A Summary of Results and Problems Related to the Caccetta-Häggkvist Conjecture

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1 Introduction

This paper is an attempt to survey the current state of our knowledge on the Caccetta-Haggkvist conjecture and related questions. In January 2006 there was a workshop hosted by the American Institute of Mathematics in Palo Alto, on the Caccetta-Haggkvist conjecture, and this paper partly originated there, as a summary of the open problems and partial results presented at the workshop. Our thanks to the many participants who helped with this paper.

Notations & Definitions

- A *graph* $G = (V(G), E(G))$ is a collection of vertices $V(G)$ and edges $E(G)$, where each edge is an unordered pair of distinct vertices u, v.

- A *digraph* $D = (V(D), E(D))$ is a collection of vertices $V(D)$ and edges $E(D)$, along with two incidence relations $h, t : V(D) \times E(D) \to 0, 1$. We let $t(u, e) = 1$ iff u is the tail of e e (i.e. the edge is directed from u to another vertex v), and similarly $h(v, e) = 1$ iff v is the head of the edge. Note this allows multiple edges from u to v. We will assume that $|V(D)|$ and $|E(D)|$ are finite throughout this paper, unless explicitly stated otherwise.

- A *simple digraph* is a directed graph G such that for all $u, v \in V(G)$, at most one edge from u to v appears in $E(G)$ (i.e. no parallel directed edges).

- In a digraph G, for $u, v \in V(G)$, the distance $d(u, v)$ from u to v is the length of the shortest directed path from u to v. We say v is at *out-distance* $d(u, v)$ from u, and u at *in-distance* $d(u, v)$ from v.

- For integers $j > 0$, $N_j^+(v)$ is the set of vertices at out-distance exactly j from v, and $N_j^-(v)$ is the set of vertices at in-distance exactly j from v. We may abbreviate $N_1^+(v)$ and $N_1^-(v)$ to $N^+(v)$ and $N^-(v)$, respectively.

- In a digraph G, δ^+_G and δ^-_G denote the minimum out-degree and in-degree of G, respectively. For a given vertex v, $\delta^+_G(v), \delta^-_G(v)$ denote the out-degree and in-degree of the vertex v. In cases where the graph being referenced is clear, we may write $\delta^+(v)$ and $\delta^-(v)$.

2 The Caccetta-Häggkvist Conjecture

Conjecture 2.1. *(L. Caccetta, R. Häggkvist [5]) Every simple n-vertex digraph with minimum out-degree at least r has a cycle with length at most $\lceil \frac{n}{r} \rceil$.*

This can be restated in the following way: Let A be an $n \times n$ 0-1 matrix such that $a_{ij} = 1$ implies $a_{ji} \neq 1$ for all $i \neq j$. Let $a_{ii} = 1$ for all i. If the sum of every row in A is at least $r + 1$, $A^{\lceil n/r \rceil}$ has trace greater than n.

2.1 Partial Results

The C-H conjecture has been proved for:

- $r = 2$ by Caccetta and Häggkvist [5]
- $r = 3$ by Hamidoune [17]
- $r = 4$ and $r = 5$ by Hoáng and Reed [19]
- $r \leq \sqrt{n/2}$ by Shen [30]. For the exact statement of his result, see Theorem 5.1. This shows that for any given r, the number of counterexamples to the conjecture (if any) is finite.
- Cayley graphs (which implies all vertex transitive graphs using coset representations) by Hamidoune [15]. This proof uses a lemma of Kemperman [21] (Lemma 5.9).

Also, Shen [32] proved that if $deg^+(u) + deg^+(v) \geq 4$ for all $(u, v) \in E(G)$, then $g \leq \lceil n/2 \rceil$, where g denotes the girth of G. This is an average local outdegree version for the $r = 2$ case of the Caccetta-Häggkvist conjecture.

2.2 Approximate Results I - Additive Constant

Another approach is to show that if $\delta^+_G \geq r$, then there is a cycle of length at most $\frac{n}{r} + c$ for some small c. This has been proved for some values of c, as follows:

- $c = 2500$ by Chvátal and Szemerédi [9]
- $c = 304$ by Nishimura [27]
- $c = 73$ by Shen [31].

