Models for Crystals

Cristian Lenart

State University of New York at Albany


Includes joint work with A. Postnikov (MIT).
Crystal graphs

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A crystal

\[ B_3 : \Lambda_2 + \Lambda_1 \]
Models for crystals

- **tableaux** - type specific: Kashiwara-Nakashima
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I will present the alcove path model (L. and Postnikov).
Weyl group:

\[ W = \langle s_\alpha : \alpha \in \Phi \rangle = \langle s_i : i = 1, \ldots, r \rangle. \]

Length: \( \ell(w) = \min \{ k : w = s_{i_1} \ldots s_{i_k} \} . \)
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Length: \( \ell(w) = \min \{ k : w = s_{i_1} \ldots s_{i_k} \} \).

Bruhat graph: directed graph on \( W \) with labeled edges

\[ w \xrightarrow{\alpha} ws_\alpha \text{ if } \ell(ws_\alpha) = \ell(w) + 1. \]
Alcoves

Hyperplanes $H_{\alpha,k} = \{ \lambda : \langle \lambda, \alpha^\vee \rangle = k \}$ ($k \in \mathbb{Z}$).

Reflection in $H_{\alpha,k}$ denoted by $s_{\alpha,k}$. 

**Alcoves**

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**Alcoves:** connected components of $V \setminus (\bigcup H_{\alpha,k})$. 
Alcoves

Hyperplanes \( H_{\alpha,k} = \{ \lambda : \langle \lambda, \alpha^\vee \rangle = k \} \) \((k \in \mathbb{Z})\).

Reflection in \( H_{\alpha,k} \) denoted by \( s_{\alpha,k} \).

Alcoves: connected components of \( V \setminus (\bigcup H_{\alpha,k}) \).

Fundamental alcove:

\[
A_\circ = \{ \lambda \in V : 0 < \langle \lambda, \alpha^\vee \rangle < 1 \text{ for } \alpha \in \Phi^+ \}.
\]
Given $\lambda \in \Lambda^+$, let
\[
(A_0 = A_0, A_1, \ldots, A_l = A_0 - \lambda)
\]
be a shortest sequence of adjacent alcoves (alcove path).
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Let $F_i \subset H_{\beta_i, k_i}$: common wall of $A_{i-1}$ and $A_i$, where $\beta_i \in \Phi^+$.
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Let \( \hat{r}_i := s_{\beta_i, k_i} \).
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$\lambda$-chain (of roots): $\Gamma = (\beta_1, \ldots, \beta_l)$.
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Indexing set $\mathcal{A}(\lambda) = \mathcal{A}(\lambda, \Gamma)$ for a basis of $V_\lambda$; consists of subsets $J = \{j_1 < j_2 < \ldots < j_s\} \subseteq \{1, \ldots, l\}$ such that we have the following path in the Bruhat graph:
\[1 \xrightarrow{\beta_{j_1}} w_1 \xrightarrow{\beta_{j_2}} w_2 \ldots \xrightarrow{\beta_{j_s}} w_s =: \kappa(J) \text{ (key)}.\]

Such subsets will be called admissible subsets.
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\]

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Weight of an admissible subset:

\[
\mu(J) := -\widehat{r}_{j_1} \ldots \widehat{r}_{j_s} (-\lambda).
\]
Example. Type $A_2$, $\lambda = 3\varepsilon_1 + \varepsilon_2$. 

\[ J = \{3, 6\}, \text{saturated chain} 
\] 

\[ e = 123 < t_23 = 132 < t_{23} = 231. \] 

\[ \hat{r}_6 = s\alpha_{13}, -2 \] 

\[ \hat{r}_3 = s\alpha_{23}, 0 \]
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$J = \{3, 6\}$, saturated chain $e = 123 < t_{23} = 132 < t_{23}t_{13} = 231$.

