 2\mu, 2\nu) \in C_{\text{Rep}} \qquad c_{2\lambda,2\mu,2\nu} \neq 0$

The goal of this talk is to generalize this argument to other groups (root systems).

How do we generalize the number 2 (the multiplication factor)?
Given any root system R, there is a k_R such that for all vertices v in an apartment, $k_R \cdot v$ is a special vertex. Recall that a vertex is special if and only if it lies in the weight lattice $P^\vee$.
The number k_R can be computed from the Dynkin diagram: If

$$\theta = \sum_{i=1}^{\ell} m_i \alpha_i,$$

then

$$k_R = \mathrm{LCM}(m_1, \ldots, m_\ell).$$

Now, given $G^\vee$, compute its root system R and the factor k_R.

Theorem (Kapovich, Millson)**.** For each triple $(\lambda, \mu, \nu) \in (P_+^\vee)^3$ such that $\lambda + \mu + \nu \in Q^\vee$, if there is an $N \in \mathbb{N} = \mathbb{Z}^+$ such that $c_{N\lambda,N\mu,N\nu} \neq 0$, then $c_{k_R^2\lambda,k_R^2\mu,k_R^2\nu} \neq 0$.

Remark. If $R = A_\ell$, then $k_R = 1$, and we obtain saturation "on the nose."

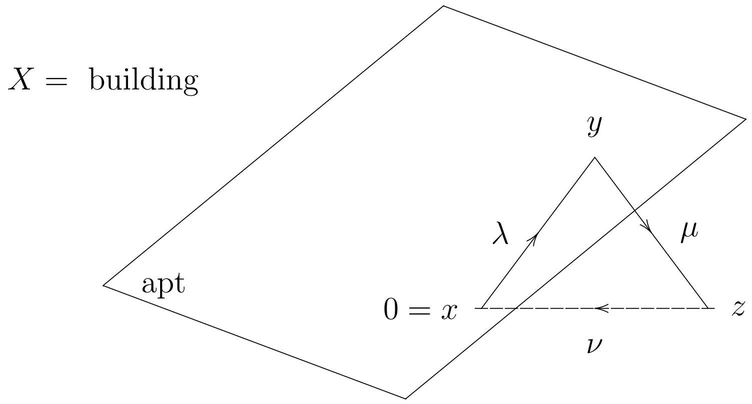
Conjecture (Kapovich, Millson)**.** We can drop the squares.

The proof of the above theorem breaks into two parts:

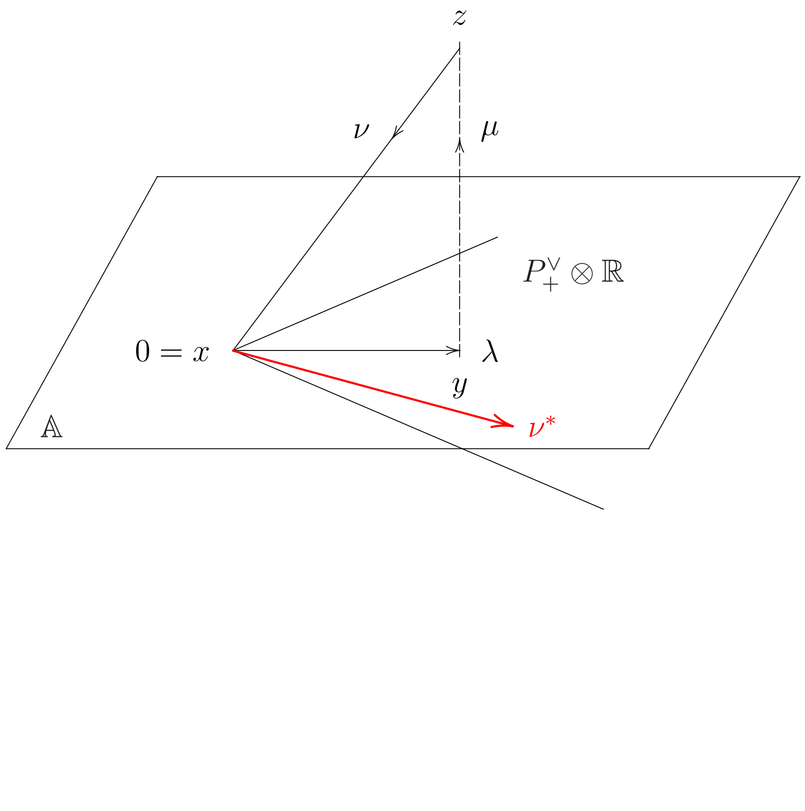
1. analysis (responsible for the first stretching of λ, μ, ν by the factor of k_R)
2. combinatorics (responsible for the second stretching by the factor of k_R)