2.3 Approximate Results II - Special Case n/3

The case $r = n/2$ is trivial, but $r = n/3$ has received much attention. Research has sought the minimum constant c such that $\delta_G^+ \geq cn$ in an n-vertex simple digraph G forces a directed cycle of length at most 3. The conjecture is that $c = 1/3$, and the current results are:

- $c \leq (3 - \sqrt{5})/2 = 0.382$ by Caccetta and Häggkvist [5]
- $c \leq (2\sqrt{6} - 3)/5 = 0.3797$ by Bondy in a neat subgraph counting argument [4]
- $c \leq 3 - \sqrt{7} = 0.3542$ by [29]

Similarly, Seymour, Graaf, and Schrijver [12] asked for the minimum value of β so that when the minimum in- and out-degrees of G are at least βn, G has a directed cycle of length at most 3. They proved that $\beta \leq 0.3487$ and gave a formula relating β and c. Shen applied this formula to his 1998 result to get a slight improvement to $\beta \leq 0.3477$ [29].

3 Seymour's Second Neighborhood Conjecture

This conjecture implies the special case of Caccetta-Häggkvist when both in- and out-degrees are at least $n/3$, and has received much attention of its own.

Conjecture 3.1. *(Seymour) Any simple digraph with no loops or digons has a vertex v whose second neighborhood is at least as big as its first neighborhood, i.e.* $|N_2^+(v)| \geq |N^+(v)|$.

The following is known for Seymour's second neighborhood conjecture:

- When G is a tournament, this is Dean's conjecture, and was proved by Fisher [13] using probabilistic methods. There is also a combinatorial proof by Havet and Thomassé [18].
- It was proved for digraphs with minimum outdegree ≤ 6 by Kaneko and Locke [20].
- There is a vertex v where $|N_2^+(v)| \geq \gamma |N^+(v)|$ and $\gamma = 0.657298...$ is the unique real root of $2x^3 + x^2 - 1 = 0$. (Note the conjecture is that $\gamma = 1$) by Chen, Shen, and Yuster [7]. They also claim a slight improvement to $\gamma = 0.67815$ (proof unpublished).
- Godbole, Cole, and Wright [14] showed that the conjecture holds for almost all digraphs.

4 r-Regular Digraphs

A digraph G is *r-regular* if every vertex v has $\delta_G^+(v) = \delta_G^-(v) = r$.

Conjecture 4.1. *(Behzad, Chartrand, Wall [2]) The minimum number of vertices in an r-regular digraph G with girth g is $r(g-1)+1$.*

Behzad, Chartrand and Wall give an example achieving this by placing $r(g-1)+1$ vertices on a circle with each vertex having edges to the next r vertices in clockwise order. The Caccetta-Häggkvist Conjecture is a generalization of this earlier conjecture.
The Behzad-Chartrand-Wall conjecture was proved for the following special cases:

- $r = 2$ by Behzad [1]
- $r = 3$ by Bermond [3]
- Vertex-transitive graphs by Hamidoune [16]
- If $\delta_G^+ \geq r$, then $g \leq 3\lceil \frac{n}{r} \ln(\frac{2+\sqrt{7}}{3}) \rceil \approx \frac{1.312n}{r}$ by Shen [31].

5 Related Results

Theorem 5.1. *(Shen [30]) For a digraph G on n vertices, if $\delta_G^+ \geq r$, and $n \geq 2r^2 - 3r + 1$, then G has a cycle of length at most $\lceil \frac{n}{r} \rceil$.*

In a graph G, for $u, v \in V(G)$, $\kappa(u, v)$ denotes the maximum number of internally disjoint paths between u and v. If G is a digraph, κ counts the maximum number of internally disjoint directed paths from u to v.
In a graph G, for $u, v \in V(G)$, $\lambda(u, v)$ denotes the maximum number of edge-disjoint paths between u and v. If G is a digraph, λ counts the maximum number of edge-disjoint directed paths from u to v.

Theorem 5.2. *(Thomassen [34]) For all positive integers r, there is a digraph D without digons with $\delta_D^+ \geq r$ and $\delta_D^- \geq r$ such that:*

i. no vertex $v \in V(D)$ is contained in three openly disjoint circuits (that is, three circuits which pairwise share only v)

ii. no edge $(x, y) \in E(D)$ has $\kappa(y, x) \geq 3$.

