1. Sums of squares of polynomials. (Proposed by John D'Angelo) Let $\Omega \subset \mathbf{C}^n$ be a strongly pseudoconvex domain with compact algebraic boundary. Let $R(z, \overline{z})$ be a real polynomial which is positive on the boundary of Ω. Does there exist a positive integer k and polynomials $p_1(z), p_2(z), \ldots, p_k(z)$ such that

$$R(z, \overline{z}) = \sum_{j=1}^{k} |p_j(z)|^2$$

on Ω? The answer is known to be yes if Ω is the unit ball in $\mathbf{C}^n$. (See [CD].)

2. The radical of the set of squared norms of polynomial mappings. (Proposed by John D'Angelo and Dror Varolin) Let P denote the set of Hermitian symmetric polynomials $R(z, \overline{w})$, and let $S \subset P$ denote the set of squared norms of holomorphic polynomial mappings. For $R \in P$, give a necessary and sufficient condition for the existence of an integer N such that $R^N \in S$.

In other words, when do there exist an N and holomorphic polynomials p_j such that:

$$R(z, \overline{z})^N = \sum_{j=1}^{k} |p_j(z)|^2.$$

It is known that, if $R^N \in S$, then R satisfies the inequality

$$|R(z, \overline{w})|^2 \leq R(z, \overline{z}) R(w, \overline{w}).$$

This inequality can hold without R being a squared norm.

The following is a necessary and sufficient condition for $R \in S$: For all choices of points $\{z_j\} \in \mathbf{C}^n$, and for all k, the k by k matrix with (i, j) entry equal to $R(z_i, \overline{z}_j)$ is nonnegative definite.

See [CD] and [DV] for related additional information.

Another necessary and sufficient condition in the case $N = 1$: write $[A] = (A_1, A_2, \ldots, A_n)$ where the A_i are $r \times r$ commuting matrices. Then R can be written as a sum of square norms of polynomials if and only if $R([A], [A^*]) \geq 0$ holds for all possible choices of such commuting $A_j's$. (Convention: write adjoints first in the calculation of $R([A], [A^*])$.)

3. Finite mappings. (Proposed by Linda Rothschild) Suppose $f : (\mathbf{C}^n, 0) \to (\mathbf{C}^n, 0)$ is a finite mapping, that V is a complex variety. Suppose that $f^{-1}(V)$ is a smooth manifold. Does this imply that V is a manifold?

The following counterexample in characteristic 2 shows that an "algebra-only" argument will not do. Let $k = 3$, $f(x) = (z_1^2, z_2^2, z_3 + z_1 z_2)$ and $V = \{(w_1, w_2, w_3) \in \mathbf{C}^3 : w_3^2 = w_1 w_2\}$. Then $f^{-1}(V) = \{(z_1, z_2, z_3) \in \mathbf{C}^3 : z_3 = 0\} \cup \{(z_1, z_2, z_3) \in \mathbf{C}^3 : z_3 + 2z_1 z_2 = 0\}$ which is the complex hyperplane where $z_3 = 0$ in the event that $2 = 0$.

If the dimension of V is 1 then the answer is yes.

Some motivation for this: Given a (germ of a) real analytic submanifold $M \subset \mathbf{C}^n$ near 0, and f as above. A natural question to ask is: when is $f(M)$ a manifold? See [ER].

4. Mappings between Riemann surfaces. (Proposed by Dror Varolin) Let X, Y be compact Riemann surfaces and $\{a_j\} \in X$ points considered to have finite multiplicity $m_j > 0$. Does there exist $f : X \to Y$ so that near a_j, $f(z) = z^{m_j+1}$, and f has no other critical points?

Next, assume $\phi : X \to \mathbf{R}$ is C^∞, $\Delta\phi \geq 0$, $\Delta\phi > 0$ away from the a_j and

$$\frac{\Delta\phi}{|z - a_j|^{2m_j}}$$

near the a_j. Then does there exist such an f?

5. Holomorphic continuation of CR maps. (Proposed by Francine Meylan) Let $(M, p) \subset \mathbf{C}^n$ be a generic real analytic submanifold, and suppose $f : M \to \partial B_N$, $n \leq N$, is a CR and continuous mapping that extends as a meromorphic function in a neighborhood of p. Then when does f extend as a holomorphic function near p? In particular, does f extend if M is minimal?

Comment. When M is of codimension 1, this is always true. (See [Ch].)

6. Determination by finite order jets. (Proposed by Bernhard Lamel) Let M be a generic real analytic manifold, holomorphically nondegenerate, of finite type and connected. Then for all $p \in M$ there exists $k(p)$ such that

$$Aut\, M(p) \to G^k(\mathbf{C}^n)$$

$$f \mapsto j_p^k f$$

where $k(p)$ is the least k such that the above mapping is injective. Does there exist an M in $\mathbf{C}^2$ with a sequence $\{p_j\}$ in M such that $k(p_j) \to \infty$ as $j \to \infty$?

