CD(4) HAS BOUNDED WIDTH

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Abstract. We prove that the constraint languages invariant under a short sequence of Jónsson terms (containing at most three non-trivial ternary terms) are tractable by showing that they have bounded width. This improves the previous result by Kiss and Valeriote [15] and presents some evidence that the Larose-Zádori conjecture [19] holds in the congruence-distributive case.

1. Introduction

In recent years, universal algebra has proven to be very useful in the study of the computational complexity of the constraint satisfaction problem. For every relational structure **B**, the constraint satisfaction problem (CSP) associated to **B**, CSP(**B**), is the following computational problem: given a finite structure **A**, determine whether **A** is homomorphic to **B**. Many computational problems, coming from areas as diverse as artificial intelligence, scheduling, graph theory, database theory, and others can be formulated, in a natural way, as a constraint satisfaction problem. From a computational complexity point of view the importance of the CSP was first pointed out by Feder and Vardi [10] who have shown that if the class of constraint satisfaction problems, in its logic formulation, is slightly generalized in several different ways then we obtain a class of problems which is essentially as rich as the whole of NP. This fact motivates the dichotomy question "are there CSPs that are not solvable in polynomial time nor NP-complete?" which despite considerable effort still remains open.

The groundbreaking work of Jeavons, Cohen, and Gyssens [13] successively developed and refined by Bulatov, Jeavons, and Krokhin [6] and Larose and Tesson [18] has shown strong ties between CSP and universal algebra. In particular, it has been shown that the computational complexity of CSP(**B**) is uniquely determined by the algebra $\mathcal{A}_{\mathbf{B}}$ which has the same universe as **B** and whose basic operations are the polymorphisms of the relations in **B**. A good deal of recent results on the complexity of the CSP are due to this link (see the survey of Bulatov, Jeavons and Krokhin [17] for an overview). It is worth mentioning that all this activity has

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spurred development of universal algebra itself, witnessed mostly in the development of all sorts of new Mal'cev-style conditions. Some examples of results of this sort are [20], [2] and [1].

There are basically two algorithms, or rather algorithmic principles, for CSPs. The first one is linked to the few subalgebras property studied in [8] and [12].

The second one, central to this paper, is called the k-consistency algorithm and gives rise to the notion of bounded width (see the recent survey [7] by Bulatov, Krokhin, and Larose for a nice overview of bounded width). In a nutshell, for every fixed $k > 0$, the k-consistency algorithm is an iterative, polynomial-time, algorithm that computes a set H of partial homomorphisms from $\mathbf{A}$ to $\mathbf{B}$ satisfying the condition that every complete homomorphism (if it exists) from $\mathbf{A}$ to $\mathbf{B}$ must have all its k-ary projections in H (see section 2 for precise definitions). When the set H returned by the k-consistency algorithm is empty we have a guarantee that there is no homomorphism from $\mathbf{A}$ to $\mathbf{B}$. Those relational structures $\mathbf{B}$ for which there exists some $k > 0$ such that the converse also holds are said to have bounded width. Consequently, if $\mathbf{B}$ has bounded width then it is possible to use the k-consistency algorithm for some $k > 0$ to solve CSP($\mathbf{B}$) correctly in polynomial time. A most important question in the area is to determine which structures $\mathbf{B}$ have bounded width, which is equivalent to delineate the reach of the k-consistency algorithm as a tool to solve CSPs. In this study, universal algebra has played a major role. Larose and Zádori [19] have shown that if $\mathbf{B}$ has bounded width then its associated algebra $\mathcal{A}_{\mathbf{B}}$ generates a variety that omits Hobby-McKenzie types **1** and **2** (also Bulatov has proved an essentially equivalent statement in [4]). It has been conjectured in [19] that this condition is also sufficient. The Larose-Zádori conjecture would imply, in particular, that any structure $\mathbf{B}$ whose associated algebra $\mathcal{A}_{\mathbf{B}}$ is in a congruence-distributive variety (for short, CD) has bounded width. This has only been verified for algebras containing a near-unanimity term by Feder and Vardi [10] and for algebras in $CD(3)$ by Kiss and Valeriote [15]. In this paper we generalize the latter result to algebras in $CD(4)$.

2. Preliminaries

2.1. **Constraint Satisfaction Problems and Bounded width.** Most of the terminology introduced in this section is fairly standard. A *vocabulary* is a finite set of relation symbols or predicates. In what follows, τ always denotes a vocabulary. Every relation symbol R in τ has an arity $r \geq 0$ associated to it. We also say that R is an r-ary relation symbol.

A τ-structure $\mathbf{A}$ consists of a set A, called the *universe* of $\mathbf{A}$, and relations $R^{\mathbf{A}} \subseteq A^r$ for every relation symbol $R \in \tau$ where r is the arity of R. All structures in this paper are assumed to be *finite*, i.e., structures with a finite universe. Throughout the paper we use the same boldface and slanted capital letters to denote a structure and its universe, respectively.

A homomorphism from a τ-structure $\mathbf{A}$ to a τ-structure $\mathbf{B}$ is a mapping $h : A \to B$ such that for every r-ary $R \in \tau$ and every $(a_1, \ldots, a_r) \in R^{\mathbf{A}}$, we have $(h(a_1), \ldots, h(a_r)) \in R^{\mathbf{B}}$. We will write $\mathbf{A} \to \mathbf{B}$, meaning that there exists a homomorphism from $\mathbf{A}$ to $\mathbf{B}$ and say $\mathbf{A}$ *is homomorphic to* $\mathbf{B}$.