Theorem 5.3. *(Mader [25]) For every integer $k \geq 0$, If G has $|V(G)| > k^2(k+1)$ and at most $k^2(k+1)$ vertices of G have out-degree at most $k^3(k+1)$, then there are vertices $x \neq y$ so that $\kappa(x, y) > k$.*

5.1 Undirected Graph Theorems

Theorem 5.4. *(Mader [22]) Every graph G with $\delta_G \geq r$ contains vertices x, y with $\kappa(x, y) \geq r$ when $r \geq 1$.*

Theorem 5.5. *(Mader [23]) Every graph G with $\delta_G \geq r$ contains $r + 1$ vertices $v_1, \ldots, v_{r+1}$ with $\lambda(v_i, v_j) \geq r$ for all $i \neq j$.*

5.2 Additive Number Theory Results

Given an additive group Γ, and sets $A, B \subseteq \Gamma$, let $A + B := \{a + b \mid a \in A, b \in B\}$, and $A \hat{+} B$ $:= \{a + b \mid a \in A, b \in B, a \neq b\}$. Finally, for a positive integer r, let $rB := \{b_1 + \cdots + b_h |$ all $b_i \in B$, not necessarily distinct$\}$.

Theorem 5.6. *(Cauchy [6] and Davenport [10], [11]) Let p be a prime, and $A, B \subseteq \mathbb{Z}/p\mathbb{Z}$ be nonempty. Then $|A + B| \geq \min(p, |A| + |B| - 1)$.*

Theorem 5.7. *(I. Chowla [8]) Let m be a positive integer, and $A, B \subseteq \mathbb{Z}/m\mathbb{Z}$ such that $0 \in B$ and $gcd(b, m) = 1$ for all nonzero $b \in B$. Then $|A + B| \geq \min(m, |A| + |B| - 1)$.*

Theorem 5.8. *(Dias de Silva and Hamidoune [33]) The Erdös-Heilbronn Conjecture: Let $A \subseteq \mathbb{Z}/p\mathbb{Z}$, with p prime. Then $|A \hat{+} A| \geq \min(2|A| - 3, p)$.*

For a multiplicative group Γ and sets $A, B \subseteq \Gamma$, let $AB := \{ab \mid a \in A, b \in B\}$.

Lemma 5.9. *(Kemperman [21]) Given a group Γ and finite non-empty subsets $A, B \subseteq \Gamma$, if $1 \in A, B$ but $(1, 1)$ is the only pair (a, b) with $a \in A, b \in B$ such that $ab = 1$, then $|AB| \geq |A| + |B| - 1$.*

We say G is a *layered digraph* if G is a digraph with $V(G) = \cup_{i=0}^{h} V_i$ with $V_i \neq \emptyset$, and $V_i \cap V_j = \emptyset$ for all $i \neq j$, and $(u, v) \in E(G)$ implies $u \in V_{i-1}, v \in V_i$ for some $i \in \{1, \dots, h\}$.
A *Plünnecke graph* is a layered digraph G with the following two properties:

1. If $u, v, w_1, \dots, w_k$ are vertices of G with $(u, v), (v, w_1), \dots, (v, w_k) \in E(G)$, then there are distinct vertices $v_1, \dots, v_k$ so that $(u, v_i), (v_i, w_i) \in E(G)$ for $i = 1, \dots, k$.

2. If $v, w, u_1, \dots, u_k$ are vertices of G with $(v, w), (u_1, v), \dots, (u_k, v) \in E(G)$, then there are distinct vertices $v_1, \dots, v_k$ so that $(u_i, v_i), (v_i, w) \in E(G)$ for $i = 1, \dots, k$.

Let G be a digraph, and X, Y nonempty subsets of $V(G)$. Then
Im(X, Y) $:= \{y \in Y |$ there is a directed path from X to $y\}$. The *magnification ratio D(X, Y)* is

$$D(X, Y) := \min_{Z \subseteq X, Z \neq \emptyset} \left\{ \frac{|Im(Z, Y)|}{|Z|} \right\}.$$

Theorem 5.10. *(Plünnecke [28]) In a Plünnecke graph, let $D_i = D(V_0, V_i)$. Then*

$$D_1 \geq D_2^{\frac{1}{2}} \geq \dots \geq D_h^{\frac{1}{h}}.$$

The following are consequences of applying Plünnecke's Inequalities to a special graph created from subsets A, B of a group:

