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Deligne  Mixed Hodge Theory

$U$ "open" smooth \$/\mathbb{C}

$= X \setminus \bigcup_i U D_i$

smooth divisor in $X$

normal crossings

$j: U \hookrightarrow X$

$H_c(U, \mathbb{Q}) = H(X, j! \mathbb{Q}_U)$

$j: \mathbb{Q}_U \to \mathbb{Q}_X \to \bigoplus_i \mathbb{Q}_{D_i} \to \cdots$

$j! \mathbb{Q}_U = [\mathbb{Q}_U \to \mathbb{Q}_X \to \bigoplus \mathbb{Q}_{D_i} \to \cdots ]$

E.g. simplest case

$U = X \setminus D$

$j! U: \mathbb{Q}_X \to \mathbb{Q}_D$

$\cdots \to H^i_c(U) \to H^i_c(X) \to H^i_c(D) \to H^i_c(U)$

$E^{a, b}_{\alpha, \beta} = \bigoplus_{\beta} H^b(D_{\alpha, \beta} \cap \cdots \cap D_{\alpha, \beta}) \Rightarrow H^i_c(U)$

$a$-fold $\cap$ is $X$ by defn
simplest case example
\[ 0 \to H^{i-1}(D)/H^{i-1}(x) \to H^i_c(U) \to \ker(H^i_X \to H^i_D) \to 0 \]
smooth normal crossing divisors
\[ \Rightarrow \quad \text{a-fold } \cap \]
\[ D_1 \cap \ldots \cap D_n \]
is smooth projective
spectral seq. degenerates \( d_r = 0 \)
for \( r \geq 2 \)
\( H^i_c(U, \mathbb{Q}) \) has an increasing filtration
\[ W_{-1} = 0 \subseteq W_0 \subseteq W_1 \subseteq \ldots \subseteq W_i \]
\[ W_j/W_{j-1} \otimes \mathbb{Q} \text{ is a pure Hodge structure weight} \]

\underline{Miracle:}
(i) Structure is indep. of the auxiliary choices of \( X, D_i \) used to define it.
(ii) Functionality
Assume we define MHS for all varieties already.

\[ V \subset Y \text{ open } Z := Y \setminus V \]

\[ \cdots \to H_c^i(V) \to H_c^i(Y) \to H_c^i(Z) \to H_c^i(U) \to \]

respects W-filtration and \( \text{gr}^W(\cdots) \) is an exact sequence of pure Hodge structures of weight 0.

\[ \text{ordinary cohom} \]

\[ U = X \setminus U \text{ Di}_i \]

\[ \text{Leray spectral seq.} \]

\[ E_2^{a,b} = H^2(X, R^b j_* \mathcal{O}_U) \]

\[ \Rightarrow H^{a+b}(U, \mathcal{O}) \]

\[ \text{gives filtration} \]

\[ W_0 = 0 \subset W_1 \subset \cdots \text{ on } U \]

\[ \Omega_X (\log D) \]

\[ \text{filter this according to } \mathcal{H} \]

\[ H(\cdots) = H(U, \mathcal{O}) \]
E - polynomial

\[ E(U; x, y) \in \mathbb{Z}[x, y] \]

\[ = \sum_i (-1)^i \left\{ \sum_{a, b} x^a y^b \operatorname{gr}_w(H_c(U)) \right\} \]

Excision property \Rightarrow

∀ open in \( Y \), \( Z := Y \setminus V \)

\[ E(V; x, y) = E(Y; x, y) - E(Z; x, y) \]

\( K(\text{sch}/\mathbb{C}) \)

free abelian group on \([T]\)

\( T \) separated \( \mathbb{C} \)-scheme of finite type

\[ [T] \sim [T \text{red}] \]

\[ [V] \sim [Y] - [Z] \]

N.B. In \( K(\text{sch}/\mathbb{C}) \) every element is

\[ [\text{proj nonsing } X] - [\text{proj nonsing } Y] \]

(not necessarily connected)
pf reduce to affine
\[ Z \subseteq A^n \]
\[ [A^n] - [A^n \setminus Z] = [Z] \]

inclusion-exclusion
\[ U = \bigcap_{\mathcal{X}} - \sum_{i} [D_i] + \sum_{i,j} [D_i \cap D_j] - \ldots \]

Deligne MHS \( E \) is well defined on \( K(\mathfrak{sch}/\mathbb{C}) \) (write it as
\[ E (X; x, y) = E (Y; x, y) \]
for \( [X] - [Y] \)

polynomial count varieties

arbitrary \( X/\mathbb{F}_q \) is polynomial-count

if there exists \( p (t) \in \mathbb{C} [t] \) s.t.

for all finite fields \( k/\mathbb{F}_q \)

\[ \# X (k) = p (\# k) \]

e.g.
\[ X = \mathbb{A}^n, \quad p (t) = t^n \]
\[ X = \mathbb{P}^n, \quad p (t) = 1 + t + \ldots + t^n \]
Ex. X polynomial count P has \( \mathbb{Z} \)-coeffs. (Note P is uniquely determined).

Use \( \mathbb{Z}(X, T) \) is a rational fctns can we do it without this?

\( X/C \) is polynomial count if there exists a spreading out \( X/R \subset C \)
and \( P \in C[T] \) s.t. finitely generated \( \mathbb{Z} \)-algebra

\[ R \xrightarrow{\phi} k \]
finite field

\[ X \otimes k \text{ is polynomial with } \]
\[ R \xrightarrow{\phi} \text{ polynomial P.} \]

Defn In \( K \) group say:

\[ [X] \sim [Y] \]
if there exist spreading out \( X \) and \( Y \)
over some \( R \) s.t. for all

\[ R \xrightarrow{\phi} k \]
\[ \# X_R(k) = \# Y_R(k) \]
(egviv. they have the same \( \zeta \) fctn)
$K(\text{schemes} / R) \to K(\text{schemes} / C)$

In this language, polynomial count means zeta equivalent to $\sum an \left\{ x^n \right\}$

$P(t) = \sum an t^n$)

or zeta equivalent to $\sum bm \left\{ P^n \right\}$

**THM** If $x, y \in K(\text{sch} / C)$ are zeta equiv then

$E(x, x, y) = E(y, x, y)$

**PF**

$[x] = [S] - [T]$, $s, t, s_1, t_1$

$[y] = [S_1] - [T_1]$ proj. smooth

$[S] + [T_1] \sim [S_1] + [T]$

Fontaine-Messing $\Rightarrow E(S \sqcup T_1; x, y)$

Faltings $E(S \sqcup T_1; x, y)$

(same Hodge numbers)

For polynomial count $X$

$E(x, x, y) = \sum bm E(P^n; x, y)$

Cor $E(x, x, y) = P(xy)$. 


Example

$E_1, E_2$ two isogenous elliptic curves

$\mathbb{P}^2 \setminus E_1 \cong \mathbb{P}^2 \setminus E_2 \sim \mathbb{P}^2$

$\mathbb{P}^N$

Let $X = \mathbb{P}^N \setminus \text{image } j$
polynomial count not paved by affines

$# E(\mathbb{F}_q) = q + 1 - aq$