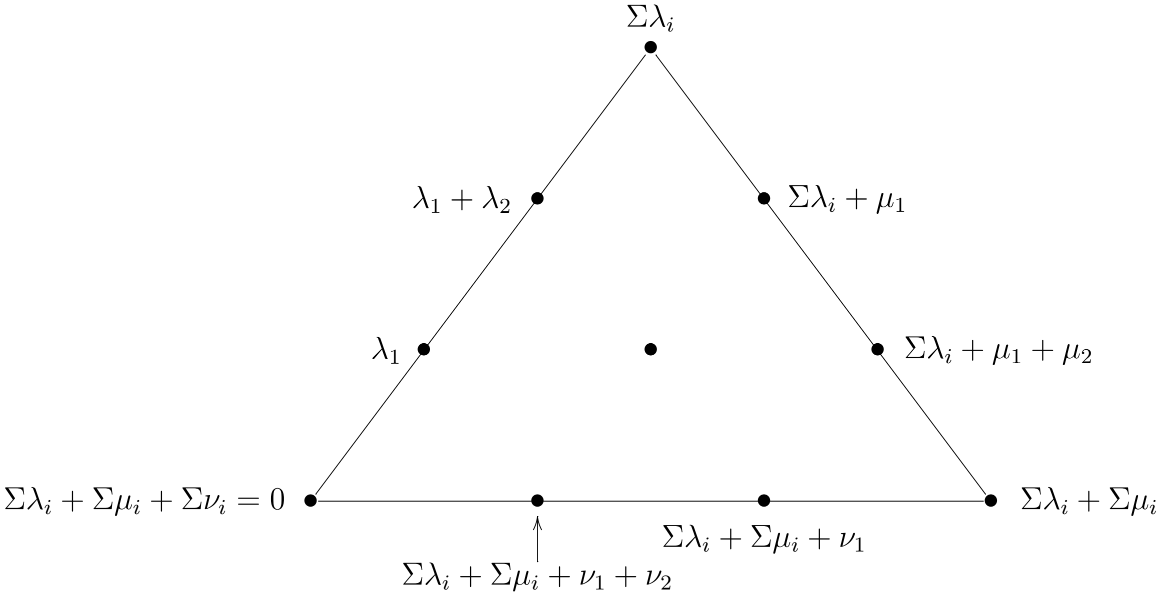
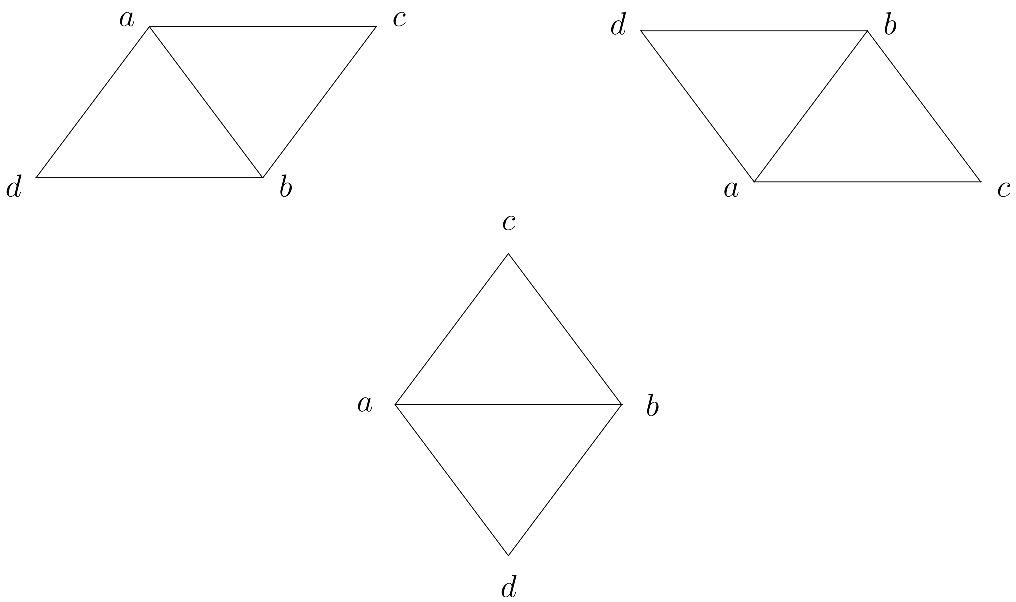
Honeycombs

Allen Knutson (with Terry Tao)

Definition (Johnson, 1970; Berenstein-Zelevinsky, 1993)**.** A *hive* for GL_n is an $(n+1)$-size triangle ($n+1$ dots on each side of the triangle) of real numbers or integers



such that for all rhombi



$a+b \geq c+d$ (convexity of hive mound). These inequalities are "rhombus inequalities," and $a+b-c-d$ is a "rhombus functional."

