Our goal is to extend our work down the sequence

\[ G \text{ reductive group}/\mathbb{C} \rightarrow \text{numbers, rings, \ldots, categories} \]

We want to describe numbers, rings, etc. in terms of combinatorics. A solution should be a Chevalley description such as root data.

The geometric Satake is a geometric construction of the dual group \( G^\vee \).

Consider the affine Grassmannian

\[ \text{Gr}_G = G(\mathbb{C}((t)))/G(\mathbb{C}[[t]]) =: G(K)/G(\mathcal{O}), \]

an infinite-dimensional variety. Note that \( G(\mathbb{C}((t))) \) is the loop group and \( G(\mathbb{C}[[t]]) \) is the arc group.

We have the Cartan decomposition

\[ G(\mathcal{O})\backslash \text{Gr}_G \simeq \Lambda_G/W_G \simeq \Lambda_G^+ \simeq \check{\Lambda}_{G^\vee}^+, \]

where \( \simeq \) denotes equivalence, \( \Lambda_G^+ \) is a set of dominant weights for \( G \), and \( \check{\Lambda}_{G^\vee}^+ \) is a set of dominant weights for the dual group \( G^\vee \).

The combinatorics of the representation theory of \( G^\vee \), denoted \( \text{Rep}(G^\vee) \), is encoded in

\[
\begin{align*}
\text{convolution} & \quad \longleftrightarrow \quad \otimes \\
\text{MV cycles} & \quad \longleftrightarrow \quad \text{weight multiplicities}
\end{align*}
\]

All of this can be encoded as a categorical statement—namely, that the perverse sheaves \( \text{Perv}_{G(\mathcal{O})}(\text{Gr}_G) \) are equivalent, at least as abelian categories, to the representations of \( G^\vee \)

\[ \text{Rep}(G^\vee) : \]

as abelian categories.

What goes in the ellipses above are abelian categories.

**Example.** Let \( G = \text{GL}_1 \).
Then $\text{Gr}_G \cong \mathbb{Z}$ as a topological space.

\[ \text{punctured disk} \quad \xrightarrow{\quad} \quad G = \mathbb{C}^x \]

Since we only need to worry about the order of the pole,

\[ \cdots \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \cdots \]

we assign a (finite-dimensional) vector space to each point.

\[ \text{Perv}_{\text{GL}_1(\mathcal{O})}(\text{Gr}_G) \cong \mathbb{Z} - \text{graded vector spaces} \cong \text{Rep}(G) \]

But we’re supposed to be doing $G(\mathcal{O})$-equivariant geometry on $\text{Gr}_G$. In other words, we should be doing geometry on $G(\mathcal{O})\backslash \text{Gr}_G$ (note that as a set, $G(\mathcal{O})\backslash \text{Gr}_G$ is $\Lambda^+_G$).

What should replace the $\mathbb{Z}$-graded vector spaces?

For $\text{GL}_1$, it should be $\text{GL}_1(\mathcal{O}) \backslash \mathbb{Z}$ (stack, groupoid, “Artin orbifold,” . . .).

To a topologist who has never met an algebraic geometer, $\text{GL}_1(\mathcal{O}) = \mathbb{C}^x \cong S^1$, so

\[ \text{GL}_1(\mathcal{O})\backslash \bullet^0 = S^1\backslash \bullet \cong S^1\backslash S^\infty = \mathbb{C}P^\infty \]

What is $\mathbb{C}P^\infty$?

We replace $\mathbb{Z}$ with

\[ \cdots \mathbb{C}P^\infty \mathbb{C}P^\infty \mathbb{C}P^\infty \mathbb{C}P^\infty \mathbb{C}P^\infty \cdots \]

A vector space is a $\mathbb{C}$-module. We had

\[ \bullet \quad \leadsto \quad \text{algebra } \mathbb{C} = H^*(\text{pt}) \quad \leadsto \quad \text{category of } \mathbb{C}\text{-modules} \]

In the new picture,

\[ \mathbb{C}P^\infty \leadsto H^*(\mathbb{C}P^\infty) = \mathbb{C}[u] \leadsto \text{category of differential graded } \mathbb{C}[u]\text{-modules} \]

\[ \deg u = 2 \]
We thus have a new Satake category: differential graded (dg) \(\mathbb{C}[u]\)-modules (two gradings). What is the spectral description of this? In other words, what should we consider instead of \(\text{Rep}(\text{GL}_1)\)?

Earlier, we had \(\text{Rep}(\text{GL}_1)\).

