Contact topology in higher dimensions

The American Institute of Mathematics

The following compilation of participant contributions is only intended as a lead-in to the AIM workshop "Contact topology in higher dimensions." This material is not for public distribution.

Corrections and new material are welcomed and can be sent to workshops@aimath.org

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A.1 Fraser, Maia

In contrast to the situation for the group $\mathrm{Ham}(M, \Omega)$ of Hamiltonian symplectomorphisms of a symplectic manifold (M, Ω), where Hofer's metric has been intensely studied, no non-trivial bi-invariant metrics were known on $\mathrm{Cont}_0(V, \lambda)$, the identity component of the group of contactomorphisms of a contact manifold (V, λ), until the work of Sandon in 2009. In her thesis [Sandon2009] she defined a bi-invariant *integer-valued* metric for compactly supported contactomorphisms in the case $V = T^*\mathbb{R}^n \times S^1$, via her generating function-based spectral invariants $c()$, $d_S(\phi, \psi) := \lfloor c(\phi\psi^{-1}) \rfloor + \lfloor c(\psi\phi^{-1}) \rfloor$. For this metric, distance to the identity $d_S(\phi, id)$ of ϕ with support in a domain $\mathcal{V}$ is bounded above by $\lfloor c(\psi) \rfloor$ for any ψ which displaces $\mathcal{V}$ (and so by $E(\mathcal{V})$, the infimum of such $\lfloor c(\psi) \rfloor$). Analogously one may define $d_{S,\max}(\phi, \psi) := \max(\lfloor c(\phi\psi^{-1}) \rfloor, \lfloor c(\psi\phi^{-1}) \rfloor)$, in which case it is easy to see that $d_{S,\max}(\phi, id)$ takes on all integer values from 0 up to the displacement energy $E(\mathcal{V})$ for ϕ supported in $\mathcal{V}$. Sandon's norm (d_S or $d_{S,\max}$) is therefore bounded on $\mathrm{Cont}_0(\mathcal{U} \times S^1)$ for every bounded $\mathcal{U} \subset T^*R^n$ but is unbounded on the whole $\mathrm{Cont}_0(T^*R^n \times S^1)$.

Recently, three works have independently addressed extensions to the above: Zapolsky also uses generating functions and defines a spectral-invariant based conjugation-invariant norm on $V = T^*X \times S^1$ for arbitrary compact X; Colin and Sandon define a conjugation-invariant norm, the *discriminant norm*, on the universal cover of the group $\mathrm{Cont}_0(V, \lambda)$ and use generating functions to show it is unbounded in the cases $V = T^*\mathbb{R}^n \times S^1$ and $\mathbb{R}P^{2n-1}$.

Recently, together with Polterovich, we have been considering the setting of (V, λ) for which the Reeb flow is 1-periodic. We define a conjugation-invariant norm on the universal cover of the identity component of the group G of contactomorphisms of the form $e_t\phi$ where ϕ is compactly supported and e_t is the Reeb flow. Specifically, we use the notion of dominance developed in [Eliashberg-Polterovich2000] and let $d(\{\phi_t\}, id) := \max(\nu(\{\phi_t\}), \nu(\{\phi_t^{-1}\}))$, where $\nu(\{f_t\})$ is the lowest power of $e = \{e_t\}_{t\in[0,1]}$ which dominates $\{f_t\}$. We can then show that in the presence of the "stable intersection property" (which holds for example in $V = T^*X \times S^1$ via standard Floer theory when X is closed) one may obtain lower bounds on the bi-invariant norm $d(\{\phi_t\}, id)$, which allow to conclude that this norm has infinite diameter on the subgroup G_0 of contactomorphisms generated by isotopies with support in a bounded tube containing the zero section. In particular, this norm is unbounded on certain cyclic subgroups $\{f^n\}$ and hence potentially one can study the interplay between the geometry of such subgroups and the dynamics of f.

Since Sandon and Colin as well as Polterovich and myself will all be participants at AIM, I suggest forming a mini-session to discuss these recent advances and explore future steps.

A.2 Geiges, Hansjoerg

My main interest in this workshop is to learn about the recent work of Giroux on the existence of contact structures in higher dimensions, and the contribution of Murphy to this work of Giroux. I also plan to continue work with van Koert (on a joint project with Fan Ding) on diagrammatic representations of contact 5-manifolds.

A.3 Massot, Patrick

This is a description of some problems I'd like to discuss during the workshop.

Developping tools to find bLobs.