Definition 1. *(Constraint Satisfaction Problems) Let* $\mathbf{B}$ *be a finite relational structure.* $\mathrm{CSP}(\mathbf{B})$ *is defined to be the set of all structures* $\mathbf{A}$ *such that* $\mathbf{A} \to \mathbf{B}$*. Alternatively, we view* $\mathrm{CSP}(\mathbf{B})$ *as the computational problem asking to decide whether a given* τ*-structure* $\mathbf{A}$ *(the input) is homomorphic to* $\mathbf{B}$*.*

The notion of bounded width has several equivalent formulations. In this paper we shall base our definition on a variant of the existential k-pebble game [16] due to Feder and Vardi [10].

Definition 2. *Let* $0 \le j < k$ *be integers and let* $\mathbf{A}$ *and* $\mathbf{B}$ *be relational structures. A* partial homomorphism *from* $\mathbf{A}$ *to* $\mathbf{B}$ *is any mapping from some subset of the universe of* $\mathbf{A}$ *to* $\mathbf{B}$ *that preserves all tuples of* $\mathbf{A}$ *entirely contained in its domain. Given two mappings* f, g *we say that* g extends f*, denoted by* $f \subseteq g$ *if the domain of* f *is a subset of that of* g *and both coincide over the domain of* f*. A* winning strategy for the duplicator in the existential (j,k)-pebble game on $\mathbf{A}$ and $\mathbf{B}$ *- or a* (j,k)-strategy *or even just a* strategy *if the rest is clear - is a nonempty set* H *of partial homomorphisms satisfying the following two conditions:*

- Closure under subfunctions. *If* $g \in H$ *and* $f \subseteq g$ *then* $f \in H$
- (j,k)-forth property. *If* $I \subseteq J \subseteq A$ *with* $|I| \le j$ *and* $|J| \le k$ *and* $f \in H$ *with domain* I*, then there exists* $g \in H$ *with domain* J *such that* $f \subseteq g$*.*

There is a standard procedure [19], called (j,k)-consistency (in [19] they called it the (j,k)-algorithm), that given two relational structures $\mathbf{A}$ and $\mathbf{B}$ returns, if it exists, a (j,k)-winning strategy. The (j,k)-consistency algorithm starts by throwing initially in H all partial homomorphisms with domain size $\le k$. Once this is done the procedure removes all those mappings that falsify one of the two conditions that define a winning strategy. At the end of this iterative process we either get a winning strategy or an empty set, implying that such a strategy does not exist. It is not difficult to verify that this process runs in time exponential on k, but polynomial if k is fixed.

Observe that every satisfiable instance has a winning strategy that consists of all subfunctions of the solution with domain size $\le k$. The converse is not true. A structure $\mathbf{B}$ has width (j,k) if the opposite always holds. More formally,

Definition 3. *A* σ*-structure* $\mathbf{B}$ *has* width (j,k) *if for every* σ*-structure* $\mathbf{A}$*, if there exists a winning* (j,k)*-strategy then* $\mathbf{A}$ *is homomorphic to* $\mathbf{B}$*. Furthermore,* $\mathbf{B}$ *is said to be of* width j *if it has width* (j,k) *for some* k *and to be of* bounded width *if it has width* j *for some* j*.*

It is a major open problem in the area to characterize all structures with bounded width. Up to the present moment only width 1 has been characterized [10] (see also [9]). Another very important question is the existence of an infinite hierarchy, i.e., whether for any j there are structures of bounded width but that do not have width j. This is known to be true for $j = 1$ but open even for $j = 2$.

2.2. **Congruence distributive varieties.** In this subsection we are going to define the algebraic notions used in the paper. We assume that the reader is familiar with basic notions and results of universal algebra, such as algebras, varieties, congruences, clones, and so on. Good textbooks are [3] and [21]. We do not use the results of Tame Congruence Theory (see [11]) in this paper, except for its mention in the Introduction, but it is fair to say that iteration of Jónsson terms at the beginning of Section 4 was in part inspired by its basic methods. Contrary to the standard notation, we are using $\mathcal{A}$, $\mathcal{B}$ etc. to denote algebras, as we reserved the boldfaced letters for relational structures.

It is well known that congruences of any algebra form an algebraic lattice under the inclusion order. A famous result by B. Jónsson [14] states that an algebra $\mathcal{A}$ lies in a congruence-distributive variety (the congruence lattice of any algebra in the variety is distributive) if and only if there exists some $n > 0$ such that $\mathcal{A}$ has ternary term operations $p_0, p_1, \dots, p_n$ that satisfy the following identities:

$$\begin{array}{lcll} p_0(x,y,z) & \approx & x & \\ p_n(x,y,z) & \approx & z & \\ p_i(x,y,x) & \approx & x & \\ p_i(x,x,y) & \approx & p_{i+1}(x,x,y) & \text{for all even } i \\ p_i(x,y,y) & \approx & p_{i+1}(x,y,y) & \text{for all odd } i \end{array}$$

We will say that an algebra $\mathcal{A}$ lies in $CD(n)$ if it has fundamental operations $p_1, p_2, \dots, p_{n-1}$ satisfying the above identities.

We define two operators which provide a Galois connection between algebras and relational structures: Let $\mathbf{A}$ be a relational structure. Then $\mathrm{Pol}(\mathbf{A})$ is the clone of all operations on the universe of $\mathbf{A}$ which preserve each of the relations in $\mathbf{A}$. Let $\mathcal{A}$ be an algebra. Then $\mathrm{Inv}(\mathcal{A})$ is the relational clone (set of relations closed under constructions via primitive positive formulas) of all relations on the universe of $\mathcal{A}$ which are preserved by each of the operations of $\mathcal{A}$. In other words, the relations of $\mathrm{Inv}(\mathcal{A})$ are all subuniverses of all finite powers of $\mathcal{A}$.