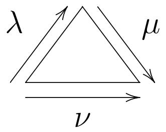
Note that convexity implies $\lambda_1 \geq \cdots \geq \lambda_n$, $\mu_1 \geq \cdots \geq \mu_n$, and $\nu_1 \geq \cdots \geq \nu_n$.

Theorem (Berenstein-Zelevinsky)**.** The number of integer hives with boundary data (λ, μ, ν) is $\dim(V_\lambda \otimes V_\mu \otimes V_\nu)^{\mathrm{GL}_n}$. Moreover,

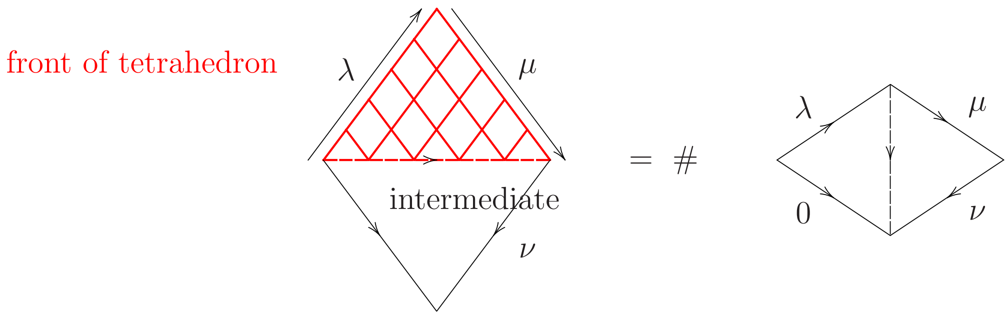
$$(\lambda, \mu, \underbrace{-w_0 \cdot \nu}_{\text{reverse}}) = \dim(\mathrm{Hom}_{\mathrm{GL}_n}(V_\nu, V_\lambda \otimes V_\mu) = c^\nu_{\lambda\mu}.$$

Definition. A *hive* (not necessarily associative) *ring* Hive_n has $\mathbb{Z}$-basis $\{b_\lambda : \lambda = (\lambda_1 \geq \cdots \geq \lambda_n)\}$ and

$$b_\lambda \cdot b_\mu := \sum_\nu \ b_\nu \cdot \#\ \text{hives}$$



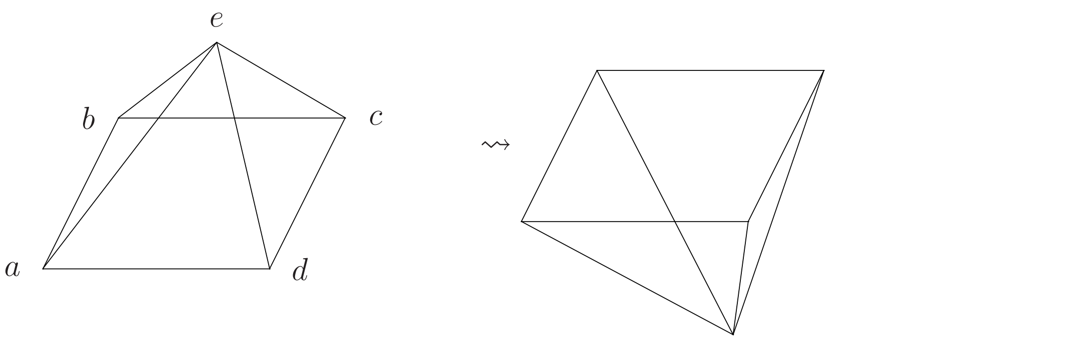
This has an obvious bijection with $\mathrm{Rep}(\mathrm{GL}_n(\mathbb{C}))$ via $b_\lambda \mapsto [V_\lambda]$.
(joint with Chris Woodward) To show this is a ring homomorphism (and hence an isomorphism), it is enough to prove associativity and the Pieri formula $\lambda = (1, 1, \ldots, 1, 0, \ldots, 0)$.
For associativity, consider