Consider the combinatorics part of the proof; i.e., assume that we have already stretched by a factor of k_R and have a triangle in the Euclidean building X with special vertices.
Suppose we are given a triangle $T = T(x, y, z)$ in the Euclidean (Bruhat-Tits) building corresponding to $G(\mathbb{K})$ ($\mathbb{K} = \mathbb{Q}_p$ or $\mathbb{K} = \mathbb{C}((z))$) such that the vertices of Δ are *special* and have side-lengths $\vec{\lambda}, \vec{\mu}, \vec{\nu} \in P^\vee$ with $\vec{\lambda} + \vec{\mu} + \vec{\nu} \in Q^\vee$.
We want to show that $c_{k_R\lambda,k_R\mu,k_R\nu} \neq 0$.

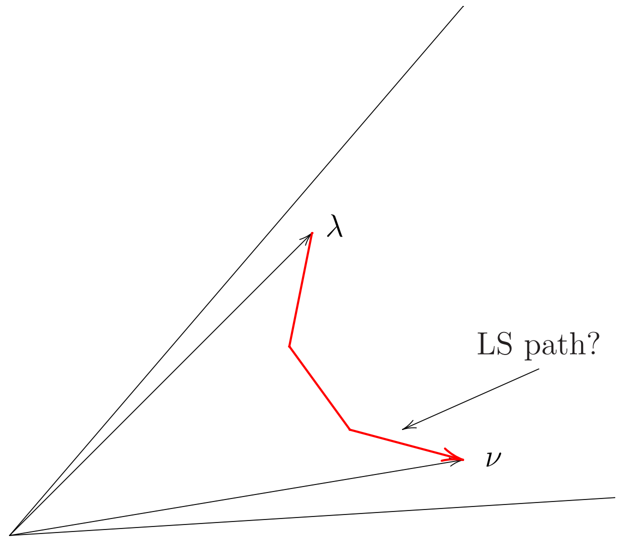
First consider the special case $\vec{\mu} \in \mathbb{N}\omega_i$, where ω_i is a fundamental co-weight. Choose the apartment $\mathbb{A} \subset X$ to contain the segment $\overline{xy}$. Since x is a special vertex, we can identify it with the origin 0 in $\mathbb{A}$. Pick $\Delta \subset \mathbb{A}$ containing $\overline{xy}$.