Theorem 5.11. *For sets $A, B \subset \Gamma$:*

1. *$|iB|^{\frac{1}{i}} \geq |hB|^{\frac{1}{h}}$ for all $0 \leq i < h$.*

2. *If $|B| = k$, and $|B + B| \leq ck$, then $|hB| \leq c^h k$.*

3. *If $|A| = n$, and $|A + B| < cn$ then for all $k, \ell \in \mathbb{Z}^+$, we have that $|kB - \ell B| \leq c^{k+\ell} n$ where $kB - \ell B$ denotes the set of all elements expressable as $(b_1 + \dots + b_k) - (b'_1 + \dots + b'_\ell)$ where all b_i, b'_i are in B.*

Theorem 5.12. *(generalization of Erdös-Heilbronn (Thm. 5.8)) Let $A, B \subseteq \mathbb{Z}/p\mathbb{Z}$, with p prime and $|A| \neq |B|$. Let $C = A \hat{+} B$. Then $|C| \geq \min(|A| + |B| - 2, p)$.*

6 Open Problems and Conjectures

6.1 Rainbow Conjectures

6.1.1 A Colored Generalization of Seymour's Second Neighborhood

Given a digraph $G = (V, E)$ with each edge $e \in E$ having a set S_e of labels in $\{1, 2, \ldots, k_G\}$, a *rainbow* structure H in G (such as a path or cycle) means one in which there is a way to assign each edge $e \in E(H)$ a label $\ell(e) \in S_e$ so that $ell(e) \neq \ell(f)$ for all edges $e \neq f$ in $E(H)$.

Conjecture 6.1. *(Seymour, Sullivan) Let G be a simple digraph on the vertex set V, and $E_1, \ldots, E_k \subseteq E(G)$. Say an edge $e \in E(G)$ has label set $S_e \subseteq \{1, \ldots, k\}$ where $i \in S_e$ if and only if $e \in E_i$. Finally, let $G_i = (V, E_i)$.*

i. There exists a rainbow (di)cycle in G or

ii. There exists a vertex v such that $|\{w \mid \text{there exists a rainbow path from } v \text{ to } w\}| \geq \sum_{i=1}^{k} \delta^+_{G_i}(v)$.

Notes: This is false if you require that the colors appear in an increasing order (cyclic on the cycle). We have been able to show that this conjecture holds when $G_1, \ldots, G_k$ are Cayley graphs on a common group Γ (using induction on Lemma 5.9), and when $\delta^+_{G_i}(v) \leq 1$ for all v and all i except $i = 1$, where we allow the outdegrees to be unbounded (but finite).

6.1.2 Implications of Conjecture 6.1

Conjecture 6.2. *Seymour's Second Neighborhood Conjecture*

Notes: To see this, for a digraph H, let $k = 2, G = H$, and $E_1 = E_2 = E(H)$.

Conjecture 6.3. *Caccetta-Häggkvist Conjecture (general case)*

Notes: For a digraph H, take $G = H$, $k = \lceil \frac{n}{\delta^+_H} \rceil$, and $E_1 = \cdots = E_k = E(H)$. We must get a rainbow cycle because the sum of the outdegrees at each vertex is $\geq n$. This corresponds to a dicycle of length at most $\lceil \frac{n}{\delta^+_H} \rceil$ in H, as desired.

Conjecture 6.4. *Any simple digraph with no loops or digons has a vertex v such that $|N^+_2(v)| + |N^+(v)| \geq 2|N^-(v)|$.*

Notes: Recall, for comparison, SSN can be written as $|N^+_2(v)| + |N^+(v)| \geq 2|N^+(v)|$. To see how Conjecture 6.1 implies 6.4, take a simple digraph H with no loops or digons, and set $G = H$ and $E_1 = E_2 = E(H)$. G cannot have a rainbow cycle by definition of H. Define $N^{+*}_G(u) = \{$vertices you can reach by a rainbow path in G from $u\}$, and $N^{-*}_G(u) = \{$vertices that have a rainbow path in G to $u\}$. Let E_3 be the edges $\{(u, v) \mid v \text{ is not in the set } N^{-*}_G(u)\}$. Let $G' = H$, and have subsets $E_1, E_2, E_3 \subseteq E(G')$ giving rise to label sets $S'_e \subseteq \{1, 2, 3\}$ for $e \in G'$. We can see from these definitions that for any u in $V(G')$,

$$\sum_{i=1}^{3} \delta^+_{G_i}(u) = 2|N^+(u)| + ((n-1) - |N^-(u)| - |N^-_2(u)|),$$

where all neighborhoods referenced on the RHS are in H. We can rewrite this as:

$$\sum_{i=1}^{3} \delta^+_{G_i}(u) = ((n-1) - (|N^-(u)| + |N_2^-(u)| - 2|N^+(u)|)).$$