Then

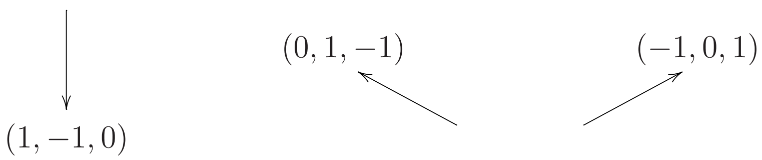
$$\mathrm{Hom}_{\mathrm{GL}_n}(V_0, (V_\lambda \otimes V_\mu) \otimes V_\nu) \cong \mathrm{Hom}_{\mathrm{GL}_n}(V_0, V_\lambda \otimes (V_\mu \otimes V_\nu)).$$

The rule ("octahedron recurrence") for labelling new lattice points in the interior upon excavating an octahedron is as follows:

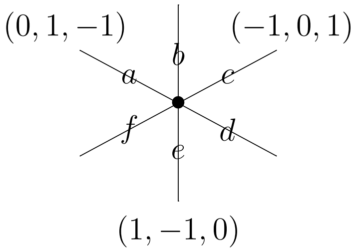


$e' = \min\{a + c, b + d\} - e$

Definition. A *honeycomb* is a picture in $R^3_{\Sigma=0}$ (the sum of the coordinates is 0) that is a union of finitely many line segments (sometimes semi-infinite line segments) with $\mathbb{N}_+$-multiplicities all parallel to

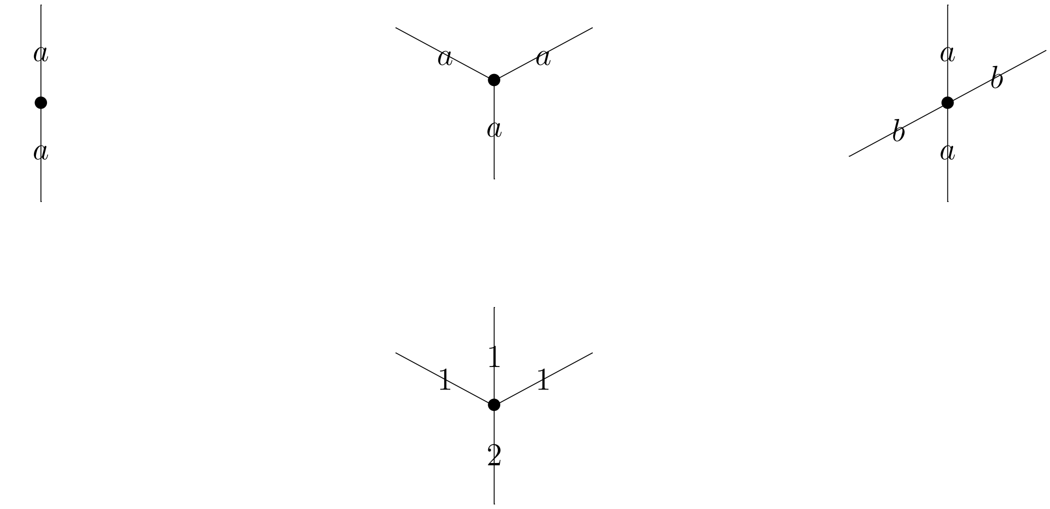


Each edge has a "constant coordinate." There is a local condition called "zero tension": The total vector sum with multiplicites must be $\vec{0}$:

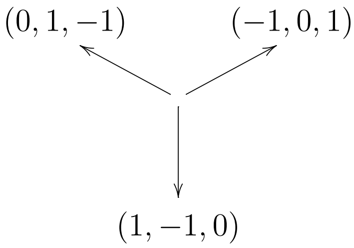


$$(-b-c+e+f, a+b-d-e, -a+c+d-f) = (0,0,0)$$

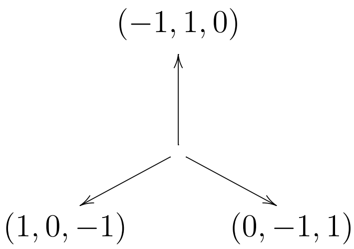
Example.