$\overline{xy} \subset$ apt $\mathbb{A}$

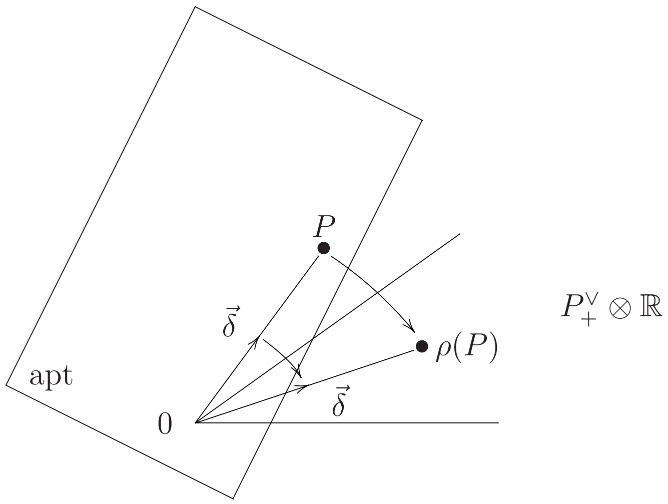


Recall that we want



positively folded path

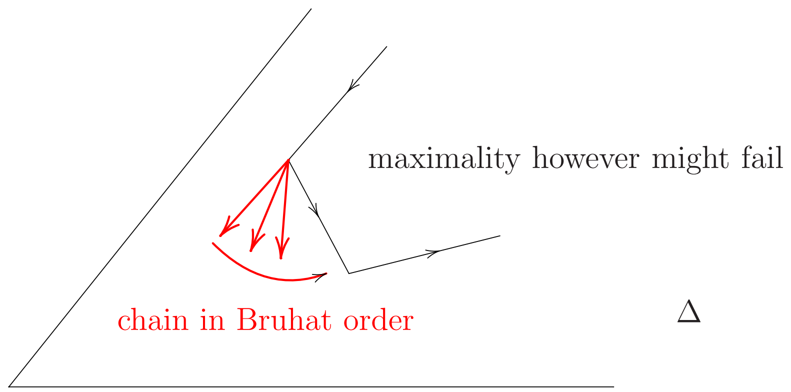
Retract X to the positive chamber $P_+^\vee \otimes \mathbb{R}$. Namely, given $P \in X$ with $\vec{d}(0, P) = \vec{\delta}$, send P to the point $\rho(P) = \vec{\delta} \in \Delta$.



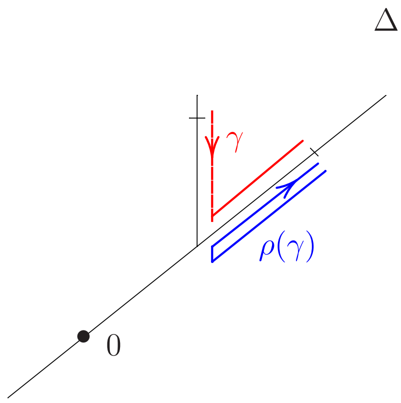
$$\rho : X \to P_+^\vee \otimes \mathbb{R} = \Delta \text{ is a simplicial map}$$

This retraction is similar to the one used by Gaussent-Littelmann (but they retract from "$-\infty$"; i.e., they move the point 0 to $-\infty$).

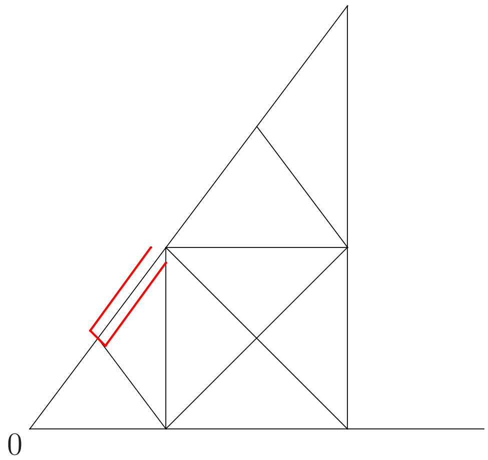
Theorem. For any geodesic segment $\gamma \in X$, $\rho(\gamma)$ is a positively folded path (i.e., a Hecke path) in Δ.



Example. X is a tree



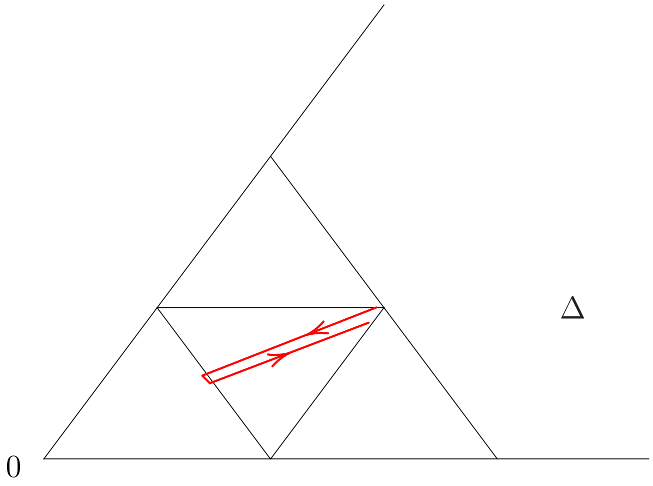
Our assumption that $\vec{\mu} \in \mathbb{N}\omega_i$, where ω_i is a fundamental co-weight implies that $\pi := \rho(\gamma)$ travels along the *edges* of $\mathbb{A}$.