The well-known result of Bulatov, Jeavons and Krokhin [6] states that, when $\mathbf{B}_1$ is a relational structure all of whose relations lie in $\mathrm{Inv}(\mathrm{Pol}(\mathbf{B}))$—or even in $\mathrm{Inv}(\mathrm{Pol}_{id}(\mathbf{B}))$, where $\mathrm{Pol}_{id}(\mathbf{B})$ denotes the idempotent subclone of $\mathrm{Pol}(\mathbf{B})$—then the problem $\mathrm{CSP}(\mathbf{B}_1)$ is not harder than $\mathrm{CSP}(\mathbf{B})$. Therefore, we may say that an algebra $\mathcal{B}$ is tractable, meaning that the problem $\mathrm{CSP}(\mathbf{B})$ is tractable for any relational structure $\mathbf{B}$ with relations in $\mathrm{Inv}(\mathcal{B})$. Having in mind the reduction to idempotent subclone, in this paper we are interested only in finite idempotent algebras and the varieties they generate, that is finite algebras in which each fundamental operation f satisfies the identity $f(x, x, \dots, x) \approx x$.

3. Main Theorem

An algebra $\mathcal{B}$ has *bounded width* if every structure $\mathbf{B}$ with relations in $\mathrm{Inv}(\mathcal{B})$ has bounded width. We are now ready to state the main result of this paper:

Theorem 1. *Every algebra in $CD(4)$ has bounded width.*

Clearly, our result proves also that every algebra which has non-trivial Jónsson terms p_1, p_2 and p_3 has bounded width, as adding operations to an algebra reduces the set of compatible relations, making the set of possible inputs of the related constraint satisfaction problem smaller. Therefore, as in most papers in the area, when an algebra satisfies a Mal'cev-style condition, we immediately assume that the term(s) guaranteed by this condition are all basic operations of the algebra.

The proof of Theorem 1 spans the next two sections. In Section 4 we prove some results concerning ideal free (defined below) simple algebras in $CD(4)$. In Section 5 we start by reducing a k-strategy in $\mathrm{Inv}(\mathcal{B})$ (with $\mathcal{B}$ in $CD(4)$) to one whose components are ideal free. Next, we use the nice properties of the ideal free algebras in $CD(4)$ to reduce the strategy to one whose components are singletons.

4. The structure of relations

Recall that any algebra $\mathcal{B}$ in $CD(4)$ has three basic operations p_1, p_2, p_3. We shall denote $p_2(y,x,x)$ by $l(x,y)$ and $p_2(x,x,y)$ by $r(x,y)$. Note that the Jónsson equations imply that $p_1(y,x,x) = l(x,y)$ and $p_3(x,x,y) = r(x,y)$.

Definition 4. *Let $\mathcal{D}$ be a member of $CD(4)$ and C a nonempty subuniverse of $\mathcal{D}$. We say that C is an* l-ideal *(resp.* r-ideal*) of $\mathcal{D}$ if for every $x, y \in D$, $l(x,y) \in C$ (resp. $r(x,y) \in C$) whenever $x \in C$. The algebra $\mathcal{D}$ is said to be* l-ideal free *if its only l-ideal is itself, and it is said to be* ideal free *it it is l-ideal free and r-ideal free. The* l-ideal of $\mathcal{D}$ generated by an element $a \in D$ *is the smallest l-ideal of $\mathcal{D}$ containing a. An element $a \in D$* generates a minimal l-ideal *if the generated ideal contains no proper subuniverses which are l-ideals of $\mathcal{D}$. Analogous notions can be defined for r-ideals.*

Let $\mathcal{B}$ be a finite algebra in $CD(4)$ and let p_1, p_2, p_3 be the Jónsson terms of $\mathcal{B}$. We shall do some preprocessing over these operations. In particular we want to guarantee that p_1, p_2, and p_3 besides obeying the Jónsson identities satisfy a few more equations. More precisely, we need that $l(x, l(x,y)) = l(x,y)$ and $r(x, r(x,y)) = r(x,y)$.

We shall see how to obtain from p_1, p_2, and p_3, a new family of terms p'_1, p'_2, and p'_3 that satisfies all required identities.

For every x consider the function l_x that maps every element y to $l(x,y)$. There exists some natural n_x such that composing l_x with itself n_x times we obtain a retraction.

We define inductively the sequence of operations $q_1^i(x,y,z)$, $i \geq 0$ with rules: (i) $q_1^0(x,y,z) = x$ and (ii) $q_1^{i+1}(x,y,z) = q_1^i(p_1(x,y,z), y, z)$. It is easy to verify by induction that for every i, q_1^i satisfies the identities: $q_1^i(x,x,y) = q_1^i(x,y,x) = x$, and $q_1^i(y,x,x) = (l_x)^i(y)$. Let us fix p'_1 to be $q_1^{n_1}$ with $n_1 = \prod_{x \in A} n_x$. Similarly define p'_3 and n_3. Finally, define $p'_2(x,y,z)$ as $p_2(q_1^{n_1-1}(x,y,z), y, q_3^{n_3-1}(x,y,z))$. It is easy to verify that p'_1, p'_2 and p'_3 satisfy the required identities.

From now on we fix the finite algebra $\mathcal{B}$ in $CD(4)$ and the variety $\mathcal{V} = \mathcal{V}(\mathcal{B})$. We stipulate that the Jónsson terms p_1, p_2 and p_3 satisfy the additional equations $l(x, l(x,y)) = l(x,y)$ and $r(x, r(x,y)) = r(x,y)$ in $\mathcal{V}$. The following observation is going to be used a good number of times.