$J = \{6\}$ not admissible: $e < t_{13} = 321$. 
Theorem. (L. and Postnikov) The irreducible character $ch(V_{\lambda})$ of $\mathfrak{g}$ can be expressed as

$$ch(V_{\lambda}) = \sum_{J \in A(\lambda)} e^{\mu(J)}.$$
Theorem. (L. and Postnikov) The irreducible character \( ch(V_\lambda) \) of \( \mathfrak{g} \) can be expressed as

\[
ch(V_\lambda) = \sum_{J \in \mathcal{A}(\lambda)} e^{\mu(J)}.
\]

Remark. There is a similar Demazure character formula.
Crystal graphs

Theorem. (L. and Postnikov) The crystal graph structure corresponding to $V_\lambda$ can be defined combinatorially on $\mathcal{A}(\lambda)$ by directed edges

$$J \mapsto (J \setminus \{m\}) \cup \{k\}.$$
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$$J \mapsto (J \setminus \{m\}) \cup \{k\}.$$ 

There is a corresponding poset structure on $A(\lambda)$. Minimum $J_{\text{min}} = \emptyset$ and maximum $J_{\text{max}}$. 
$B_3: \Lambda_2 + \Lambda_1$
Fact. (Lusztig) $A(\lambda)$ is a self-dual poset, i.e. there is a bijection $\eta : A(\lambda) \to A(\lambda)$ such that

\[ J \leq J' \iff \eta(J) \geq \eta(J') . \]

In particular, $\eta : J_{\min} \leftrightarrow J_{\max}$. 

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The map $\eta$ is given by the action of $w_\circ$ (longest element of $W$) on the canonical basis, so

$$\mu(\eta(J)) = w_\circ(\mu(J)).$$
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$$\mu(\eta(J)) = w_\circ(\mu(J)).$$

Goal: describe $\eta$ explicitly.
Fact. (Lusztig) \( \mathcal{A}(\lambda) \) is a self-dual poset, i.e. there is a bijection \( \eta : \mathcal{A}(\lambda) \to \mathcal{A}(\lambda) \) such that

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In particular, \( \eta : J_{\min} \leftrightarrow J_{\max} \).

The map \( \eta \) is given by the action of \( w_\circ \) (longest element of \( W \)) on the canonical basis, so

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Goal: describe \( \eta \) explicitly.

In type \( A \), it is given by Schützenberger’s evacuation on semistandard Young tableaux (Berenstein and Zelevinsky).
Schützenberger’s evacuation

\[
\begin{array}{cccc}
1 & 1 & 2 & 3 \\
2 & 3 & 3 & \\
4 & 5 & & \\
\end{array}
\quad \xrightarrow{\text{REVERSE}} \quad
\begin{array}{ccc}
5 & 4 & \\
3 & 3 & 2 \\
3 & 2 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{cccc}
1 & 2 & & \\
3 & 3 & 3 & 4 \\
4 & 5 & 5 & \\
\end{array}
\quad \xrightarrow{\text{SLIDE}} \quad
\begin{array}{ccc}
1 & 2 & \\
3 & 3 & 4 \\
3 & 4 & 5 & 5 \\
\end{array}
\]

\[
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1 & 2 & 4 & \\
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1 & 2 & 3 & 4 \\
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\]
Generalizing Schützenberger’s evacuation

Assume that \( \lambda \) is regular, for simplicity (i.e., \( \langle \lambda, \alpha^\vee \rangle > 0 \) for all \( \alpha \in \Phi^+ \)).
Generalizing Schützenberger’s evacuation

Assume that $\lambda$ is regular, for simplicity (i.e., $\langle \lambda, \alpha^\vee \rangle > 0$ for all $\alpha \in \Phi^+$).

Consider the $\lambda$-chain

$$\Gamma := (\beta_1, \ldots, \beta_m, \beta_1, \ldots, \beta_l),$$

where $\{\beta_1, \ldots, \beta_m\} = \Phi^+$. 

Fact: $\Gamma^\text{rev} := (\beta_1, \ldots, \beta_m, \beta_l, \beta_l - 1, \ldots, \beta_1)$ is also a $\lambda$-chain.

STEP 1 (REVERSE-COMPLEMENT)

Define a bijection $J \in A(\lambda, \Gamma) \mapsto J^\text{rev} \in A(\lambda, \Gamma^\text{rev})$, such that $\mu(J^\text{rev}) = w \circ \mu(J)$. 

Generalizing Schützenberger’s evacuation

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Fact:

$$\Gamma^{\text{rev}} := (\beta_1, \ldots, \beta_m, \beta_l, \beta_{l-1}, \ldots, \beta_1)$$

is also a $\lambda$-chain.
Generalizing Schützenberger’s evacuation

Assume that \( \lambda \) is regular, for simplicity (i.e., \( \langle \lambda, \alpha^\vee \rangle > 0 \) for all \( \alpha \in \Phi^+ \)).