Thus, all the breaks of π occur at vertices in $\mathbb{A}$.
After stretching π by k_R, all the breaks of π occur at *special* vertices in $\mathbb{A}$.
Therefore $k_R \cdot \pi$ satisfies maximality at every break point. Hence, $k_R \cdot \pi$ is an LS path.

For a general Hecke path π, of course, $k_R \cdot \pi$ is not an LS path.

Example. In the case of A_2, $k_R = 1$, but there are Hecke paths that are not LS paths.

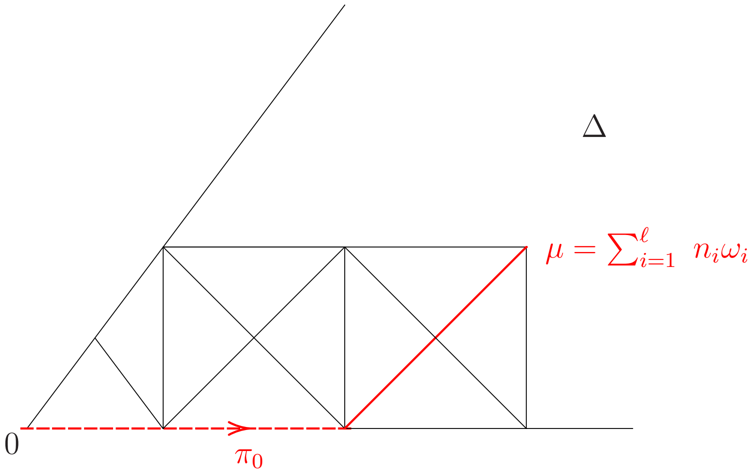


Hecke but not LS

How do we get rid of our assumption that $\vec{\mu} \in \mathbb{N}\omega_i$, where ω_i is a fundamental co-weight? We need to modify our path model. Rather than starting with the geodesic path connecting 0 to μ, we will take a piecewise-linear path π_0 connecting 0 to μ. If

$$\mu = \sum_{i=1}^{\ell} n_i \omega_i,$$

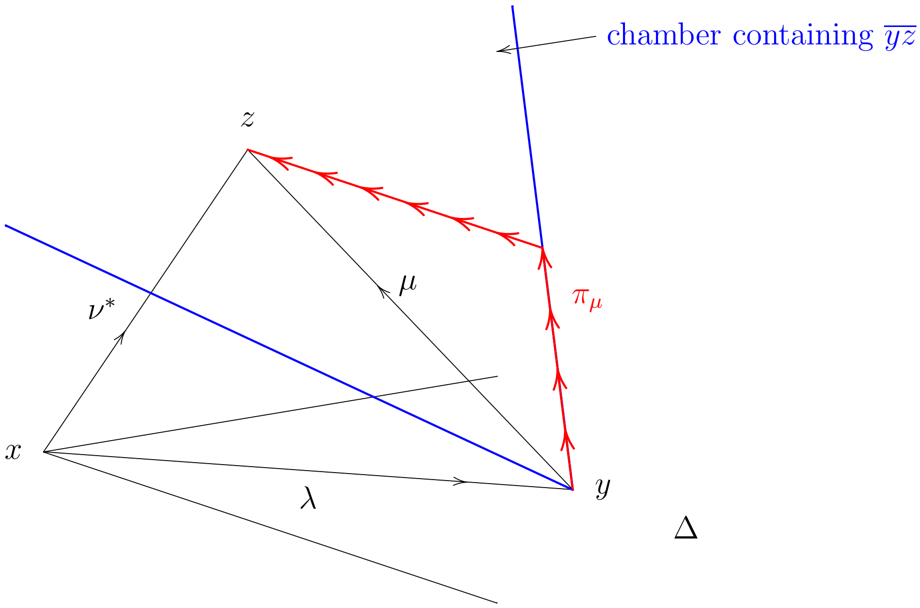
we first travel along $n_1 \varpi_1$, then parallel to $n_2 \varpi_2$, etc.