Observation 1. *Let $\mathcal{B}_1$ be a finite algebra in $\mathcal{V}$ and let X be a subuniverse of $\mathcal{B}_1$. If X is not an l-ideal of $\mathcal{B}_1$, we can always find some x in X and some x' in $B_1 \setminus X$ such that $l(x,x') = x'$. Same applies to r-ideals.*

Lemma 1. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be finite algebras in $\mathcal{V}$, $\mathcal{B}_1$ l-ideal free, $\mathcal{D}$ be a minimal r-ideal of $\mathcal{B}_2$, and let $\mathcal{R} \leq \mathcal{B}_1 \times \mathcal{D}$ be subdirect. If $B_1 \times \{d\} \subseteq R$ for some $d \in D$ then $R = B_1 \times D$. The same statement holds with l and r changing places.*

Proof. Put $E = \{ e \in D : B_1 \times \{e\} \subseteq R \}$. By our assumption E contains d, and our goal is to show that E is an r-ideal of $\mathcal{B}_2$. Clearly E is a subalgebra. Suppose that E is not an r-ideal. Then there exists $e \in E$ and $e' \in B_2 \setminus E$ such that $p_2(e,e,e') = e'$. Since $e \in D$ and D is an r-ideal of $\mathcal{B}_2$, we get that $e' \in D$. Put $C = \{ c \in B_1 : (c,e') \in R \}$. As R is subdirect and $e' \notin E$, $C \neq \emptyset$ is a proper subuniverse of B_1. We show that C is an l-ideal of $\mathcal{B}_1$. Take $c \in C$ and $a \in B_1$. Then $(a,e), (c,e), (c,e') \in R$ and therefore $(p_2(a,c,c), p_2(e,e,e')) = (l(c,a), e') \in R$, that is $l(c,a) \in C$. This proves that $C = B_1$ which is a contradiction. $\square$

In this section, $\mathcal{B}_1$ and $\mathcal{B}_2$ will always denote finite algebras in $\mathcal{V}$ and $\mathcal{R}$ will be a subdirect product of $\mathcal{B}_1$ and $\mathcal{B}_2$. We shall define G_1, as the subuniverse of $\mathcal{B}_1^2$ that consists of all tuples $(a, a') \in B_1^2$ such that there exists some b such that both (a, b) and (a', b) are in R. We shall regard G_1 as a reflexive graph.

Lemma 2. *If $\mathcal{B}_1$ is simple and R is not the graph of a homomorphism $\mathcal{B}_2 \to \mathcal{B}_1$ then G_1 is connected.*

Proof. Indeed, by composing G_1 with itself a large enough number of times we obtain a graph G_1^* that has an edge precisely in those elements that are connected in G_1. G_1^* is a congruence and hence trivial. If G_1^* is the identity then R is a homomorphism $\mathcal{B}_2 \to \mathcal{B}_1$, which is impossible. Hence we can conclude that G_1^* is $B_1 \times B_1$ and hence G_1 is connected. □

Let $X \subseteq B_1$ and $Y \subseteq B_2$, we shall say that X *sees* Y if for every $y \in Y$ there exists a tuple $(x, y) \in R$ with $x \in X$ and similarly that Y sees X if for every $x \in X$ there exists some tuple $(x, y) \in R$ with $y \in Y$. We shall also say that a sees a set Y meaning that $\{a\}$ sees Y.

An element a of B_1 is said to be 2-fan if it can see two different elements of B_2. Similarly we define 2-fan elements of B_2. Obviously, R is not the graph of homomorphism from $\mathcal{B}_2$ to $\mathcal{B}_1$ iff B_2 contains a 2-fan element.

Lemma 3. *Let $\mathcal{B}_1$ be simple, $\mathcal{R} \leq \mathcal{B}_1 \times \mathcal{B}_2$ be subdirect, and $\mathcal{S}$ be an r-ideal of $\mathcal{R}$. Assume that $\mathcal{R}$ is not the graph of a homomorphism from $\mathcal{B}_2$ onto $\mathcal{B}_1$, and that $\mathcal{S}$ is the graph of a homomorphism of $\mathcal{D} = \pi_2(\mathcal{S})$ onto $\mathcal{C} = \pi_1(\mathcal{S})$. Then $r(c, a) = c$ for all $c \in C$ and $a \in B_1$. The analogous statement with l replacing r everywhere also holds.*

Proof. Since $\mathcal{R}$ is not the graph of a homomorphism and $\mathcal{B}_1$ is simple, G_1 is connected. Clearly, $r(c, c) = c$ for any $c \in C$. By using the connectivity of G_1 it is enough to show that $r(c, a') = c$ whenever $r(c, a) = c$ and $(a, a') \in G_1$. As $c \in C$, there exists $d \in D$ so that $(c, d) \in S$. Let $b \in B_2$ be such that $(a, b), (a', b) \in R$. Then $r((c, d), (a', b)) = (r(c, a'), r(d, b)) \in S$ and $r((c, d), (a, b)) = (r(c, a), r(d, b)) = (c, r(d, b)) \in S$. Since S is the graph of homomorphism we get that $r(c, a') = c$. □

Lemma 4. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be finite algebras in $\mathcal{V}$, where $\mathcal{B}_1$ is simple and ideal free, and let $\mathcal{R}$ be a subdirect product of $\mathcal{B}_1$ and $\mathcal{B}_2$. If B_2 contains a 2-fan element then it also contains an element that sees the whole of B_1.*

Proof. Let us assume that $\mathcal{R}$ is a counterexample to the statement with $|B_1| + |B_2|$ as small as possible. We shall do a separate analysis depending on whether B_2 has a proper ideal and/or congruence. In all cases we shall reach a contradiction.

CASE 1: $\mathcal{B}_2$ is not ideal free. Let Y be a proper l-ideal of $\mathcal{B}_2$. We first prove that Y sees the whole of B_1 by showing that the subset Z of B_1 that contains all elements seen by Y is an l-ideal of B_1. Indeed, let a, a' be any elements of B_1 with $a \in Z$. Let b, b' be elements of B_2 seen by a, a' respectively. Since $a \in Z$ we can assume that $b \in Y$. By applying l to (a, b) and (a', b') we obtain $(l(a, a'), l(b, b'))$. Since $l(b, b') \in Y$ we conclude that $l(a, a') \in Z$.

