Finitary abstract elementary classes

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A **Finitary AEC** $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is an abstract elementary class with

- amalgamation, joint embedding and arbitrarily large models
- $\mathrm{LS}(\mathbb{K}) = \aleph_0$ and
- *finite character*.

We say that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ has **finite character**, if the following holds:

Let $\mathscr{A}, \mathscr{B} \in \mathbb{K}$, $\mathscr{A} \subset \mathscr{B}$ and for each finite $\bar{a} \in \mathscr{A}$

$$\mathrm{tp}^g(\bar{a}/\emptyset, \mathscr{A}) = \mathrm{tp}^g(\bar{a}/\emptyset, \mathscr{B}),$$

then $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathscr{B}$.

Corollary of finite character: Let $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathscr{B}$. An embedding $f : \mathscr{A} \to \mathscr{B}$ is a $\mathbb{K}$-embedding if and only if it is **type-preserving**, i.e. for each finite $\bar{a} \in \mathscr{A}$,

$$\mathrm{tp}^g(\bar{a}/\emptyset, \mathscr{B}) = \mathrm{tp}^g(f(\bar{a})/\emptyset, \mathscr{B}).$$

Remark: If $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is a class of structures and there is a logic L with

- finitely many free variables in one formula and
- $\preccurlyeq_{\mathbb{K}}$ is the L-elementary substructure -relation,

then $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ has finite character.

The main example: Excellent classes

Let $\mathfrak{M}$ denote the **monster model**. We say that $\mathscr{A}$ is a *model* if $\mathscr{A} \in \mathbb{K}$ and $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathfrak{M}$.

Recall Galois type:

$$\mathrm{tp}^g(\bar{a}/A) = \mathrm{tp}^g(\bar{b}/A) \text{ iff}$$

$$\text{there is } f \in \mathrm{Aut}(\mathfrak{M}/A) \text{ such that } f(\bar{a}) = \bar{b}.$$

We **define weak type** ("syntactic type") such that

$$\mathrm{tp}^w(\bar{a}/A) = \mathrm{tp}^w(\bar{b}/A) \text{ iff}$$

$$\mathrm{tp}^g(\bar{a}/B) = \mathrm{tp}^g(\bar{b}/B) \text{ for each } \textit{finite } B \subset A.$$

Lemma 1 (ω-union of types) *Assume that A_n, $n < \omega$ are increasing such that $\bigcup_{n<\omega} A_n \preccurlyeq_{\mathbb{K}} \mathfrak{M}$ and let $\bar{a}_n$, $n < \omega$ be such that*

$$\mathrm{tp}^g(\bar{a}_{n+1}/A_n) = \mathrm{tp}^g(\bar{a}_n/A_n)$$

for each $n < \omega$. Then there is $\bar{a} \in \mathfrak{M}$ such that

$$\mathrm{tp}^g(\bar{a}/A_n) = \mathrm{tp}^g(\bar{a}_n/A_n)$$

for each $n < \omega$.

$\aleph_0$-stable finitary classes

We define $\aleph_0$-stability respect to weak types.

We say that the type $\mathrm{tp}^w(\bar{a}/A)$ **splits** over finite $E \subset A$ if there are $\bar{c}, \bar{d} \in A$ such that

$$\mathrm{tp}^g(\bar{c}/E) = \mathrm{tp}^g(\bar{d}/E) \text{ but}$$

$$\mathrm{tp}^g(\bar{c}/E \cup \bar{a}) \neq \mathrm{tp}^g(\bar{d}/E \cup \bar{a}).$$

We write that

$$\bar{a} \downarrow^s_A B$$

if there is finite $E \subset A$ such that $\mathrm{tp}^w(\bar{a}/B)$ does not split over E.

Theorem 2 (Local character) *Assume that* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is an* $\aleph_0$-*stable finitary class. Let* $\mathscr{A}$ *be a model and* $\bar{a}$ *a tuple. There is finite* $E \subset \mathscr{A}$ *such that* $\mathrm{tp}^w(\bar{a}/\mathscr{A})$ *does not split over* E.

Proof: A binary tree construction (with finite levels) and Lemma 1.

for an $\aleph_0$-stable finitary class we get **several properties** for $\downarrow^s$ over $\aleph_0$-saturated models:

Invariance, monotonicity, finite character, transitivity, stationarity (for weak types) and $\aleph_1$-extension.