Consider the \( \lambda \)-chain

\[
\Gamma := (\beta_{\overline{m}}, \ldots, \beta_m, \beta_l, \ldots, \beta_{\overline{1}}),
\]

where \( \{\beta_{\overline{1}}, \ldots, \beta_m\} = \Phi^+ \).

Fact:

\[
\Gamma^{\text{rev}} := (\beta_{\overline{1}}, \ldots, \beta_m, \beta_l, \beta_{l-1}, \ldots, \beta_{\overline{1}})
\]

is also a \( \lambda \)-chain.

**STEP 1 (REVERSE-COMPLEMENT)**

Define a bijection

\[
J \in \mathcal{A}(\lambda, \Gamma) \mapsto J^{\text{rev}} \in \mathcal{A}(\lambda, \Gamma^{\text{rev}}),
\]

such that

\[
\mu(J^{\text{rev}}) = w_\circ(\mu(J)).
\]
Example.

Type $A_2$, $\lambda = 4\varepsilon_1 + 2\varepsilon_2$, $J = \{2, 4\}$,

$$\Gamma = (\alpha_{12}, \alpha_{13}, \alpha_{23}, \alpha_{13}, \underline{\alpha_{12}}, \alpha_{13}, \underline{\alpha_{23}}, \alpha_{13})$$
Example.

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\]

\[
J^{\text{rev}} = \{\underline{1}, 2, 4\}
\]
STEP 2 (SLIDE)
Yang-Baxter moves. Let $\Gamma$, $\Gamma'$ be $\lambda$-chains related as follows:

\[
\begin{align*}
\Gamma &= (\beta_1, \ldots, (\beta_i, \beta_{i+1}, \ldots, \beta_j), \ldots \beta_l) \\
\Gamma' &= (\beta_1, \ldots, (\beta_j, \beta_{j-1}, \ldots, \beta_i), \ldots \beta_l),
\end{align*}
\]

where $\{\beta_i, \beta_{i+1}, \ldots, \beta_j\} = \Phi^+$ of rank 2.
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$$\Gamma = (\beta_1, \ldots, (\beta_i, \beta_{i+1}, \ldots, \beta_j), \ldots \beta_l) \mapsto$$

$$\Gamma' = (\beta_1, \ldots, (\beta_j, \beta_{j-1}, \ldots, \beta_i), \ldots \beta_l),$$

where $\{\beta_i, \beta_{i+1}, \ldots, \beta_j\} = \Phi^+$ of rank 2.

Theorem. (L.) There is a bijection

$$J \in A(\lambda, \Gamma) \mapsto J' \in A(\lambda, \Gamma')$$

such that $J \setminus [i, j] = J' \setminus [i, j]$, $\kappa(J) = \kappa(J')$, $\mu(J) = \mu(J')$. 

Let $\Gamma_{\text{rev}} = \Gamma_1, \Gamma_2, \ldots, \Gamma_k = \Gamma$ be related as above. We have

$$J \in A(\lambda, \Gamma) \mapsto J_{\text{rev}} = J_1 \in A(\lambda, \Gamma_1) \mapsto \cdots \mapsto J_k = J_{\ast} \in A(\lambda, \Gamma).$$

Theorem. (L.) We have $J_{\ast} = \eta(J)$. 
STEP 2 (SLIDE)

Yang-Baxter moves. Let $\Gamma, \Gamma'$ be $\lambda$-chains related as follows:

\[
\Gamma = (\beta_1, \ldots, (\beta_i, \beta_{i+1}, \ldots, \beta_j), \ldots \beta_l) \mapsto \Gamma' = (\beta_1, \ldots, (\beta_j, \beta_{j-1}, \ldots, \beta_i), \ldots \beta_l),
\]

where $\{\beta_i, \beta_{i+1}, \ldots, \beta_j\} = \Phi^+$ of rank 2.

