THE EXISTENCE PROBLEM

Contact Geometry in High Dimensions

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Contact forms

A contact form on a manifold V is a non-vanishing 1-form α whose differential $d\alpha$ at any point is non-degenerate on the hyperplane $\xi = \operatorname{Ker} \alpha$.

Equivalently, α is a contact form if V has odd dimension $2n + 1$ and $\alpha \wedge (d\alpha)^n$ nowhere vanishes.

Notation

$\mathcal{CF}(V)$: the space of contact forms on V.

Almost contact forms

An almost contact form on a manifold V is a pair (α, ω) consisting of a non-vanishing 1-form α and a 2-form ω which is non-degenerate on $\xi = \operatorname{Ker} \alpha$ at any point.

An almost contact form (α, ω) is a contact form iff $\omega = d\alpha$.

Notation

$\widehat{\mathcal{CF}}(V)$: the space of almost contact forms on V.

Forms *vs.* structures

Any hyperplane field ξ on a manifold V has a curvature, namely the bilinear pairing

$$\xi \times \xi \to TV/\xi, \quad (v_p, w_p) \mapsto [v, w]_p/\xi,$$

which is the intrinsic version of $d\alpha$ for $\xi = \operatorname{Ker} \alpha$.

A contact structure is a hyperplane field ξ with non-degenerate curvature.

An almost contact structure is a hyperplane field ξ given with an auxiliary non-degenerate pairing $\xi \times \xi \to TV/\xi$.

The existence problem

Question (Chern 1966)

Under which topological conditions does a manifold V admit a contact form ?

A necessary condition is that V admits an almost contact form.

Equivalent condition

V admits an almost contact form iff $V \times \mathbf{R}$ admits an almost complex structure.

The existence problem and the h-principle

Question

What is the homotopy type of the pair $\widehat{\mathcal{CF}}(V) \supset \mathcal{CF}(V)$? First of all, is the map

$$\pi_0 \mathcal{CF}(V) \to \pi_0 \widehat{\mathcal{CF}}(V)$$

(induce by the inclusion) surjective ? injective ?

Theorem (Gromov 1969)

If V is an open manifold, the map

$$\pi_k \mathcal{CF}(V) \to \pi_k \widehat{\mathcal{CF}}(V)$$

is bijective for any $k \geq 0$.

Closed manifolds

Two contact structures ξ, ξ' on V are isotopic if $\xi' = \phi_* \xi$ for a diffeomorphism $\phi\colon V \to V$ isotopic to the identity.

Theorem (Gray 1958)

If V is a closed manifold, the isotopy classes of cooriented contact structures on V are parametrized by $\pi_0 \mathcal{CF}(V)$.

Theorem (Bennequin 1982, ...)

If V is a closed manifold, the map

$$\pi_0 \mathcal{CF}(V) \to \pi_0 \widehat{\mathcal{CF}}(V)$$

is not injective in general.

status of the existence problem

Conjecture

If V is a closed manifold of arbitrary dimensio, the map

$$\pi_0 \mathcal{CF}(V) \to \pi_0 \widehat{\mathcal{CF}}(V)$$

is surjective.

Theorem (Casals-Pancholi-Presas 2012)

If V is a closed 5-dimensional manifold without 2-torsion in $H^2(V; \mathbf{Z})$ then the map

$$\pi_0 \mathcal{CF}(V) \to \pi_0 \widehat{\mathcal{CF}}(V)$$

is surjective.

Classical examples

1. The space of contact elements over a manifold.

2. The space of 1-jets of functions on a manifold.

3. The contactization of an exact symplectic manifold.

4. The prequantization of an integral symplectic manifold.

5. The boundary of a Stein (or Liouville) domain — in particular, the link of an isolated singularity in a complex hypersurface.

Some historical steps

Lutz 1979 : construction of a contact structure on the 5-torus.

Thomas-Geiges 1977-2001 : construction of contact structures on some "simple" manifolds using topological classification results (Barden, Wall. . .) and the surgery technique of Eliashberg-Weinstein.

Bourgeois 2003 : construction of contact structures on $S_g \times V$ with $g > 0$ (where V is a contact manifold and S_g the surface of genus g) using open books and contact connections.

Partial integration

Let $\widehat{\mathcal{CF}}_0(V) \subset \widehat{\mathcal{CF}}(V)$ denote the space of almost contact forms for which the 2-form ω is exact.

Lemma (Gromov 1969)

For any manifold V, the inclusion map

$$\widehat{\mathcal{CF}}_0(V) \hookrightarrow \widehat{\mathcal{CF}}(V)$$

is a homotopy equivalence.

Basic obstruction

$(\alpha, d\beta)$: an almost contact form ;
v : the (unique) vector field defined by $d\beta(v, .) = 0$ and $\alpha(v) = 1$.

Remarks

If $\beta(v) > 0$ everywhere, then $(\beta, d\beta)$ is a contact form.

If there exists a function f such that $df(v) > -\beta(v)$ everywhere, then $(\beta + df, d\beta)$ is a contact form.

But such an f cannot exist if v, for example, has a periodic orbit along which $\beta(v) < 0$.

Cut-and-paste model

$(W, d\lambda)$: an exact symplectic manifold ;
$\phi\colon W \to W$: a symplectic diffeomorphism such that
$\phi^*\lambda - \lambda = -du$ with $u \geq 0$.

The diffeomorphism

$$\widetilde{\phi}\colon W \times \mathbf{R} \to W \times \mathbf{R}, \quad (z, t) \mapsto (\phi(z), t + u(z))$$

preserves the contact form $\lambda + dt$.