So, the projection S of R to $B_1 \times Y$ is subdirect and a proper l-ideal of $\mathcal{R}$. As $|B_1| > 1$, the ideal freeness of $\mathcal{B}_1$ implies that there exist elements $a, b \in B_1$ such that $l(a, b) \neq a$. According to Lemma 3, this means that S contains an element which sees two elements of B_1, a contradiction with minimality of (B_1, B_2, R). Analogously we prove that B_2 can have no r-ideals.

CASE 2: $\mathcal{B}_2$ is not simple. Let θ be a non trivial congruence of B_2. Consider now the relation S defined as

$$\{(a, b/\theta) \mid (a, b) \in R\}$$

If b is any element in B_2 that has 2-fan in R, then b/θ has 2-fan in S. By the minimality of R and Lemma 1, $S = B_1 \times B_2/\theta$. Let Y be b/θ (now regarded as a subset of B_2). We have that Y sees the whole of B_1 and contains a 2-fan element, namely b. By the minimality of R, B_2 contains an element that sees the whole of B_1, a contradiction.

CASE 3: $\mathcal{B}_2$ is ideal free and simple. Select any 2-fan element b of B_2 and construct the sequence $Y^0, Y^1, \ldots$, defined inductively by the following rules: (i) $Y^0 = \{b\}$, and (ii) Y^{i+1} is the set of elements seen by Y^i. By Lemma 2 G_1 is connected and therefore the sequence reaches at some point B_1 (and hence also B_2). Let Y^i be the latest element of the sequence before any of B_1 or B_2 occurs. We consider two possibilities: If $Y^i \subseteq B_2$, then it must contain b. Consider the relation S defined as $R \cap (B_1 \times Y^i)$. By the minimality of R, the relation S must contain some element that sees the whole of B_1. The very same element should also see the whole of B_1 in R, a contradiction. If $Y^i \subseteq B_1$ we see that if Y^i has no 2-fan elements, then $Y^{i-1} = Y^{i+1} = B_2$, which contradicts the choice of Y^i. Hence we conclude that there exists an element of B_1 that sees the whole of B_2. By Lemma 1 $R = B_1 \times B_2$ and we are done. □

The next lemma is quite trivial, but its use is going to be essential in the construction of substrategies.

Lemma 5. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be algebras in $\mathcal{V}$ and let $\mathcal{R}$ be a subdirect product of $\mathcal{B}_1$ and $\mathcal{B}_2$.*

1. *If $(a, b) \in R$ generates a minimal l-ideal in $\mathcal{R}$, then a generates a minimal l-ideal in $\mathcal{B}_1$ and b generates a minimal l-ideal in $\mathcal{B}_2$.*
2. *If $a \in B_1$ generates a minimal l-ideal in $\mathcal{B}_1$, then there exists $b \in B_2$ such that $(a, b) \in R$ and (a, b) generates a minimal l-ideal in $\mathcal{R}$.*

Same statements holds for r-ideals.

Proof. To prove statement (1), assume that $(a, b) \in R$ generates a minimal l-ideal in $\mathcal{R}$. Let $\mathcal{C}$ be the ideal of $\mathcal{B}_1$ generated by a. If $\mathcal{C}$ is not minimal, then there exists an element $c \in C$ that does not generate a. However, as $\mathcal{R}$ is subdirect, there is $d \in B_2$ so that $(c, d) \in R$ is generated by (a, b) (just follow the steps in generation of c by a, start from (a, b) and whenever a constant $x \in B_1$ is used, replace it by $(x, y) \in R$). Therefore (c, d) cannot generate (a, b) in $\mathcal{R}$, which is a contradiction with the choice of (a, b).

To prove statement (2), assume that $a \in \mathcal{B}_1$ generates a minimal ideal C of $\mathcal{B}_1$. Since $\mathcal{R}$ is subdirect, there exists $b \in B_2$ such that $(a, b) \in R$. Let $\mathcal{S}$ be the ideal of $\mathcal{R}$ generated by (a, b), and choose $(c, d) \in S$ that generates a minimal ideal $\mathcal{S}'$ of $\mathcal{R}$ (any element of a minimal ideal $S' \subseteq S$ generates it, of course). Since $(c, d) \in S$, then $c \in C$. Then since C is minimal, we see that C is also the ideal of $\mathcal{B}_1$ generated by c. So, as a is in the ideal generated by c, we can generate an element $(a, b') \in S'$. This element generates S', a minimal ideal in $\mathcal{R}$. □

Lemma 6. *Let $\mathcal{B}_1$ be simple, ideal free member of $\mathcal{V}$ and let $\mathcal{B}_2 \in \mathcal{V}$. Let $\mathcal{R} \leq \mathcal{B}_1 \times \mathcal{B}_2$ be a subdirect product that is not the graph of a homomorphism from $\mathcal{B}_2$*

onto $\mathcal{B}_1$. *If* $\mathcal{S}$ *is a minimal* l*-ideal of* $\mathcal{R}$ *or a minimal* r*-ideal of* $\mathcal{R}$*, then* $S = B_1 \times \pi_2(\mathcal{S})$.

