AIM Workshop on Calibrations
Problem Discussion Sessions

compiled by Spiro Karigiannis

August 24, 2006

The following is a compilation of the problems which were discussed during the two problem sessions at the AIM Workshop on Calibrations. These were held on June 26 and 27, 2006. In the following, the abbreviation SLAG means special Lagrangian submanifold, and the abbreviation CY means Calabi-Yau. Also, a subscript n on a manifold M^n will always mean the *real* dimension of M. The other participants should contact me if there are changes or additions that they think I should make to this document.

---

1. Elliptically fibred CY manifolds and their SLAG submanifolds
   - What can be said about SLAGs in a CY 3-fold that happens to be elliptically fibred or K3 fibred?
   - Are there nice characterizations of such situations?
   - Do we know anything special about the SLAGs in these types of CY manifolds? For example, are they fibred? Even if they are not fibred in general, maybe we can make a nice theory out of the ones that are fibred. Can we make a nice derived category using the Lagrangians that respect the elliptic structure?
   - More specifically, suppose M^6 is a CY 3-fold fibreing over $\mathbb{CP}^1$ with K3 fibres. Imagine that L^3 is a SLAG in M that fibres over a curve C in $\mathbb{CP}^1$, with fibres which are real T^2 torii in the K3's. We understand torii in K3, so perhaps we can assemble L as 1-parameter families?
   - The question probably makes the most sense in a family of CY manifolds or in a limit of such a family. Here is a related question. Consider a family M_t of such fibred CY's, fibreing over the same base B with K3 fibres, which collapse in some "adiabatic limit" as $t \to 0$. That is, $M_0 = B$. Suppose we have a family L_t of SLAGs in M_t, fibreing over a curve C in B. What is the curve B in the limit as $t \to 0$?
   - Can we find a way to make the limiting geometry yield information about the SLAGs in the family of K3-fibred CY 3-folds?

2. Yau-Zaslow number for K3 manifolds
   - Let X be a K3 surface. Then a SLAG L in X is a holomorphic curve with respect to some other complex structure J on X.
   - Can we use this fact to compute the Yau-Zaslow number? (The number of holomorphic curves representing a given homology class in X.)

3. Backlund transformations and integrable systems in calibrated Geometry

    - Are there Backlund type transformations that relate different calibrated geometries? In other words, are there equivalences or relations among PDE systems for calibrated objects?
    - For example, there is a relationship between pseudo-holomorphic curves in S^6 and SLAGS fibred by great circles. Then the cone on them is coassociative in $\mathbb{R}^7$.
    - More generally, what is the role of integrable systems in calibrations?
    - In the case of SLAGs in $\mathbb{C}^3$, it might be important. Can integrable systems be used to investigate the index problems for SLAGs?
    - A related question: Given a calibration, what is the best way to understand what it calibrates? How can we express the calibrated condition in a useful way? For example, as an exterior differential system, or as completing inequalities to equalities.

4. Analogue of Floer homology in G_2 geometry

    - Consider two coassociative submanifolds N and N' in a G_2 manifold M^7 which are sufficiently close. How many associative submanifolds A of M with boundary in both N and N' are there? This problem might be equivalent to Seiberg-Witten data on the coassociatives.
    - Another way to phrase this is to consider two coassociatives N and N' with associative "necks" joining them. In general, N and N' will intersect in circles. We expect the associative necks joining them to visit the circles.
    - It might be more appropriate to generalize this question to the context of *closed* G_2 structures, not necessarily co-closed. This might be the right area for transversality arguments to work.
    - Assume that N' is essentially the graph of a small closed self-dual 2-form on N and consider only associative necks of small volume. Do these compute the Seiberg-Witten invariants of N? This is clear by Taubes "GW = SW" if N and N' are non-intersecting. (If the self-dual 2-form has no zeroes.)
    - It would also be interesting to investigate the quantum field theory analogue of this situation in M-theory.

5. Globally defined calibrated submanifolds in Euclidean spaces

    - Describe the graphs of $h : \mathbb{R}^3 \to \mathbb{R}^4$ which are associative, and globally defined. They do not have to be linear. For example, the Cartesian product of the graph of a holomorphic function with a line.
    - These are Bernstein-type questions. Perhaps we can consider graphs with growth conditions, such as polynomial volume growth?
    - For example, three-dimensional special Lagrangian graphs (submanifolds) in $\mathbb{R}^6 \subset \mathrm{Im}(\mathbb{O})$ are always associative, thus one has plenty of associative global graphs if the potential function of three variables for the special Lagrangian graphs are taken as harmonic functions of the first two variables.

- We can formulate the Liouville-Bernstein question: Are all associative global graphs with linear growth affine? The answer is yes, at least if one has the gradient bound. One may also formulate the growth condition in terms of the volume.

6. Geometric Measure Theory and Calibrations

   - Do singularities of calibrated currents simplify if we impose some genericity conditions on the calibrated manifold?
   - Can we close the gap between general calibrated cycles and more well-behaved ones?
   - For example, is the generic calibrated cycle in an infinite family of cycles for a family of calibrations?
   - Another question: Consider a finite-dimensional family of calibrated currents in an infinite dimensional family of calibrations. For a generic calibration, is the generic current nicely structured? (Well behaved in some sense?)
   - A generic calibration here could mean, for example, Joyce's notion of almost Calabi-Yau, or a closed but not necessarily co-closed G_2 structure.
   - Suppose that for a family of calibrations, we get a family of calibrated submanifolds existing. What does this mean?
   - What about uniqueness of tangent cones for calibrated currents? Rivière proved the epiperimetric inequality in complex dimension $n \leq 2$ (for almost complex submanifolds.) There are some other results known (Tian-Rivière). Uniqueness of tangent cones is true in the complex case.

7. Special Lagrangian cones

   - An interesting question is the analyticity of special Lagrangian cones. There exist examples of SLAG cones over torii which are not analytic. Are these cones going to be semi-analytic? (This means that if they were reflection invariant, they would be analytic. So they are "half" of an analytic cone.)
   - We can consider semi-analyticity of cones in calibrated geometry in general. For example, let Σ be an almost complex curve in S^6. The cone $C(\Sigma)$ is associative in $\mathbb{R}^7$. Is $C(\Sigma)$ a component of an algebraic variety?
   - One question is to classify all the SLAG T^2 cones in $\mathbb{C}^3$ which are semi-analytic. The fact that they are cones over T^2 allows you to use integrable systems. The construction involves some algebraic map from a generalized Jacobi variety to projective space.
   - It would be useful to prove that the index of a cone was proportional to area. We would need a more effective proof of a 2-sided bound on area in terms of the index.
   - Are there any higher genus special Legendrian submanifolds in S^5 with $\lambda_1 = 2$, where λ_1 is the first eigenvalue of the Laplacian?

8. Regularity of calibrated cycles satisfying extra conditions

   - Do we have better regularity results for calibrated cycles satisfying some transversality condition? For example, calibrated submanifolds whose tangent planes lie in some geometrically defined open subset of the full calibrated Grassmanian.