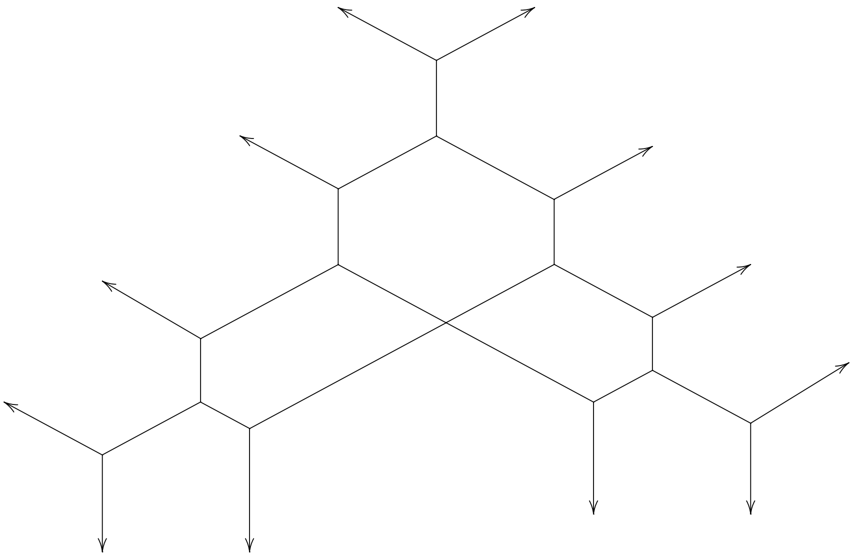
There is also a global condition: The semi-infinite line segments only go in the directions

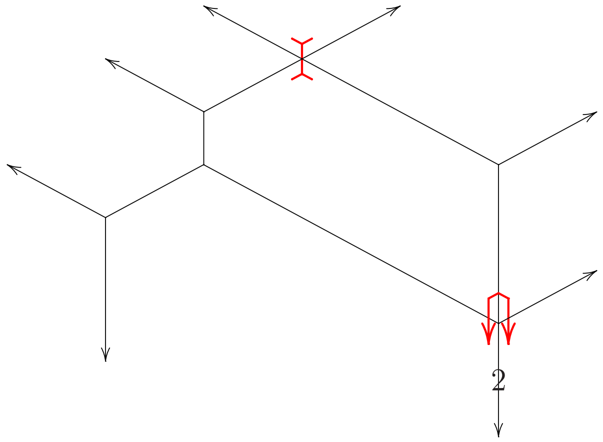


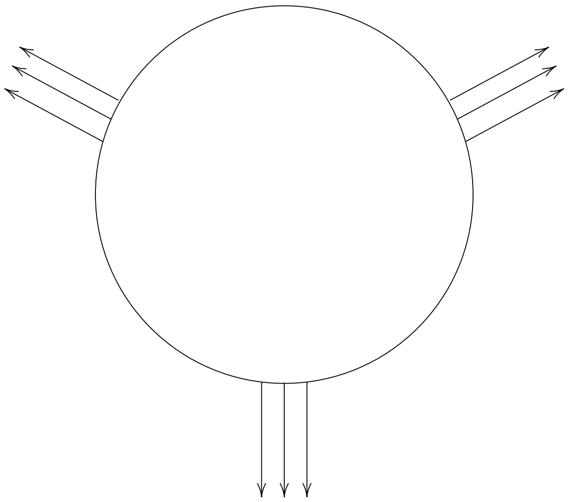
not



Example. All the line segments in the following honeycomb have multiplicity 1.