Then $\sum_{i=1}^{3} \delta^+_{G_i}(u) \geq n$ whenever $|N^-(u)| + |N_2^-(u)| < 2|N^+(u)|$. If this were true for all vertices u, then no vertex could have $|N^{+*}_{G'}(u)| \geq \sum_{i=1}^{3} \delta^+_{G_i}(u) \geq n$, so we must have a rainbow cycle in G', by Conjecture 6.1. However, by construction, since G has no rainbow cycle, G' has no rainbow cycle. Thus there is a vertex $v \in V(H)$ so that $|N^-(u)| + |N_2^-(u)| \geq 2|N^+(u)|$. If we reverse all edges in H, this gives $|N^+(u)| + |N_2^+(u)| \geq 2|N^-(u)|$, as claimed.

Conjecture 6.5. *(Seymour) Under the hypotheses of Conjecture 6.1, if* $|V| = d$ *and* $\sum_{i=1}^{k} \delta_{G_i}(v) \geq d$ *for all* v*,* G *must have a rainbow cycle.*

Note: This conjecture is false if $|V| = d$ is replaced by $|V| = d + 1$.

6.1.3 Other Conjectures Inspired by (or related to) Conjecture 6.1

If we believe Seymour's second neighborhood conjecture and Conjecture 6.1 (specifically 6.4), we might be led to ask if the following holds:

Conjecture 6.6. *("Compromise Conjecture") Any simple digraph with no loops or digons has a vertex* v *such that* $|N_2^+(v)| \geq |N^-(v)|$.

Conjecture 6.7. *Any simple digraph with no loops or digons has a vertex* v *such that* $|N_2^+(v)| + |N^+(v)| \geq 2\min(|N^-(v)|, |N^+(v)|)$.

Conjecture 6.8. *Under the hypotheses of Conjecture 6.1, if* $\delta^+_{G_i} \geq r_i$*, and* $\sum_{i=1}^{t} r_i \geq |V|$*, there is a rainbow cycle in* G.

Conjecture 6.9. *Under the hypotheses of Conjecture 6.1, if* $\sum_{i=1}^{t} \delta^+_{G_i}(v) \geq |V|$ *for all vertices* v*, there is a rainbow cycle in* G.

Conjecture 6.10. *(Devos) Under the hypotheses of Conjecture 6.1, if there is no rainbow cycle in* G *with strictly increasing edge labels, then the average number of vertices reachable from a fixed vertex* v *by label-increasing (possibly trivial) paths is at least* $1 + \sum_{i=1}^{k} \delta^+_{G_i}$.

Note: This can be proved when $G_i = \text{Cayley}(\Gamma, A_i)$ for some group Γ, using Lemma 5.9

6.2 Second & K^{th} Neighborhood Conjectures

Conjecture 6.11. *Is Seymour's Second Neighborhood true for locally finite digraphs? What if we just require the outdegrees to be finite?*

Conjecture 6.12. *(Thomassé) Let* G *be a digraph with no directed cycle of length at most three. Then there is a vertex* $v \in V(G)$ *with* $\delta^+(v)$ *at most the number of non-neighbors of* v.

The following generalization of second neighborhood to kth neighborhood was taken from a (unpublished) paper of Serge Burckel:

Conjecture 6.13. *Any simple digraph with no directed cycles of length at most k has a vertex v such that $|N_k^+(v)| \geq |N_{k-1}^+(v)|$.*

Serge Burckel also asked the following structural question:

Conjecture 6.14. *For any k and any digraph G, define G^* to be those vertices with $|N_2^+(v)| \geq |N^+(v)|$. Then any vertex of out-degree k is at distance at most k from a vertex in G^*.*

Notes: He motivates this with the following remark: "If a vertex x has one successor y, then if y has no successors, $y \in G^*$, otherwise $x \in G^*$. This property seems to generalize for any out-degree, and if it is true, is optimal considering 'pyramids' where any vertex of out-degree k is at distance exactly k from the (unique) solution." His 'pyramids' are formed by placing one vertex, then two in the row beneath it, and so forth (i vertices in row i) to form a triangle. The vertices in row i are then completely joined to all vertices in row $i-1$ for $i \geq 2$.