Apply the root operators e_α, f_α to π_0 to get a family $\mathscr{P}_{\pi_0}$ of paths in $\mathbb{A}$ such that all segments are edges of $\mathbb{A}$. Then

Theorem (Littelmann)**.** $c_{\lambda\mu\nu}$ equals the number of paths $\pi \in \mathscr{P}_{\pi_0}$ contained in Δ and connecting λ to ν^*.

Note that the Retraction Theorem still works.



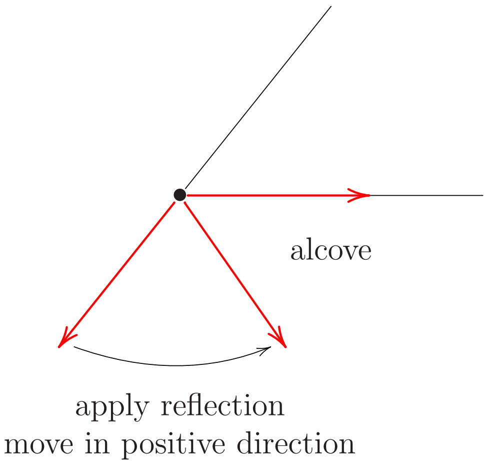
Theorem. The path $\rho(\pi_\mu)$ belongs to $\mathscr{P}^+_{\pi_0}$, the set of all positively folded paths of type π_μ.

But $\rho(\pi_\mu)$ can fail the maximality condition. Note that $\rho(\pi_\mu)$ has breaks only at vertices on $\mathbb{A}$.

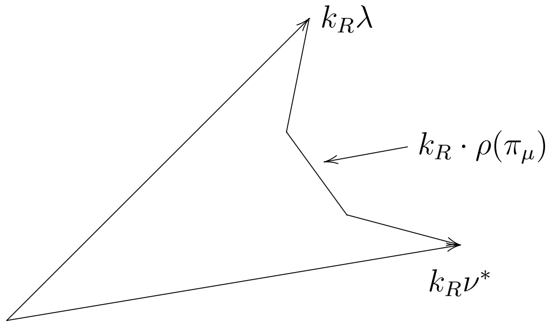
Stretch $\rho(\pi_\mu)$ by k_R. Then $k_R \cdot \rho(\pi_\mu)$ has breaks only at special vertices.

It follows that $k_R \cdot \rho(\pi_\mu)$ satisfies the maximality condition and is still positively folded, and $k_R \cdot \rho(\pi_\mu) \in \mathscr{P}_{\pi_{k_R \cdot \mu}}$.

Remark. Positively-folded condition:



Thus, $c_{k_R\lambda, k_R\mu, k_R\nu} \neq 0$.



This concludes the proof of the combinatorial part of the proof.