Comment. Such an M does exist in higher dimensions.

Next, if M is only generically of finite type, does there exist an M as above with the p_j converging to a point of M?

7. Approximation of formal mappings with convergent mappings. (Proposed by Nordine Mir) Let $0 \in M \subset \mathbf{C}^n$, $0 \in M' \subset \mathbf{C}^N$ be real analytic manifolds with real analytic defining functions ρ, ρ'. Suppose also that both are of (Bloom-Graham-Kohn) finite type, $f : (\mathbf{C}^n, 0) \to (\mathbf{C}^N, 0)$ a formal holomorphic mapping such that $f(M) \subset (M')$ formally. (This means that $f(z) = (f_1[z], f_2[z], \ldots, f_N([z]))$ is an N-tuple of formal power series where for some formal power series $a(z, \overline{z})$, $\rho'(f(z), \overline{f(z)}) = a(z, \overline{z})\rho(z, \overline{z})$ as formal power series.) Then the question is: does there exist an Artin-type approximation theorem, i.e., for each $\ell > 0$ can we find a (convergent) holomorphic mapping $f^\ell : (\mathbf{C}^n, 0) \to (\mathbf{C}^n, 0)$ such that $f(M) \subset M'$ and $j_0^\ell f^\ell = j_0^\ell f$?

If M' is real algebraic, the answer is yes. (See [MMZ].)

Suppose that $0 \in A_0 \subset \mathbf{C}^n \times \mathbf{C}^N$, the graph of f lies in A_0 (formally?) and we let $c(f) := dim(A_0) - N$ (the "complexity" of f). Then $c(f) = 0$ if and only if f is convergent.

8. Rational mappings of balls. (Proposed by Han Peters) Let $R : B_n \to B_N$, $n \geq 3$ be a rational proper mapping: $R(z) = p(z)/q(z)$. Let $d = d^0(R)$ be the degree of R. Question 8a: is $d \leq \frac{N-1}{n-1}$? Question 8b: Is $d \leq \frac{N-1}{n-1}$ if we also assume that $n \geq 2d^2 + 2d$? Characterize all $R(z)$ such that this inequality is sharp.

For monomials the answer to question 8b is yes. (See [DLP].) For $n = 2$ it is known for monomial maps that $d \leq 2N - 3$ and that this is sharp. (See [DKR].)

A further question: let $f : B_n \to B_N$ be proper, and suppose f is rational, polynomial or monomial with $d := d^0(f) > 1$. Does there exist a proper $g : B_n \to B_N$ which is rational, polynomial or monomial such that $d^0(g) = d^0(f) - 1$?

9. Finite jet determination. (Proposed by Bernhard Lamel) Suppose that $R, \tilde{R} : B_n \to B_N$ are proper rational functions. Find the smallest integer r such that if $j_0^r R = j_0^r \tilde{R}$ then $R = \tilde{R}$. The same question could be asked if $R, \tilde{R}$ are merely proper (and not rational.)

10. Deformation of varieties. (Proposed by Dmitri Zaitsev) Let $n \geq 3$ and suppose that $V \subset \mathbf{C}^n$ is a complex analytic subvariety of codimension 1. Assume that $0 \in V$ and that 0 is an isolated singularity of V. Suppose that ∂V is compact and smooth. Can V be 'deformed' into $B_{n-1} \subset \mathbf{C}^{n-1}$? A smooth deformation $(V_t, \partial V_t)$ of such varieties with t in an interval I is defined to satisfy:

(1) In a neighborhood of every (p_0, t_0) with $t_0 \in I$, $p_0 \in V_{t_0}$, there exists a smooth complex function $F(p,t)$, holomorphic in p, such that $V_t = \{p : F(p,t) = 0\}$.

(2) In a neighborhood of every (p_0, t_0) with $t_0 \in I$, $p_0 \in \partial V_{t_0}$, there exists a smooth complex function $F(p,t)$ as above together with a smooth real function $r(p,t)$, such that $V_t = \{p : F(p,t) = 0, r(p,t) < 0\}$ and such that the partial derivative vectors $\partial F/\partial p$ and $\partial r/\partial p$ are linearly independent.

11. Solutions to $\overline{\partial}$. (Proposed by Mei-Chi Shaw) Let $\Omega \subset \mathbf{CP}^n$ be pseudoconvex with C^∞ boundary. Let us solve $\overline{\partial} u = f$ in Ω. If f is a (p,q) form such that $\overline{\partial} f = 0$, $f \in C^\infty(\overline{\Omega})$, then does there exist $u \in W^1(\Omega)$ such that $\overline{\partial} u = f$?

It is known that there exists such a u in $L^2(\Omega)$, and also that there exists such a u in $W^\epsilon(\Omega)$ for some $\epsilon > 0$, where ϵ depends on Ω. See [CSW].