Conjecture 6.15. *(Seymour &/or Jackson) If G is an Eulerian digraph with no loops or digons, then*

$$\sum_{v \in V(G)} |N_2^+(v)| \geq \sum_{v \in V(G)} |N^+(v)|.$$

Conjecture 6.16. *(Thomassé & Kral) Let G be an Eulerian digraph on n vertices, so that $|E(G)| \geq n^2/3$. Then G has a directed cycle of length at most three.*

6.3 Matrices

Conjecture 6.17. *The following were all presented by Seymour, with no other attributions given: For the following questions, matrices are assumed to be $n \times n$ 0-1 matrices such that $a_{ij} = 1$ implies $a_{ji} \neq 1$, and all diagonal elements equal to 1.*

1. *Let $A_1, A_2, \ldots, A_{\lceil n/r \rceil}$ be matrices (not necessarily distinct) so that the row sums of A_i are at least $r+1$ for all i. Does $A_1 A_2 \cdots A_{\lceil n/r \rceil}$ have trace $> n$? This is a special case of Conjecture 6.5.*

2. *Let $A_1, A_2, \ldots, A_t$ be matrices (not necessarily distinct) so A_i has row sums at least $r_i + 1$ and $\sum_{i=1}^{t} r_i \geq n$. Does $A_1 A_2 \cdots A_t$ have trace $> n$? This is equivalent to Conjecture 6.5.*

3. *Form a digraph from the matrices $A_1, \ldots, A_t$ by putting $t+1$ copies of the vertex set V in a row, connecting copy k of v to copy $k+1$ of v with a horizontal edge for all vertices v and all k, and then from copy k to copy $k+1$ put in the edge from copy k of v_i to copy $k+1$ of v_j precisely when $a_{ij} = 1$ in matrix A_k for $k = 1, \ldots, t$. The question now is whether there exists a vertex u so that there is a non-trivial (i.e. not all horizontal edges) path from copy 1 of u to copy $t+1$ of u.*

4. *If we "squish" all the bipartite graphs from the previous items so they live on a single copy of V, marking edges from copy i to copy $i+1$ with color i, then we have the sum of the (colored) outdegrees at each vertex is at least n, and we're asking for a non-trivial rainbow cycle which has the colors appearing in increasing order.*

For a matrix A, the *spectral radius of A* is defined to be $\max\{|\lambda| : \lambda \text{ an eigenvalue of } A\}$.

Conjecture 6.18. *(Charbit) Let A be the adjacency matrix of a digraph G with zero on the diagonal and $a_{ij} = 1$ if and only if the edge $(i, j) \in E(G)$. If the spectral radius of A is at least n/k, then G has a cycle of length $\leq k$.*

6.4 Disjoint Cycles

Conjecture 6.19. *(Bermond-Thomassen) In a digraph D with $\delta_D^+ \geq 2k - 1$, there are k vertex disjoint cycles.*

Notes: Open for $k \geq 3$, and the proof for $k = 2$ by Thomassen is not intuitive.

Conjecture 6.20. *(Hoáng & Reed) If G is a digraph with minimum outdegree r, then there are directed cycles $C_1, \ldots, C_r$ such that for all ℓ,*

$$|V(C_\ell) \cap (\cup_{i=1}^{\ell-1} V(C_i))| \leq 1.$$

Notes: The related conjecture that given minimum outdegree at least r, there should be a vertex v with r cycles through v which are otherwise vertex-disjoint is false. A counterexample for $r = 3$ was given by Thomassen in 1985 (see Theorem 5.2 for his complete result). Adding the condition that the minimum indegree is also at least r does not improve the veracity of the conjecture, though it is still open when indegree and outdegree are identically r everywhere.

6.5 Connectivity

Conjecture 6.21. *(Hamidoune, 1981) Let D be a digraph with $\delta_D^+ \geq r$, $\delta_D^- \geq r$, and $r \geq 1$. Then there is an edge (x, y) such that $\kappa(y, x) \geq r$.*

Note that a counterexample to the above conjecture for all $r \geq 3$ is given by Thomassen in Theorem 5.2.