Theorem 3 *Let* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *be finitary and* $\aleph_0$*-stable. Let* $\mathscr{A}$ *be a countable model (*$\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathfrak{M}$*). Then*

$$\mathrm{tp}^w(\bar{a}/\mathscr{A}) = \mathrm{tp}^w(\bar{b}/\mathscr{A})$$

if and only if

$$\mathrm{tp}^g(\bar{a}/\mathscr{A}) = \mathrm{tp}^g(\bar{b}/\mathscr{A}).$$

Proof is a "primary model" construction using Lemma 1, finite character and $\aleph_0$-stability.

Recall: $\aleph_0$-**tameness** says that Galois types over models are determined by Galois types over countable submodels.

Theorem 4 *Let* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *be finitary,* $\aleph_0$*-stable and* $\aleph_0$*-tame. Let* $\mathscr{A}$ *be a model. Then*

$$\mathrm{tp}^w(\bar{a}/\mathscr{A}) = \mathrm{tp}^w(\bar{b}/\mathscr{A})$$

if and only if

$$\mathrm{tp}^g(\bar{a}/\mathscr{A}) = \mathrm{tp}^g(\bar{b}/\mathscr{A}).$$

Assume that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is an $\aleph_0$-stable $\aleph_0$-tame finitary class. Then

1. Then notion $\downarrow^s$ has full extension and symmetry over $\aleph_0$-saturated models.

2. The class is Galois-stable in each cardinal.

3. In each cardinality there is a model, which is saturated respect to Galois types.

4. Unions of coherent types $\{\mathrm{tp}^g(\bar{a}_B/B) : B \subset \mathscr{A} \text{ finite}\}$ are realized ($\mathscr{A}$ a model).

Assume that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is a finitary class, categorical above the Hanf number (hence also $\aleph_0$-stable). Then

1. Then notion $\downarrow^s$ has full extension and symmetry over $\aleph_0$-saturated models.

2. The class in weakly stable in each cardinal.

3. In each cardinality there is a model, which is saturated respect to weak types.

Question: Is there a finitary class which is $\aleph_0$-stable but not $\aleph_0$ tame?

Lascar strong splitting and simplicity.

We say that a sequence $(\bar{a}_i)_{i<\alpha}$ is **strongly E-indiscernible** if

- for any ordinal β we can extend the sequence to $(\bar{a}_i)_{i<\beta}$ such that
- any order-preserving partial $f : \beta \to \beta$ induces $F \in \mathrm{Aut}(\mathfrak{M}/E)$ mapping $\bar{a}_i$ to $\bar{a}_{f(i)}$ for each $i \in \mathrm{dom}(f)$.

We say that tp$^w(\bar{a}/A)$ **Lascar splits** over finite $E \subset \mathscr{A}$ if there is a strongly E-indiscernible sequence $(\bar{a}_i)_{i<\omega}$ such that $\bar{a}_0, \bar{a}_1 \in A$ and

$$\mathrm{tp}^g(\bar{a}_0/E \cup \bar{a}) \neq \mathrm{tp}^g(\bar{a}_1/E \cup \bar{a}).$$

We write that

$$\bar{a} \downarrow_A B$$

if there is finite $E \subset A$ such that for each $D \supseteq B$ there is $\bar{b}$ realizing tp$^w(\bar{a}/A \cup B)$ such that tp$^w(\bar{b}/D)$ does not Lascar split over E.

We say that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is **simple** if for each tuple $\bar{a}$ and *finite* A we have that

$$\bar{a} \downarrow_A A$$

.

Assume simplicity and $\aleph_0$-stability. Then $\downarrow$ has all the usual properties over **sets**:

Invariance, monotonicity, finite character, transitivity, stationarity (for weak types over $\aleph_0$-saturated models), extension and symmetry.

Assume that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is $\aleph_0$-stable and $\aleph_0$-tame (**or** categorical above the Hanf number). Then

- When $\mathscr{A}$ is an $\aleph_0$-saturated model,

$$\bar{a} \downarrow_{\mathscr{A}} B \text{ iff } \bar{a} \downarrow^{s}_{\mathscr{A}} B.$$

- We can define U-rank and show that

  if $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ has finite U-rank, then it is simple.

Denote as $\mathbb{K}^\omega$ the ω-saturated models of $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$.

Theorem 5 (Categoricity transfer) *Let* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *be a tame simple finitary AEC, categorical in some uncountable* κ*. Then*

- $(\mathbb{K}^\omega, \preccurlyeq_{\mathbb{K}})$ *is categorical in all uncountable cardinals and*
- $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is categorical in every* $\lambda \geq \max\{\kappa, \beth_{(2^{\aleph_0})^+}\}$.