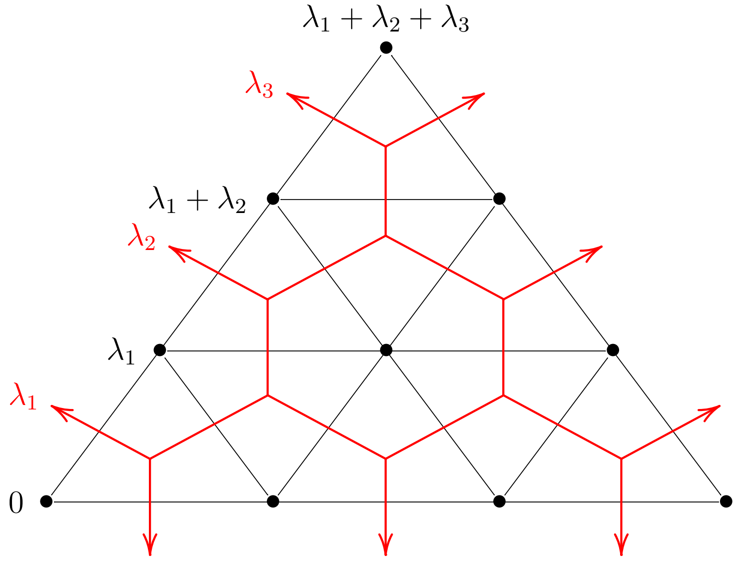




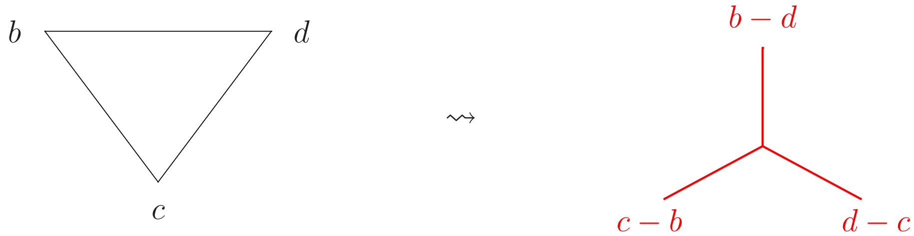
same number (with multiplicity) of semi-infinite line segments in the three directions

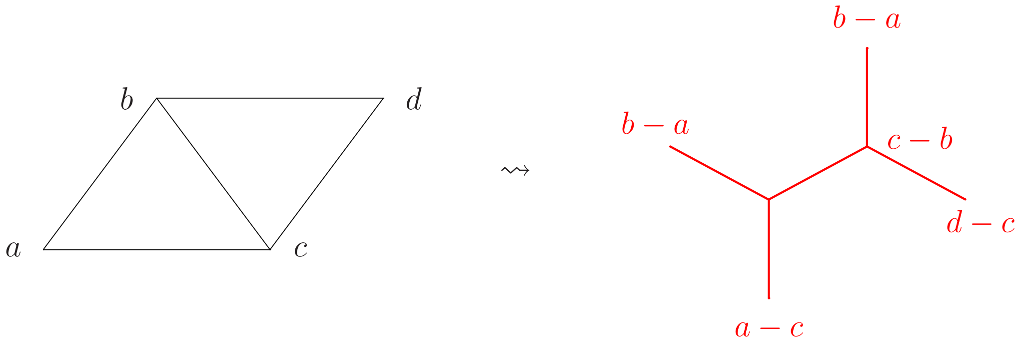
hives	$\twoheadrightarrow$	honeycombs
rhombus inequalities	$\rightsquigarrow$	edges have non-negative length

Example.



honeycomb is dual graph to hive





Theorem. This is a bijection.

What inequalities are true of λ, μ, ν arising as the boundary data of a real hive?
Obvious inequalities ("chamber inequalities"): $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$, and $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n$
An example of a more interesting inequality is $\lambda_1 + \mu_1 \geq -\nu_n$.
To study these inequalities, we can assume that $\lambda_1 > \lambda_2 > \cdots > \lambda_n$, $\mu_1 > \mu_2 > \cdots > \mu_n$, and $\nu_1 > \nu_2 > \cdots > \nu_n$; i.e., λ, μ, ν are *regular* dominant.

Definition. Hives_n is the set of all size n real hives.

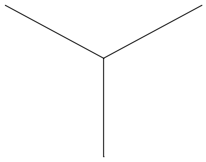
Remarks.

1. The dimension of Hives_n is $\binom{n+1}{2} - 1$.

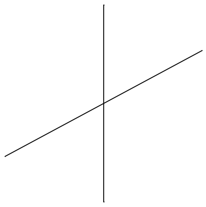
2. By taking the boundary, $\text{Hives}_n \twoheadrightarrow C$, and C has dimension $3n - 1$.