Proof. Put $\mathcal{C} = \pi_1(\mathcal{S})$ and $\mathcal{D} = \pi_2(\mathcal{S})$. By Lemma 5, $\mathcal{C}$ and $\mathcal{D}$ are minimal l-ideals of $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. However, $\mathcal{B}_1$ is l-ideal free, thus $C = B_1$. We cannot have $p_2(c, c, a) = c$ for all $a, c \in B_1$ because then every one-element subset of $\mathcal{B}_1$ would be an l-ideal of $\mathcal{B}_1$. Therefore, by Lemma 3, $\mathcal{S}$ is not the graph of a homomorphism. Now $\mathcal{S} \leq B_1 \times D$ is subdirect, then by Lemma 4, there exists $d \in D$ such that $B_1 \times \{d\} \subseteq S$. Now using Lemma 1 we get that $S = B_1 \times D$. □

5. Proof of Theorem 1

If H is a strategy and $I = \{a_1, \ldots, a_i\}$ is a subset of A with $i \leq k$ we denote by H_I the subset of H that contains precisely all the mappings with domain I. An alternative but essentially equivalent view is to fix an order on the elements of I, say $a_1 < a_2 < \cdots < a_i$, and to regard H_I, or rather $H_{a_1,\ldots,a_i}$, as the i-ary relation on B

$$\{(f(a_1), \ldots, f(a_i)) \mid f \in H, \mathrm{dom}(f) = I\}$$

In what follows we shall assume that every relation of $\mathbf{B}$ is in $\mathrm{Inv}(\mathcal{B})$. In this case, it is easy to verify -and widely known- that the strategy H returned by the (j, k)-consistency procedure satisfies the following property: for every $a_1, \ldots, a_i$, $H_{a_1,\ldots,a_i}$ is in $\mathrm{Inv}(\mathcal{B})$. We shall say, somehow abusing notation, that $H \in \mathrm{Inv}(\mathcal{B})$. We shall apply some transformations to the winning strategy returned by the (j, k)-consistency algorithm. In all our transformations this property will be maintained.

We shall also fix the values of j and k. From now on j is assumed to be $k-1$ and k is the maximum between 3 and the largest of the arities of signature σ. Observe that if $j = k-1$ then the (j, k)-forth property can be rephrased as follows: If $f \in H$ with domain $|I| < k$ and $a \in A$ then there exists some $g \in H$ defined on $I \cup \{a\}$ such that $f \subseteq g$. We will call the $(k-1, k)$-forth property the k-forth property from now on.

In order to simplify notation we shall omit the parameter j and we shall speak of k-winning strategy, k-consistency algorithm and so on.

Lemma 7. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be* σ*-structures such that every relation of* $\mathbf{B}$ *is in* $\mathrm{Inv}(\mathcal{B})$ *and let* $H \in \mathrm{Inv}(\mathcal{B})$ *be a* k*-strategy. Then there exists a* k*-strategy* $H' \in \mathrm{Inv}(\mathcal{B})$ *where for every* $a \in A$*,* H'_a *is ideal free.*

Proof. This proof shamelessly duplicates that of Lemma 3.14 in [15]. We include it here for the sake of completeness.

For simplicity of notation we shall use integers to denote the elements of A. Let us assume that H_1 has a proper ideal X. We shall obtain a new k-strategy H' such that $H'_1 = X$ in the following way:

- In the first stage we place in H' every mapping $g \in H$ with 1 in its domain such that $g(1) \in X$ and every one of its subfunctions $f \subseteq g$.
- In the second stage we include in H' all the mappings f of H such that the domain of f has exactly k elements and every one of its proper subfunctions was included in the first stage.

It is routine to verify that H' is nonempty, closed under subfunctions and in $\mathrm{Inv}(\mathcal{B})$. It has to be proved that H' has the k-forth property. We shall present the proof in the case that X is an l-ideal, which uses operation p_1. The proof for X being an r-ideal is obtained analogously by using operation p_3.

Throughout the proof we will use the following fact: if a mapping f is added to H' in the first stage then there exists an extension of f, $g \in H$, with 1 in its domain such that $g(1) \in X$.

Let $f \in H'$ be a mapping with domain I with $|I| < k$ and let i be any element of A. We have to prove that there exists some mapping $h \in H'$ defined on i that extends f. We shall observe first that the only challenging case is when $|I| = k-1$. Indeed, if $|I| < k-1$ the extension h is obtained in the following way: First observe that mapping f can only be added to H' in the first stage. Let $g \in H$ with domain $\{1\} \cup I$ be the extension of f with $g(1) \in X$. Hence by the k-forth property, there exists some extension $h \in H$ of g defined on $\{1, i\} \cup I$. Since $h(1) = g(1) \in X$, g is also included in the first stage, as is its restriction to $I \cup \{i\}$, which extends f.

So for now we shall assume that $|I| = k-1$. The case $1 \in I \cup \{i\}$ is also straightforward: f has to be included in H in the first stage, let $g \in H$ with domain $I \cup \{1\}$ be the extension of f such that $g(1) \in X$. If $i = 1$ then we are done. Otherwise, $1 \in I$ and $f = g$. Mapping f can be extended to $I \cup \{i\}$. The obtained mapping belongs to H' because it is necessarily included in the first stage.

Hence, we can assume that $1 \notin I \cup \{i\}$. This turns out to be the more complicated case. Let us set, for ease of notation, that $I = \{2, 3, 4, \ldots, k\}$ and that $i = k+1$. We shall show that there exists some b_{k+1} such that $(b_2, \ldots, b_{k+1}) \in H'_{2,3,\ldots,k+1}$.