Conjecture 6.22. *(Mader) If a digraph D has $\delta_D^+ \geq r$, then there are vertices $x \neq y$ such that $\lambda(x, y) \geq r$. Also, there is an edge (x, y) so that $\lambda(x, y) \geq r$.*

The first part of this conjecture was proven by Mader [24] for $\lambda(x, y) \geq r - 1$.

6.6 Weighted Versions

Conjecture 6.23. *(Bollobás & Scott) Let $p : E(G) \to [0, 1]$. If $\sum_{v \in N_G^+(u)} p(uv) \geq 1$ and $\sum_{v \in N_G^-(u)} p(vu) \geq 1$ for all $u \in V(G)$, there is a directed cycle in G of total weight ≥ 1.*

Note: There is a nice proof that there is a dipath of total weight at least 1.

Conjecture 6.24. *(Zhang) Let G be a digraph on n vertices, and $f : E(G) \to \{0, 1, \ldots\}$. If $\sum_{e \in E^+(v)} f(e) \geq n/k$ for all $v \in V(G)$, then there is a directed cycle C such that $\sum_{e \in C} \frac{1}{f(e)} \leq k$.*

Counterexample: (Charbit) Take the directed Cayley graph G with group $G = \mathbb{Z}_8$, and generators $\{1, 2\}$. Let the weight function f be 1 on 2-edges, and 2 on 1-edges (where r-edges are those coming from the generator r). Now let $k = 3$. We can calculate $\sum_{e \in E^+(v)} f(e) = 4 \geq 8/3$ for all vertices v, but there is no directed cycle with $\sum_{e \in C} \frac{1}{f(e)} \leq 3$.

6.7 Averaged Outdegree Conditions

Conjecture 6.25. *If D is a digraph on n vertices with*

$$\sum_{v \in V(D)} \log(1 + \frac{1}{\delta^+_D(v)}) \geq n \log(1 + \frac{1}{r}),$$

then D has a cycle of length at most $\lceil n/r \rceil$.

Counterexample: Take a transitive tournament of size $n-1$, and replace the edge from the vertex of out-degree $n-2$ to the vertex of out-degree zero with a path of length 2 in the opposite direction (thus increasing the number of vertices to n).

Conjecture 6.26. *(Shen [32]) Let G be a digraph with n vertices and minimum outdegree at least one. If $\delta^+_G(v) + \delta^+_G(u) \geq 2r$ for every edge (u, v) in G, then the girth g of G is at most $\lceil n/r \rceil$.*

Note: This was proved by Shen for $r = 2$ in [32].

6.8 UnCategorized

The following conjecture would imply that in a counterexample for Caccetta-Häggkvist for $r = n/3$, one can order the vertices so that at least 75% of the edges go from left to right:

Conjecture 6.27. *(Chudnovsky, Seymour, Sullivan) Let G be a simple digraph with k non-edges (unordered pairs $\{u, v\}$ where both uv and vu are not in $E(G)$). If G has no directed cycle of length at most 3, one can delete at most $k/2$ edges from G and obtain a graph with no directed cycle.*

Notes: We know that there are tight examples for this conjecture (transitive tournaments, C_4, and products of these). It is also known that a minimal counterexample has no source or sink vertex, and no directed cut. Kostochka recently proved all vertices in a minimal counterexample have at least 3 and at most $(n-1)/2$ non-neighbors. He has an argument using these facts to show the conjecture for all $k \leq 14$.

Conjecture 6.28. *(Devos) For any digraph G with no directed cycles of length at most three, there is a probability distribution p on $V(G)$ such that at every vertex v, $p(N^+(v)) \geq p(N^-(v))$ and $p(N^-_2(v)) \geq p(N^-(v))$, where $p(S) := \sum_{s \in S} p(s)$ for a set $S \subseteq V(G)$.*

Notes: For tournaments, such a distribution exists, and is unique. Its existence for a general digraph would imply Seymour's Second Neighborhood Conjecture, and actually it would suffice to just have a probability distribution p so that $p(N^-_2(v)) \geq p(N^-(v))$ on average in G.

Given a digraph $G = (V, E)$, we say $F \subseteq E(G)$ is a *feedback arc set* if the digraph $G' = (V, E - F)$ has no directed cycles.