Surgery construction

Cut $W \times \mathbf{R}$ along $W \times \{0\}$, and use $\widetilde{\phi}$ to glue back $W \times \mathbf{R}_+$ on top of

$$Z_u = \{(z, t) \in W \times \mathbf{R} \mid t \leq u(z)\}.$$

Contact surgery

V : a manifold with a contact form α ;

The surgery construction can be applied to any submanifold W of codimension 1 transverse to the Reeb vector field, with λ the 1-form induced by α.

Example

Near a LEgendrian submanifold S, the form α is modeled on the standard contact form $\lambda + dt$ of $T^*S \times \mathbf{R}$ near the zero-section. If S is a sphere and ϕ a Dehn twist in T^*S, the resulting surgery is the Legendrian surgery of Eliashberg-Weinstein.

Contact connections

B : a closed manifold ;
$M \to B$: a smooth bundle with contact fibers.

Lemma (LErman)

There is a one-to-one correspondence between

- *smooth connections whose holonomy maps are contact transformations, and*
- *hyperplane fields transverse to the fibers and inducing on each fiber the given contact structure.*

The connection associated to a hyperplane field is the orthogonal complement (for the curvature form) of its intersection with the fibers.

Positive contact isotopies

Definition

A contact isotopy ϕ_t generated by a time-dependent vector field v_t is :

- non-negative if $\alpha(v_t) \geq 0$;
- positive if $\alpha(v_t) > 0$.

$\mathcal{D}(V, \xi)$: the contact transformation group of (V, ξ) ;
$\widetilde{\mathcal{D}}(V, \xi)$: the universal cover of the identity component.

Partial order :

$$\phi \succeq \psi, \quad \phi, \psi \in \widetilde{\mathcal{D}}(V, \xi),$$

if $\phi\psi^{-1}$ is represented by a non-negative contact isotopy.

Orderability of Contact Manifolds

Definition (Eliashberg-Polterovich 2000)

A contact manifold (V, ξ) is orderable if the relation $\succeq$ defines a non-trivial partial order on $\widetilde{\mathcal{D}}(V, \xi)$.

Lemma (Eliashberg-Polterovich 2000)

A contact manifold is non-orderable iff its contact transformation group $\mathcal{D}(V, \xi)$ contains a positive contractible loop.

Contact connections and positivity

Consider trivial bundles over a surface with fiber a fixed closed contact manifold (V, ξ).

Proposition

A contact connection on $\partial \mathbf{D}^2 \times V$ extends to one defining a contact structure on $\mathbf{D}^2 \times V$ iff its holonomy over $\partial \mathbf{D}^2$ is a negative contact isotopy of (V, ξ).
Moreover, for a closed surface S_g of genus g, the following conditions are equivalent :

- *$S_g \times V$ admits a contact connection defining a contact structure ;*
- *there exist $2g$ element $\widetilde{\phi}_i, \widetilde{\psi}_i \in \widetilde{\mathcal{D}}(V, \xi)$, $1 \le i \le g$, such that $\prod_{i=1}^{g} \widetilde{\phi}_i \widetilde{\psi}_i \widetilde{\phi}_i^{-1} \widetilde{\psi}^{-1} \prec \mathrm{id}.$*

(V, ξ) : a contact manifold ;
(K, θ) : a supporting open book ;
$h\colon V \to \mathbf{C}$: a function such that $K = \{h = 0\}$ and $\theta = \arg h$;
α : a Pfaff equation of ξ whose Reeb vector field is transverse to the pages of (K, θ).

Claim

The contact vector fields v, w uniquely determined by

$$\alpha(v) = \operatorname{re} h, \quad \alpha(w) = \operatorname{im} h$$

satisfy $\alpha([v, w]) > 0$.

Weinstein structures

F : a compact $2n$-manifold.

Definition

A Weinstein structure on F is a symplectic form ω admitting a primitive λ such that :

- the dual vector field $\vec{\lambda}$ points transversely outward along ∂F ;
- $\vec{\lambda}$ is gradientlike for a Morse function that is constant on ∂F.

The pair (F, ω) is a Weinstein domain.

Lefschetz property

A weinstein domain retracts onto an n-dimensional subcomplex.

Open books and contact forms

$(F, d\lambda)$: a Weinstein domain ;
ϕ : a symplectic self-diffeomorphism of F relative to $K = \partial F$.

The associated open book, namely the closed manifold

$$V = \mathrm{OB}(F, \phi) = \mathrm{MT}(F, \phi) \cup_\partial (K \times \mathbf{D}^2)$$

where

$$\mathrm{MT}(F, \phi) = (F \times [0, 1])/\sim, \quad (z, 1) \sim (\phi(z), 0),$$

has a “natural” contact form.

Open books and almost contact forms

V : a closed $2n+1$-manifold with $n \geq 3$.

Proposition

Every component of $\widehat{\mathcal{CF}}(V)$ contains an almost contact form (α, ω) for which there exists an open book such that :

- *(α, ω) induces a positive almost contact form on the binding ;*
- *ω is non-degenerate on the interior of each page ;*
- *the page retracts onto an n-dimensional subcomplex.*

Speculations on Weinstein structures

F : a compact $2n$-manifold retracting onto an n-dimensional subcomplex ;
ω : a non-degenerate 2-form on F ;
ϕ : a self-diffeomorphism of F relative to ∂F and preserving the homotopy class of ω.

Question

Does F admit a Weinstein structure homotopic to ω and for which ϕ (possibly after stabilization) is isotopic to a symplectic diffeomorphism relative to the boundary ?