Sketch of the analytical part of the proof

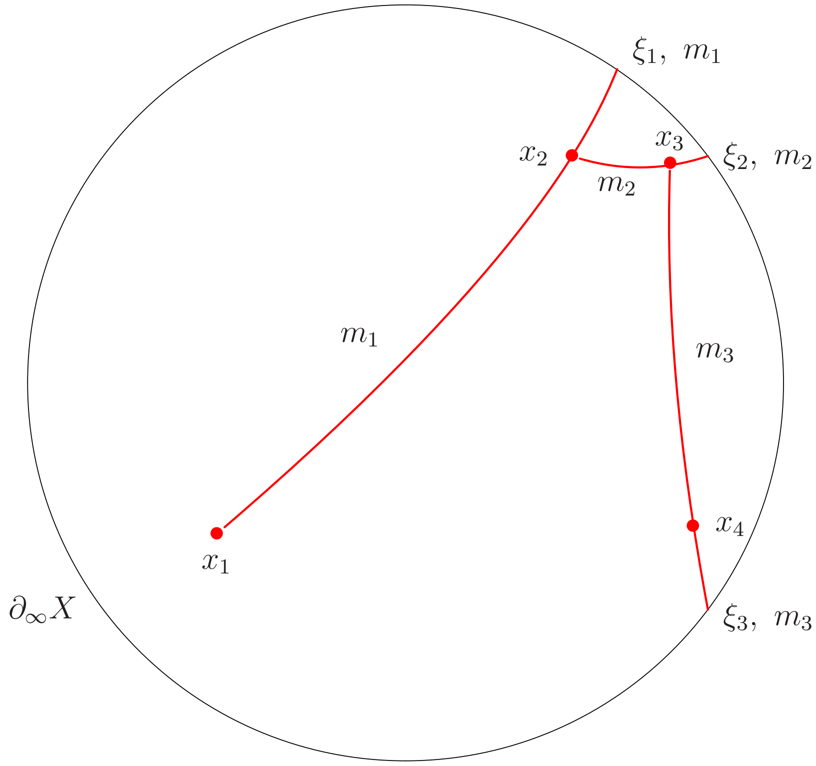
I am going to describe only some key ingredients of this part. Let X be a (nonpositively curved) symmetric space or a Euclidean building. Then $\partial_\infty X$ is its ideal boundary.
Consider a triangle $T(x_1, x_2, x_3) \subset X$ with the (usual) side-lengths $m_1, m_2, m_3 \in \mathbb{R}_+$. Define a Gauss map

$$T = T(x_1, x_2, x_3) \mapsto (m_1\xi_1, m_2\xi_2, m_3\xi_3)$$

that sends T to a weighted configuration on $\partial_\infty X$. Namely, take the first side $\overline{x_1 x_2}$ of T and extend it all way to infinity to get a point ξ_1. Place the mass m_1 at ξ_1. Repeat the same for the two other sides.

Question. How do we invert the Gauss map?

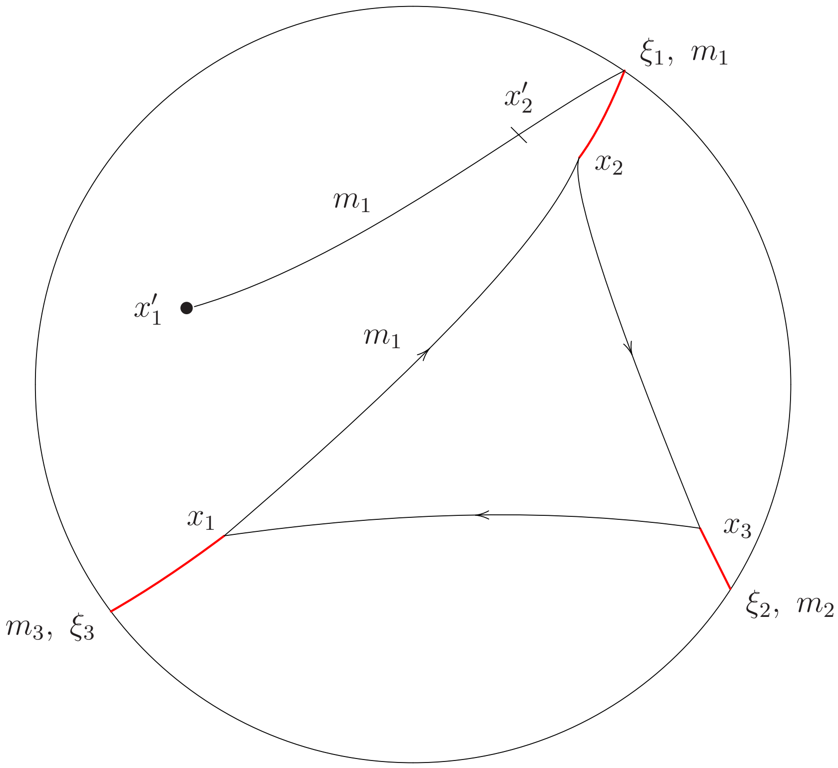
Given three points $\xi_1, \xi_2, \xi_3 \in \partial_\infty X$ and masses $m_1, m_2, m_3 \in \mathbb{R}_+$ placed at these points, we define the following map $\Phi : X \to X$, sending x_1 to x_4.
If $\Phi(x_1) = x_1$, we obtain a triangle in X and therefore we inverted the Gauss map. In general, of course, $x_1 \neq x_4$.