But now we have \(\Lambda = \mathbb{C} \oplus \mathbb{C} \cdot \lambda_{-1} (= H_*(S^1))\), where \(\deg \lambda_{-1} = -1\) and \(\lambda_{-1}^2 = 0\).

Put \(\mathbb{C}[u] := \mathbb{S}\).

graded (dg) \(\Lambda\)-modules

\[
\begin{bmatrix}
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\end{bmatrix}
\]

The “derived” geometric Satake

graded (dg) \(\Lambda\)-modules \(\approx\) graded (dg) \(\mathbb{S}\)-modules

\(\Lambda \mapsto\) trivial module

trivial module \(\mapsto\) \(\mathbb{S}\)

Note that the above equivalence is equivalence of categories.

In general (Drinfeld: “The Exercise”):

\[
D_{G(\mathcal{O})}(\text{Gr}_G) \simeq G^\vee\text{-equivariant dg } \Lambda(g^\vee[1])\text{-modules},
\]

where \(D_{G(\mathcal{O})}(\text{Gr}_G)\) is the set of all \(G(\mathcal{O})\)-equivariant sheaves on the Grassmannian.

**Remarks.**

1. This is a local statement.

2. The degree of the set of all \(G^\vee\text{-equivariant}\) dg \(\Lambda(g^\vee[1])\text{-modules}\) is \(-1\).

3. In the case of \(\text{GL}_1\),

   (a) \(D_{G(\mathcal{O})}(\text{Gr}_G)\) is \(\mathbb{S}\),

   (b) the set of all \(G^\vee\text{-equivariant}\) dg \(\Lambda(g^\vee[1])\text{-modules}\) is \(\Lambda\).

Where is \(\Lambda\) coming from?

The role of the geometric Satake in the geometric Langlands:

- harmonic analysis on locally symmetric spaces
- homological mirror symmetry (topological field theory)

Locally symmetric space: Let \(C\) be a Riemann surface
and \( \text{Bun}_G(C) \) the space of all principle \( G \)-bundles on \( C \) (when \( G = \text{GL}_1 \), then \( \text{Bun}_G(C) \) is the space of all line bundles on \( C \)).

Adelic description (Weil): remove enough points so that the \( G \)-bundle is trivial.

\[
\prod_{p \in C} G(\mathcal{O}_p) \bigg/ \prod_{p \in C} G(K_p) \bigg/ G(\mathcal{C}(C)) = \text{Bun}_G(C)
\]

**Remarks.**

1. \( G(K_p) \) is the loop group at \( p \)
2. \( \text{Bun}_G(C) \) is a global object

In terms of points, the above is

\[
\text{gauge on disks/gluing data/gauge on } C \text{ generic.}
\]

Now we want to do harmonic analysis.

Let the operator \( K \backslash G / K \) act on \( K \backslash G / \Gamma \).

Then for each \( p \in C \),

\[
G(\mathcal{O}_p) \backslash G(K_p) / G(\mathcal{O}_p).
\]

What is the conjectured spectral space for \( \text{Bun}_G(C) \) with the Satake action?

\[
G(\mathcal{O}_p) \backslash G(K_p) / G(\mathcal{O}_p)
\]

**potential eigenvalue**

\[
\text{eigenvalue: } \operatorname{Rep}(G^\vee) \rightarrow \text{vector spaces}
\]

\[
V \mapsto \mathcal{P} \times_{G^\vee} V
\]

(\( \mathcal{P} \) a principle \( G^\vee \)-bundle on the point)

The spectral space is \( \text{Conn}_{G^\vee}(C) \), the set of all \( G^\vee \)-connections on \( C \).

Finally, what is the \( \operatorname{Rep}(G^\vee) \)-action on this side? Also, where does \( \Lambda \) for \( \text{GL}_1 \) come from?

**Recall:** The Satake action on \( \text{Bun}_G(\mathbb{C}) \) at a point \( p \in C \) was “regluing.”

Let’s mimic this for connections.

Consider \( \text{Conn}_{G^\vee}(\mathbb{D}) \).

\[
\mathbb{C} \quad \mathbb{D} \text{ double punctured disk}
\]
Thus,

\[ \text{Conn}_{G^\vee}(D) = \{0\} \cap \{0\} = \text{Spec} \left( C_0 \bigotimes_{\text{Sym}(g)} C_0 \right) = \text{Spec} \left( \Lambda^{-\ast} g^\vee \right), \]

where \( \{0\} \cap g^\vee \{0\} \) is the trivial connection and \( \text{Sym}(g) \) denotes the symmetric algebra.

Bridgeland: bases for categories, mutations