In dimension 3, it is often hard to prove that a contact structure is tight. But it is usually much easier to to exhibit an overtwisted disk whose existence is suspected in an explicitely described contact manifold. In higher dimension, even this seemingly easier task has been difficult up to now. I would be very interested in new methods allowing to find bLobs, or any other generalization of overtwisted disks, especially inside contact manifolds described by a supporting open book [Giroux_ICM]. The examples I have in mind are of course negative stabilizations but also maybe contact structures constructed using flexible Stein manifolds (Stein manifolds having a handle decomposition where critical handles are attached along flexible Legendrians in the sense of [Murphy]).

Examples of flexible contact structures.

Of course, one great goal of contact topology in higher dimensions is to find a class of flexible contact structures. A more modest goal could be to understand some explicit examples of non-obvious isotopies between contact manifolds that seem overtwisted. For instance, contact structure obtained by Lutz-Mori twists [Mori_Lutz] on the same submanifold of a given contact manifold.

Bounds on fillable contact structures.

The road to the Colin-Giroux-Honda finiteness theorems [CGH] for tight contact structures in dimension 3 began with Eliashberg's bound on the Euler class of a fillable contact structure in [Eliashberg_filling]. The fact that it relied on holomorphic curves techniques gives some hope of extending it to higher dimensions. For instance, on a given 5-manifold, can you get infinitely many classes in H^2 which are first Chern classes of a fillable contact structure?

Obstructions to strong fillability.

Recently, [Massot_WSF] introduced the definition of weak fillings of contact manifolds in higher dimensions. Obstruction to weak fillability have been used to indirectly prove obstructions to strong fillability. We also have examples of contact structures on 5-manifolds which are weakly but not strongly fillable. Natural candidates in any dimensions are contact structures constructed by Bourgeois [BourgeoisTori] on $M \times T^2$ starting from a weakly fillable structure on M. It seems we need new technology to detect cases where they are not strongly fillable.

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A.4 McLean, Mark

I am very interested in studying exotic contact structures in higher dimensions especially on spheres and also exotic Legendrians on spheres. There are many many examples.

I am also interested in questions about the uniqueness/non-uniqueness of algebraic Stein fillings of contact manifolds as well in higher dimensions. Algebraic Stein fillings are Stein fillings where the associated Stein manifold is a smooth affine variety. One then has techniques from algebraic geometry to tackle these problems

A.5 Mori, Atsuhide

1) Higher dimensional Bennequin theory.

We say that a contact submanifold M of the unit $S^{2n-1} \subset \mathbb{C}^n$ is spinning if $\arg z_1|M$ defines an open-book supporting the induced contact structure on M. Generalizing Markov moves, I defined stabilizations of spinning contact submanifolds. For example the standard spinning

$$\Sigma = \{z_{k+1} = z_{k+2} = \cdots = z_n = 0\} \cap S^{2n-1}$$

can be stabilized in $S^{2k+1} = \{z_{k+2} = \cdots = z_n = 0\} \cap S^{2n-1}$ to

$$\begin{aligned} \Sigma^+ &= \{\varepsilon z_1 = z_2^2 + \cdots + z_{k+1}^2,\, z_{k+2} = \cdots = z_n = 0\} \cap S^{2n-1} \quad \text{and} \\ \Sigma^- &= \{\varepsilon \overline{z_1} = z_2^2 + \cdots + z_{k+1}^2,\, z_{k+2} = \cdots = z_n = 0\} \cap S^{2n-1}. \end{aligned}$$

It is easy to interpolate these stabilizations by smooth isotopies, and the positive stabilization by a contact isotopy. A typical Hopf plumbing can be realized as a stabilization as long as the initial contact manifold is realized as a spinning. Particularly in the case where $(n, k) = (3, 2)$, I found the following interesting path from Σ to Σ^-; first isotoping Σ till a critical position ("the first wall") where it becomes an open-book by Legendrian disks; then continuing it as an isotopy of a contact submanifold Σ' ($\approx -\Sigma$ with regarding the continuity) till another critical position ("the second wall") where it becomes the Reeb foliation on the 3-sphere realized as a union of Legendrian submanifolds of S^5; and then continuing it as an isotopy to Σ^-. This negative stabilization is a "tour" across the walls partitioning the space of contact submanifolds.

In '91 Montesinos and Morton considered a branched covering $\pi : M^3 \to S^3$ which associates an open-book O of M^3 to a braid B, where they took B as the ramification locus and O as the pull-back of the standard open-book. Moreover, given a Markov move of B, they constructed a new branched covering $\pi' : M^3 \to S^3$ which associates to it a Hopf plumbing of O. Loi and Piergallini, interpriting the Eliashberg theorem on the topology of Stein manifolds, showed that there exists a Montesinos-Morton lift of a quasipositive braid which is a positive open-book supporting a given Stein fillable contact structure.