Theorem. There is a continuous, piecewise linear section $C \to \text{Honey}_n$ called the "largest lift" such that if λ, μ, ν are regular dominant and $(\lambda, \mu, \nu) \in C$, then the largest lift of (λ, μ, ν)

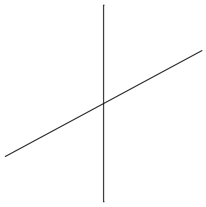
(a honeycomb) has no multiplicities. Up to rotation, all vertices look like



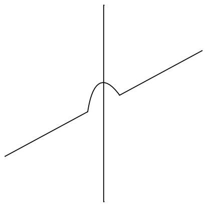
or



and if we regard

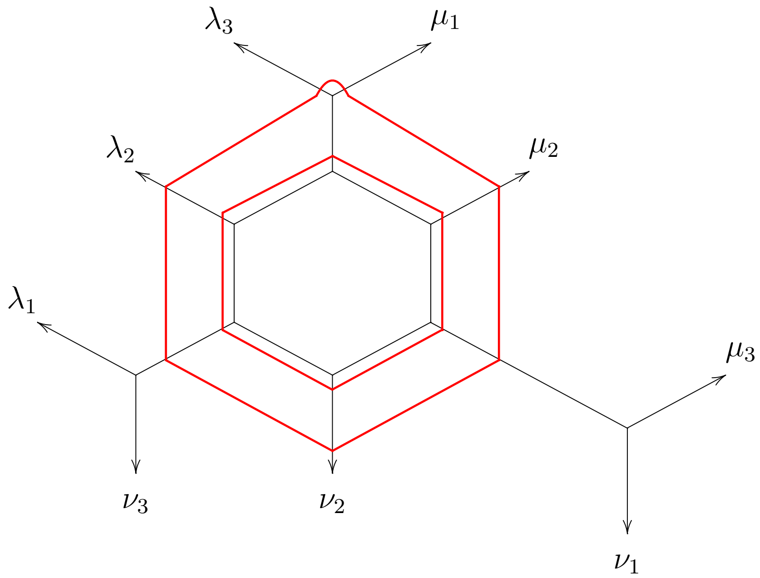


as

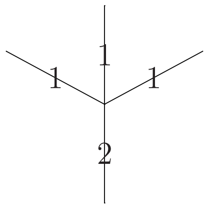


then the honeycomb is acyclic.

Example. This has a cycle but is not a largest lift.

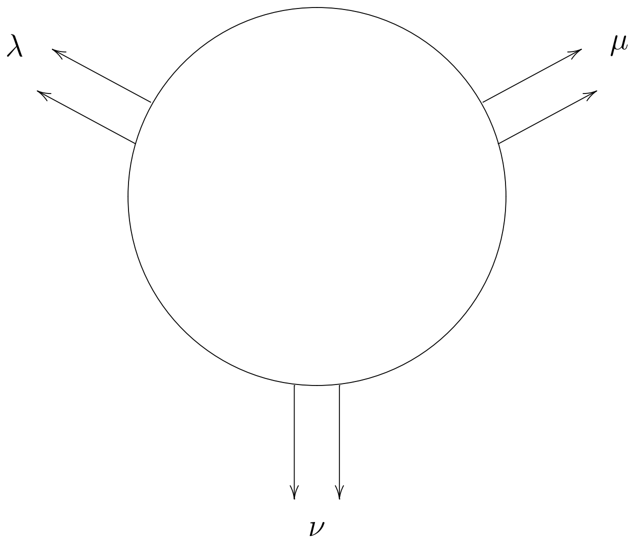


Once we have a cycle, we can use functionals to maximize. Here's the idea: the largest lift hive (associated to an element of C) is defined to be the hive that maximizes a certain linear functional. If the largest lift hive has a cycle, we can "expand" it (the cycle) and increase the functional; hence, the largest lift hive can have no cycles. It's harder if we have



Corollary. Saturation.

Proof. This "largest lift" is defined with $\mathbb{Z}$-coefficients. Thus, λ, μ, ν are integral, and there is an integer honeycomb.



$\square$