The answer is not even known if $\partial\Omega$ is real analytic.

12. Transversality. (Proposed by Peter Ebenfelt) Suppose that $0 \in M \subset \mathbf{C}^{n+1}$, $0 \in M' \subset \mathbf{C}^{N+1}$, where M, M' are hypersurfaces (C^∞ or real analytic), and $n \leq N$. Let $H : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}^{N+1}, 0)$ be holomorphic, and $H(M) \subset M'$. Suppose that H does not map a full neighborhood of 0 in $\mathbf{C}^{n+1}$ into M'. Then give conditions on M, M' such that H is transversal at 0 to M' for all such H.

The statement 'H is transversal to M' at p' means that

$$d_pH[T_p\mathbf{C}^{n+1}] + T_{H(p)}M' = T_{H(p)}\mathbf{C}^{N+1}.$$

In the case that $N = n$ and M or M' is of finite type at 0, then transversality holds under a finite map H.

If M and M' are strictly pseudoconvex, H is transversal.

Example. Consider the mapping $H : \mathbf{C}^2 \to \mathbf{C}^3$ given by

$$H(z, w) = (z + z^2 + \frac{i}{2}w, z - z^2 - \frac{i}{2}w, -2zw).$$

Let $M = \{(z, w) \in \mathbf{C}^2 : \mathrm{Im}(w) = |z|^2\}$ and

$$M' = \{(z_1, z_2, w) \in \mathbf{C}^3 : \mathrm{Im}(w) = -|z_1|^2 + |z_2|^2\}.$$

The H is transversal on $M \setminus \{(z, w) : \mathrm{Re}(z) = 0\}$.

13. Singularities of varieties. (Proposed by Xiaojun Huang) Let $M \subset \mathbf{C}^N$ be a compact real analytic spherical CR manifold of hypersurface type and suppose that M is the boundary of an analytic variety V. Then what kind of singularities can V have?

It is known that if M is algebraic then V has at most one isolated singular point. See [HJ]. See also a survey paper [H].

Example. Consider the mapping $f : B_2 \to B_3$ given by $f(z_1, z_2) = (z_1^2, \sqrt{2}z_1z_2, z_2^2)$. Then the image of f is $\{(w_1, w_2, w_3) \in \mathbf{C}^3 : Q(w_1, w_2, w_3) \equiv w_2^2 - 2w_1w_3 = 0\}$. Let $M = f(\partial B_2)$ and $V = f(B_2)$. Then V and M satisfy the conditions above. Furthermore, M has exactly one singularity: $(\frac{\partial Q}{\partial w_1}, \frac{\partial Q}{\partial w_2}, \frac{\partial Q}{\partial w_3}) = (-2w_3, 2w_2, -2w_1)$, and this is zero only at the point $(0, 0, 0)$. This example shows that the result in [HJ] is sharp.

References

[CD] Catlin, David W. and John P. D'Angelo, *A stabilization theorem for Hermitian forms and applications to holomorphic mappings*, Math. Res. Lett. **3, no. 2** (1996), 149–166.

[Ch] Chiappari, Stephen A., *Holomorphic extension of proper meromorphic mappings*, Michigan Math. J. **38, no. 2** (1991), 167–174.

[CSW] Cao, Jianguo, Mei-Chi Shaw and Lihe Wang, *Estimates for the $\overline{\partial}$-Neumann problem and nonexistence of C^2 Levi-flat hypersurfaces in* $\mathbf{CP}^n$, Math. Z. **248, no. 1** (2004), 183–221.

[DLP] D'Angelo, John P., Jiri Lebl and Han Peters, *Degree estimates for polynomials constant on a hyperplane*, Preprint.

[DKR] D'Angelo, John P., Šimon Kos and Emily Riehl, *A sharp bound for the degree of proper monomial mappings between balls*, J. Geom. Anal. **13, no. 4** (2003), 581–593.

[ER] Ebenfelt, Peter and Linda P. Rothschild, *Images of real submanifolds under finite mappings* (2006).

[H] Huang, X., *On some problems in several complex variables and CR geometry.*, First International Congress of Chinese Mathematicians (Beijing, 1998), AMS/IP Stud. Adv. Math., 20, Amer. Math. Soc., Providence, RI (2001), 383–396.

[HJ] Huang, Xiaojun and Shanyu Ji, *Global holomorphic extension of a local map and a Riemann mapping theorem for algebraic domains*, Math. Res. Lett. **5, no. 1-2** (1998), 247–260.

[MMZ] Meylan, Francine, Nordine Mir and Dmitri Zaitsev, *Approximation and convergence of formal CR-mappings.*, Int. Math. Res. Not. **4** (2003), 211–242.

[DV] D' Angelo, John P. and Dror Varolin, *Positivity conditions for Hermitian symmetric functions*, Asian J. Math. **8, no. 2** (2004), 215–231.