By the k-forth property of H there exists some extension $(b_2, b_3, b_4 \ldots, b_k, u_{k+1}) \in H$ of f, with $b_1 = g(1) \in X$, where g is the extension of f. We conclude that $(b_1, b_2, b_3, b_4 \ldots, b_k) \in H_{1,\ldots,k}$. By applying successively the closure under subfunctions and the k-forth property of H we conclude that H contains some assignments $(b_1, b_3, b_4 \ldots, b_k, v_{k+1})$, $(v_2, b_3, b_4 \ldots, b_k, v_{k+1})$, $(b_1, b_2, b_4, \ldots, b_k, w_{k+1})$, and $(b_2, w_3, b_4, \ldots, b_k, w_{k+1})$ (the domains of the mappings are implicitly indicated by the indexes). Let b_{k+1} be $p_1(u_{k+1}, v_{k+1}, w_{k+1})$. By applying p_1 to $(b_2, b_3, b_4 \ldots, b_k, u_{k+1})$, $(v_2, b_3, b_4 \ldots, b_k, v_{k+1})$, and $(b_2, w_3, b_4 \ldots, b_k, w_{k+1})$ we conclude that the tuple $(b_2, b_3, b_4 \ldots, b_k, b_{k+1})$ belongs to H. We need to show that for all $2 \leq i \leq k$, $(b_2, \ldots, b_{i-1}, b_{i+1}, \ldots, b_{k+1})$ was included in the first stage, or equivalently, that there exists some $c_1 \in X$ such that $(c_1, b_2, \ldots, b_{i-1}, b_{i+1}, \ldots, b_{k+1}) \in H$. There are a number of cases to consider:

- If $i = k+1$ then the tuple $(b_2, \ldots, b_k)$ extends to $(b_1, \ldots, b_k)$, as required.
- If $i = 2$ then by the properties of H we can conclude that H contains some tuples $(x_1, b_3, \ldots, b_k, u_{k+1})$ and $(b_1, y_3, b_4, \ldots, b_k, w_{k+1})$. Applying p_1 to these tuples along with the tuple $(b_1, b_3, b_4, \ldots, v_{k+1})$ we obtain the tuple $(l(b_1, x_1), b_2, \ldots, b_k)$. Since X is an l-ideal, $l(b_1, x_1) \in X$.
- If $i = 3$ or $3 < i < k+1$ then small variations of the previous argument will work.

By repeated application of the procedure we shall obtain the required strategy. □

Lemma 8. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be* σ*-structures such that every relation of* $\mathbf{B}$ *is in* $Inv(\mathcal{B})$ *and let* $H \in \mathrm{Inv}(\mathcal{B})$ *be a* k*-strategy. Assume that for all* $i \in A$, $|H_i| \geq 2$ *and* $\mathcal{H}_i$ *is ideal free. Then there exists a nonempty subset* $M \subseteq A$ *and maximal congruences* ϑ_m *of* $\mathcal{H}_m$ *for all* $m \in M$ *that satisfy the following statements.*

(1) *For any pair* $m_1, m_2 \in M$ *of distinct elements,* $\mathcal{H}_{m_1,m_2} / (\vartheta_{m_1} \times \vartheta_{m_2})$ *is the graph of an isomorphism* $\tau_{m_1,m_2} : \mathcal{H}_{m_1} / \vartheta_{m_1} \to \mathcal{H}_{m_2} / \vartheta_{m_2}$.

(2) *For any* $m \in M$ *and* $n \in A \setminus M$, $\mathcal{H}_{n,m} / (0_{\mathcal{H}_n} \times \vartheta_m) = \mathcal{H}_n \times (\mathcal{H}_m / \vartheta_m)$.

(3) *By setting* $\tau_{m,m}$ *to be the identity on* $\mathcal{H}_m / \vartheta_m$ *for all* $m \in M$, *then* $\tau_{m_1,m_2} \circ \tau_{m_2,m_3} = \tau_{m_1,m_3}$ *for all* $m_1, m_2, m_3 \in M$.

Proof. Let $M \subseteq A$ be of maximal size with respect to satisfying statement (1). Then M is nonempty, as any one-element subset of A satisfies that condition. Assume that statement (2) is not satisfied, that is there exist elements $m \in M$ and $n \in A \setminus M$ such that $\mathcal{H}_{n,m} / (0_{H_n} \times \vartheta_m)$ is not the direct product. Since ϑ_m is a maximal congruence of $\mathcal{H}_m$ we know that $\mathcal{H}_m / \vartheta_m$ is simple, also $\mathcal{H}_n$ and $\mathcal{H}_m$ (and consequently $\mathcal{H}_m / \vartheta_m$) have no ideals, so we can apply Lemma 6 to $\mathcal{H}_{n,m} / (0_{H_n} \times \vartheta_m)$. Hence $\mathcal{H}_{n,m} / (0_{\mathcal{H}_n} \times \vartheta_m)$ is the graph of a homomorphism $\varphi : \mathcal{H}_n \to \mathcal{H}_m / \vartheta_m$. Put $\vartheta_n = \ker \varphi$ and $\tau_{n,m} = \varphi / \vartheta_n$. Clearly, ϑ_n is a maximal congruence of $\mathcal{H}_n$ and $\tau_{n,m}$ is an isomorphism. If $|M| = 1$ then it follows, at this stage, that $M \cup \{n\}$ also satisfies statement (1). This contradicts the maximality of M. Suppose now that $|M| \geq 2$. Let $m' \in M \setminus \{m\}$ be any element. Since H is also a $(2,3)$-strategy, $\pi_{1,2}(H_{n,m,m'}) = H_{n,m}$, $\pi_{2,3}(H_{n,m,m'}) = H_{m,m'}$ and $\pi_{1,3}(H_{n,m,m'}) = H_{n,m'}$. Now the projection of $H_{n,m,m'} / (\vartheta_n \times \vartheta_m \times \vartheta_{m'})$ onto the first two and last two coordinates yield the graphs of the isomorphisms $\tau_{n,m}$ and $\tau_{m,m'}$, respectively, therefore the projection onto the first and last coordinate yields the graph of the isomorphism $\tau_{n,m'} = \tau_{n,m} \circ \tau_{m,m'}$. This proves that $M \cup \{n\}$ also satisfies statement (1), which is a contradiction. Note, that the last argument of the proof proves statement (3) as well. $\square$

With the assumptions of Lemma 8, by fixing an element $m \in M$ and a congruence class of θ_m we can obtain isomorphic congruence classes of θ_n, for all other elements $n \in M$, by means of the isomorphism $\tau_{m,n}$. For all $m \in M$, let C_m be congruence classes obtained in this way.