Proof uses primary models and "saturation transfer".

We get "weak categoricity transfer" without tameness.

Theorem 6 *Let* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *be a simple finitary AEC. Let* κ *be uncountable such that all models of size* κ *are saturated respect to weak types. Then*

- *all uncountable models of* $(\mathbb{K}^{\omega}, \preccurlyeq_{\mathbb{K}})$ *are saturated respect to weak types and*

- *all models of* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *and of size* λ *are saturated respect to weak types, when* $\lambda \geq \max\{\kappa, \beth_{(2^{\aleph_0})^+}\}$.

Superstability

We define a notion of superstability for **simple, finitary AEC**.

Definition 7 (Superstability) *We say that the class* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is* superstable *if it is weakly stable in some cardinal and the following holds.*

Let A_n *for* $n < \omega$ *be finite and increasing such that* $\bigcup_{n<\omega} A_n$ *is a model, and let* $\bar{a}$ *be a tuple. Then there is* $n < \omega$ *such that* $\bar{a} \downarrow_{A_n} A_{n+1}$.

We define **Lascar strong type** such that

$$\text{Lstp}(\bar{a}/A) = \text{Lstp}(\bar{b}/A)$$

if $E(\bar{a}, \bar{b})$ for each A-invariant equivalence relation E with a bounded number of classes.

We also define **weak Lascar strong type** such that

$$\text{Lstp}^w(\bar{a}/A) = \text{Lstp}^w(\bar{b}/A)$$

if $\text{Lstp}(\bar{a}/B) = \text{Lstp}(\bar{b}/B)$ for each finite subset $B \subset A$.

- Simple $\aleph_0$-stable finitary classes are superstable.

- Let $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ be simple, superstable and finitary. Then notion $\downarrow$ has **local character** for models, i.e. for all tuples $\bar{a}$ and models $\mathscr{A}$, we have that

$$\bar{a} \downarrow_{\mathscr{A}} \mathscr{A}.$$

- We also get the usual properties of $\downarrow$ over models and over finite sets (stationarity for weak Lascar strong types).

- Let $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ be simple, superstable and finitary. There is a cardinal κ such that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is weakly stable in all $\lambda \geq \kappa$.

We say that a model A is **a-saturated**, if all Lascar strong types over finite subsets are realized in $\mathscr{A}$.

We say that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is **a-categorical** in κ, if there is only one a-saturated model of size κ.

Theorem 8 *Assume that a simple finitary* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is a-categorical in* $\kappa \geq Hanf$ *with uncountable cofinality. Then it is superstable.*

Let a **formula** ϕ be a set of Galois types $\mathrm{tp}^g(\bar{a}/\emptyset)$. We write that

$$\mathscr{B} \models \phi(\bar{b}),$$

if $\mathrm{tp}^g(\bar{b}/\emptyset, \mathscr{B}) \in \phi$.

Let $\mathbb{S}$ be a set of formulas. We say that a set $A \subset \mathfrak{M}$ is $\mathbb{S}$-**saturated**, if the following holds:

For any finite $\bar{a} \in \mathscr{A}$, $\bar{b} \in \mathfrak{M}$ and $\phi \in \mathbb{S}$, if $\mathfrak{M} \models \phi(\bar{a}\bar{b})$, there is $\bar{d} \in A$ such that $\mathfrak{M} \models \phi(\bar{a}\bar{d})$.

We say that the class $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ has the **Tarski-Vaught property**, if there is a countable set $\mathbb{S}$ such that each $\mathbb{S}$-saturated set is a model.

Assume that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is simple finitary, superstable and has the Tarski-Vaught property. Then

- The notion $\downarrow$ has the usual properties over **all sets** with Lstp^w as the notion of type.

- When A is a countable set, $\mathrm{Lstp}^w(\bar{a}/A) = \mathrm{Lstp}^w(\bar{b}/A)$ implies $\mathrm{tp}^g(\bar{a}/A) = \mathrm{tp}^g(\bar{b}/A)$.

- With $\aleph_0$-tameness: When $\mathscr{A}$ is an arbitrary model, $\mathrm{Lstp}^w(\bar{a}/\mathscr{A}) = \mathrm{Lstp}^w(\bar{b}/\mathscr{A})$ implies $\mathrm{tp}^g(\bar{a}/\mathscr{A}) = \mathrm{tp}^g(\bar{b}/\mathscr{A})$.