When does Φ have a fixed point? Note that $\Phi : X \to X$ is a weak contraction:

$$d(\Phi(x), \Phi(y)) \leq d(x, y), \forall x, y \in X.$$

To see this, note that Φ is the composition of three weak contractions:



In order to get a fixed point we need a bounded orbit for the system

$$(\Phi^n(x_1))_{n\in\mathbb{N}}\,.$$

It turns out that we have a bounded orbit if and only if the weighted configuration $(m_i\xi_i)$ is semistable in an appropriate sense. This definition coincides with Mumford's notion of semistability provided that $m_i \in \mathbb{N}$ and all ξ_i's belong to the same flag-manifold, i.e., G-orbit in $\partial_\infty X$.
For a vector $\xi^0 \in \partial_\infty \Delta$ and a number $m \geq 0$, we will identify $m\xi^0$ with a vector in Δ, whose (usual) length equals m. Then, each weighted configuration $(m_1\xi_1, m_2\xi_2, m_3\xi_3)$ on $\partial_\infty X$ yields a triple of vectors in Δ:

$$(\lambda, \mu, \nu) = (m_1\xi_1^0, m_2\xi_2^0, m_3\xi_3^0),$$

where $\xi_i^0 \in \partial_\infty \Delta$ is the point of type ξ_i.

Question. Given a triple of vectors λ, μ, ν in Δ, when can we find a semistable configuration in $\partial_\infty X$ of type (λ, μ, ν)?

Theorem (Kapovich, Leeb, Millson)**.** There exists a system of linear homogeneous inequalities on the vectors λ, μ, ν in Δ^3 which give necessary and sufficient conditions for the existence of a weighted semistable configuration

$$(m_1\xi_1, m_2\xi_2, m_3\xi_3)$$

on $\partial_\infty X$ of type (λ, μ, ν).

Now we can give the actual sketch:

0. Suppose that $c_{N\lambda,N\mu,N\nu} \neq 0$ and $\lambda + \mu + \nu \in Q^\vee$.
1. The Satake inverse then gives a triangle T_N in X with the Δ-side-lengths $N\lambda, N\mu, N\nu$.
2. Apply the Gauss map to this triangle. The result is a semistable weighted configuration

   $$(Nm_1\xi_1, Nm_2\xi_2, Nm_3\xi_3)$$

   on $\partial_\infty X$ of type $(N\lambda, N\mu, N\nu)$.
3. Then

   $$(m_1\xi_1, m_2\xi_2, m_3\xi_3)$$

   is again semistable.
4. Hence, the corresponding map Φ has a fixed point x_1 in X.
5. Form the associated triangle $T = T(x_1, x_2, x_3)$. Then the Δ-side lengths of T are the vectors λ, μ, ν.

6. The condition $\lambda + \mu + \nu \in Q^\vee$ implies that we can choose T so that its vertices are vertices of X. However T does not necessarily have special vertices. One can "stretch" the triangle T by the factor of k_R so that the resulting triangle has special vertices.

   How do we stretch? Retract T to $\rho(T) \subset \Delta$. Then stretch $\rho(T)$ by k_R. The result is a (positively) folded triangle $k_R \cdot \rho(T) \subset \Delta$. Its Δ-side lengths are $k_R\lambda, k_R\mu, k_R\nu$ and its vertices are special. Identify it, by translation, with the triangle in Δ whose vertex is at the origin.

   Now, unfold $k_R \cdot \rho(T)$ to a geodesic triangle in X. Its Δ-side lengths are $k_R\lambda, k_R\mu, k_R\nu$ and its vertices are special.