Theorem. (L.) There is a bijection\

\[
J \in A(\lambda, \Gamma) \overset{YB}{\mapsto} J' \in A(\lambda, \Gamma')
\]

such that $J \setminus [i, j] = J' \setminus [i, j], \quad \kappa(J) = \kappa(J'), \quad \mu(J) = \mu(J').$

Let $\Gamma^{rev} = \Gamma_1, \Gamma_2, \ldots, \Gamma_k = \Gamma$ be related as above. We have

\[
J \in A(\lambda, \Gamma) \mapsto J^{rev} = J_1 \in A(\lambda, \Gamma_1) \overset{YB}{\mapsto} \\
\overset{YB}{\mapsto} J_2 \in A(\lambda, \Gamma_2) \overset{YB}{\mapsto} \ldots \overset{YB}{\mapsto} J_k = J^* \in A(\lambda, \Gamma).
\]
**STEP 2 (SLIDE)**

**Yang-Baxter moves.** Let $\Gamma, \Gamma'$ be $\lambda$-chains related as follows:

$$
\begin{align*}
\Gamma &= (\beta_1, \ldots, (\beta_i, \beta_{i+1}, \ldots, \beta_j), \ldots \beta_l) \mapsto \\
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\end{align*}
$$

where $\{\beta_i, \beta_{i+1}, \ldots, \beta_j\} = \Phi^+$ of rank 2.

**Theorem.** (L.) There is a bijection

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J \in \mathcal{A}(\lambda, \Gamma) \xrightarrow{\text{YB}} J' \in \mathcal{A}(\lambda, \Gamma')
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such that $J \setminus [i, j] = J' \setminus [i, j]$, $\kappa(J) = \kappa(J')$, $\mu(J) = \mu(J')$.

Let $\Gamma^{\text{rev}} = \Gamma_1, \Gamma_2, \ldots, \Gamma_k = \Gamma$ be related as above. We have

$$
J \in \mathcal{A}(\lambda, \Gamma) \mapsto J^{\text{rev}} = J_1 \in \mathcal{A}(\lambda, \Gamma_1) \xrightarrow{\text{YB}} \\
\xrightarrow{\text{YB}} J_2 \in \mathcal{A}(\lambda, \Gamma_2) \xrightarrow{\text{YB}} \ldots \xrightarrow{\text{YB}} J_k = J^* \in \mathcal{A}(\lambda, \Gamma).
$$

**Theorem.** (L.) We have $J^* = \eta(J)$. 

Example.
\[ J^{\text{rev}} = \{1, 2, 4\} \]
\[
\begin{array}{ccccccc}
\bar{1} & \bar{2} & \bar{3} & 1 & 2 & 3 & 4 & 5 \\
\Gamma^{\text{rev}} = (\bar{\alpha}_{12}, \bar{\alpha}_{13}, \bar{\alpha}_{23}, \bar{\alpha}_{13}, (\alpha_{23}, \alpha_{13}, \alpha_{12}), \alpha_{13}) \\
\end{array}
\]
\[
\begin{array}{ccccccc}
\bar{1} & \bar{2} & \bar{3} & 1 & 2 & 3 & 4 & 5 \\
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\end{array}
\]
Example.

\[ J^{\text{rev}} = \{ \bar{1}, 2, 4 \} \]

\[ \Gamma^{\text{rev}} = (\alpha_{12}, \alpha_{13}, \alpha_{23}, \alpha_{13}, (\alpha_{23}, \alpha_{13}, \bar{\alpha}_{12}), \alpha_{13}) \]

\[ \Gamma = (\alpha_{12}, \alpha_{13}, \alpha_{23}, \alpha_{13}, (\alpha_{12}, \alpha_{13}, \alpha_{23}), \alpha_{13}) \]
Example.

\[ J^\text{rev} = \{1, 2, 4\} \]

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\[ \Gamma = (\alpha_{12}, \alpha_{13}, \alpha_{23}, \alpha_{13}, (\alpha_{12}, \alpha_{13}, \alpha_{23}), \alpha_{13}) \]

\[ J = \{2, 4\} \iff J^* = \{1, 3, 4\} \]
Example.

$J^{\text{rev}} = \{1, 2, 4\}$

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$J = \{2, 4\} \mapsto J^* = \{1, 3, 4\}$

**Idea of proof:** Show that the map $J \mapsto J^*$ commutes with the directed edges of the crystal graphs as required.