Lemma 9. *For all* $m \in M$ *let* C_m *be the congruence of* θ_m *defined above. Let* G *be the set of all functions* $g \in H$ *that satisfy the following conditions*

(1) $g(m) \in C_m$ *for all* $m \in \operatorname{dom}(g) \cap M$, *and*

(2) g *generates a minimal* r*-ideal of* $\mathcal{H}_{\operatorname{dom}(g)}$.

Then G *is a* k*-strategy.*

Proof. Clearly G is nonempty, as for any element $m \in M$ and element $b \in H_m$ generates a minimal r-ideal of $\mathcal{H}_m$ and therefore the function g with domain $\{m\}$ with $g(m) = b$ is in G. Clearly, the set of functions satisfying condition (1) is closed under subfunctions, and also the ones satisfying condition (2) because of Lemma 5. We need to prove the k-forth property.

Let $f \in G$ with $|\operatorname{dom}(f)| < k$ and choose $i \in A \setminus \operatorname{dom}(f)$. Put $J = \operatorname{dom}(f)$ and $K = \operatorname{dom}(f) \cup \{i\}$. Note, that H_K is a subdirect product of H_J and H_i, therefore, by Lemma 5, there exists a function $h \in H_K$ such that h generates a minimal r-ideal in $\mathcal{H}_K$ and $h|_J = f$. If $i \notin M$, then $h \in G$ and we are done. So assume that $i \in M$.

If $J \cap M \neq \emptyset$, then for $j \in J \cap M$, $(h(i), h(j)) \in H_{i,j}$, and as $h(j) \in C_j$ we get that $h(i) \in C_i$ as desired. So we can assume that $J \cap M = \emptyset$.

Let $\mathcal{D} = \mathcal{H}_i$, $\mathcal{E} = \mathcal{H}_J$ and $\mathcal{R} = \mathcal{H}_{i,J}$, and $\mathcal{S}$ be the minimal r-ideal of $\mathcal{R}$ generated by the element h. Put $\hat{\mathcal{D}} = \mathcal{D} / \vartheta_i$, $\hat{\vartheta} = \vartheta_i \times 0_{\mathcal{H}_J}$, $\hat{\mathcal{R}} = \mathcal{R} / \hat{\vartheta}$ and $\hat{\mathcal{S}} = \mathcal{S} / \hat{\vartheta}$. Clearly $\hat{\mathcal{D}}$ is simple ideal free, $\hat{\mathcal{R}}$ is a subdirect product of $\hat{\mathcal{D}}$ and $\mathcal{E}$, and $\hat{\mathcal{S}}$ is a minimal r-ideal of $\hat{\mathcal{R}}$.

Assume first that $\hat{\mathcal{R}}$ is the graph of a homomorphism from $\mathcal{E}$ onto $\hat{\mathcal{D}}$ with θ as its kernel, then θ is a maximal congruence in $\operatorname{Con}(\mathcal{E})$, since $\hat{\mathcal{D}}$ is simple. If we denote the kernels of projections as η_j, for $j \in J$, then

$$\theta = \theta \vee \bigwedge_{j\in J} \eta_j = \bigwedge_{j\in J} (\theta \vee \eta_j)$$

(by the distributivity of congruences). As θ is maximal, there exists $j \in J$ so that $\theta \geq \eta_j$. Therefore, $\operatorname{pr}_{i,j} \hat{\mathcal{S}}$, which is an r-ideal of $\operatorname{pr}_{i,j} \hat{\mathcal{R}}$, is the graph of a homomorphism, while $\operatorname{pr}_{i,j} \hat{\mathcal{R}} = \mathcal{H}_{i,j} /(\vartheta_i \times 0_{\mathcal{H}_j})$ is not the graph of a homomorphism, according to the properties of M. It follows, by Lemma 3, that $r(x, y) = x$ for all $x \in \operatorname{pr}_i \hat{S}$ and all $y \in \hat{D}$. This implies that every element of $\operatorname{pr}_i \hat{S}$ is an r-ideal of $\hat{\mathcal{D}}$. This contradiction the fact that $\hat{\mathcal{R}}$ is the graph of a homomorphism. Using Lemma 6 for $\hat{\mathcal{R}}$ and $\hat{\mathcal{S}}$ we get that $\hat{S} = \hat{D} \times \pi_2(\hat{S}) = \hat{D} \times \pi_2(S)$. However, $f = h|_J \in \pi_2(S)$. Therefore there exists an element $g \in S$ such that $g(i) \in C_i$ and $g|_J = f$. As S was a minimal r-ideal, $g \in G$. □

From the strategy G obtained in this lemma we construct a k-strategy $\overline{G}$ in $\operatorname{Inv}(\mathcal{B})$, generated by G. This strategy has functions $\overline{G}_K = \operatorname{Sg}_{\prod_{k\in K} H_k}(G_K)$ for all $K \subseteq A$ of size k. Since C_m is a subalgebra for all $m \in M$, the strategy $\overline{G}$ also satisfies condition (1) of Lemma 9.

We are in a position to prove Theorem 1:

Proof of Theorem 1. By repeated application of Lemmas 7 and 9, using the strategy $\overline{G}$ generated by this lemma, (before each application of Lemma 9 we project the new H to coordinates which are not already singletons) we can construct a winning strategy where for every $i \in A$, H_i is a singleton. Since the arity of every relation is at most k the mapping s sending i to the only element in H_i is a solution. □

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