Conjecture 6.29. *(Lichiardopol's Conjecture) Every digraph D has some minimal feedback arc set F which contains a path of length δ^+_D.*

Notes: This implies Hoáng & Reed (6.20), Caccetta-Häggkvist (2.1), Bermond-Thomassen (6.19), and Thomassé (6.34).

Conjecture 6.30. *(Mader, 1985) For all $k \in \mathbb{Z}^+$, there exists $r \in \mathbb{Z}^+$ such that every r-out-regular digraph contains a subdivision of the transitive tournament on k vertices.*

Notes: This conjecture is known for $k = 3$ (where $r = 2$), and $k = 4$ ($r = 3$ proven by Mader in 1996 [26]). The existence of r for $k = 5$ is still not known, though $r = 6$ has been conjectured.

Conjecture 6.31. *(Shen) For a digraph G on n vertices of girth g, define*

$$t(G, r) = \sum_{u:\delta_G^+(u)<r} (r - \delta_G^+(u)).$$

If $\delta_G^+ \geq 1$, then $n \geq r(g-1) + 1 - t(G, r)$.

Conjecture 6.32. *(Thomassé) If G is a loopless digon-free digraph the maximum number of induced directed 2-edge paths is $n^3/15 + \mathcal{O}(n^2)$.*

Notes: First, note that $n^3/15 + \mathcal{O}(n^2)$ can be obtained by substituting C_4's inside C_4's. Next, given a digraph G, let τ be the number of induced directed 2-paths, η the number of induced edges, and θ the number of cyclic triangles. Then if $|V(G)| = n$,

$$\tau + \eta/3 + 2\theta = n^3/12 + \mathcal{O}(n^2) - \frac{1}{6} \sum_{v \in V(G)} \left((\delta^+(v) - \delta^-(v))^2 + (\frac{n}{2} - \delta^+(v))^2 + (\frac{n}{2} - \delta^-(v))^2 \right).$$

Füredi rewrote the terms on the right hand side in terms of τ, η, and θ, and showed

$$\tau \leq \frac{n^3}{12} + \mathcal{O}(n^2).$$

Bondy has a slight improvement of this result, proving that

$$\tau \leq \frac{2n^3}{25}.$$

Define an *α/β-digraph* to be a digraph D on vertex set V and edges defined by $\sigma_1, \ldots, \sigma_\beta$ permutations (linear orders) on V where the edge $(i, j) \in E \iff i < j$ in at least α of the σ_i. We say G is a *majority digraph* if $\frac{\alpha}{\beta} > 2/3$.

Conjecture 6.33. *(Thomassé & Charbit) Caccetta-Häggkvist holds for 3/4-digraphs.*

Notes: Majority digraphs have no cycles of length at most three. The class of 3/4-digraphs is stable under substitution, and contains the circular interval graph on $3k + 1$ vertices with outdegree k going clockwise. The class includes all known extremal examples for C-H, yet this class of 3/4-digraphs seems manageably small. It is still open whether or not 3/4 digraphs must have vertices x of outdegree less than $n/3$. One can more generally ask if Caccetta-Häggkvist holds for larger classes of majority digraphs.

Conjecture 6.34. *(Thomassé) Every digraph D has a path of length $\delta^+_D(g-1)$, where g is the girth of D.*

Note: This is open for $g = 3$, and implies Caccetta-Häggkvist.

Conjecture 6.35. *(Thomassé) In a digraph D,*

$$\sum_{v \in V(D)} |\delta^+_D(v) - \delta^-_D(v)| + |\{(u,v)|d(u,v) \leq 2\}| \geq 2|E(D)| + |\{v|\delta^+_D(v) > \delta^-_D(v)\}|.$$

Note: This is exact for transitive tournaments.

Conjecture 6.36. *(Thomassé) Let G be a digraph on n vertices with minimum outdegree at least $4n/15$ so that G is maximal with no cycles of length at most three, and has no homogeneous set (in other words, G cannot be obtained by substitution). Then G is a Cayley graph on $3k+1$ vertices with $1, \ldots, k$ as generators for some k (i.e. the circular interval graph with everyone joined to next k clockwise).*

Note: $\mathbb{Z}/15\mathbb{Z}$ with generators $S = \{1, 2, 4, 8\}$ gives exactly $4n/15$.

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