On the other hand, in '00, Orevkov proved that a braid is quasipositive if (and only if) its positive stabilization is quasipositive. He also pointed out that a negative stabilization of a braid is never quasipositive. This apparently corresponds to the result of Torisu and Giroux on the overtwistedness of the negative Hopf plumbings. A closed braid is a spinning 1-dimensional contact submanifold of S^3, and a 1-dimensional contact submanifold of S^3 is commonly called a transverse link. Bennequin contact-isotoped a given transverse link to a closed braid. Thus a closed braid can be used as a presentation of a transverse link. Orevkov-Shevchish in '03 and Nancy Winkle in '02 independently showed that any two presentations of a given transverse link can be related by a sequence of positive stabilizations/destabilizations (and conjugations). This apparently corresponds to Giroux's theorem which says that two supporting open-books can be related by a sequence of positive Hopf plumbings/deplumbings.

In order to account for these two correspondance phenomena, I would like to propose the above notion of stabilization of spinnings.

Problem 1. I hope someone can generalize the complementary works of Birman and Menasco (and Winkle) to the Bennequin theory to higher dimension. Here the theory does not means the inequality itself, which can be deduced from the fillability. It means his operation on Seifert surfaces spanning braids, particularly on a certain stabilization *reducing* the complexity of Seifert surfaces (essential difference from Thurston's reduction in taut foliation theory).

Problem 1'. (see my preprints arXiv:0906.3237 and 1111.0383 for the details) Show the Thurston-Bennequin inequality for convex hypersurfaces in S^{2n-1}.

Since cooriented contact $(2k-1)$-manifolds are oriented, it can be embedded in S^{4k-3} (hard Whitney embedding ($k > 2$), Hirsh embedding ($k = 2$)). I showed a realizability theorem as spinning contact submanifolds, however it was weaker than the above embedding result in dimension. Martinez Torres improved it for $k > 2$. Now any closed contact $(2k-1)$-manifold can be immersed in S^{4k-3} and embedded in S^{4k-1} as spinning contact submanifolds. Although one may consider that the immersion result is enough satisfactory, I can realize most contact 3-manifolds as embedded spinnings in S^5. This leads us to conjecture the best possible dimension, i.e., any closed contact $(2k-1)$-manifold could be embedded in S^{4k-3} as a spinning contact submanifold.

Problem 2. Embed a given closed contact 3-manifold into the standard S^5 as a spinning contact submanifold.

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2) System of PDEs defining codimension one foliation

Let us consider a smooth n-dimensional submanifold M^n of the standard S^{2n-1} which is a union of Legendrian submanifolds of S^{2n-1}. Since S^{2n-1} is a one-point compactification of the 1-jet space $J^1(n-1,1)$ of a single function with $(n-1)$ variables, the submanifold M^n can be considered as a system of $(n-1)$ first order PDEs which defines a codimension-1 foliation or similar thing. Such a submanifold is at present said to be uniquely integrable. (bLob is also a uniquely integrable submanifold, but it does not appear in the standard S^{2n-1}.) One may associate the notion of unique integrability with the above conjecture on spinning hard Whitney embedding. Indeed I isotoped the standard $S^3 \subset S^5$ through contact submanifolds to a uniquely integrable non-analytic submanifolds carrying the Reeb foliation, and then to an overtwisted negative contact submanifold. (I used it in the above interpolation between Σ and Σ^-.)

Problem 3. Find another uniquely integrable submanifold of S^5 or S^{4k-3} and perturb it to a contact submanifold.

Problem 3'. Find another uniquely integrable "wall" partitioning the space of contact submanifold.

Note that a Legendrian submanifold L^{2k-1} of S^{4k-1} can be perturbed into a contact submanifold as long as L^{2k-1} admits a contact structure. However L^{2k-1} can also be perturbed into another contact submanifold as long as L^{2k-1} admits another contact structure. Thus a Legendrian submanifold of S^{4k-1} has no contact property for $k > 1$. That is why we would like to consider uniquely integrable submanifolds rather than a single Legendrian submanifold.

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3) Singularity theory

A regular inetersection of a holomorphic object with the unit sphere is usually a spinning submanifold. This leads us to apply contact topology of spinnings to singularity theory on embedded holomorphic submanifolds.

Problem 4. Consider not only the topology (as a contact manifold) but also the braiding (as a contact spinning) of the link of a space surface singularity.

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4) Confoliations and leafwise symplectic foliations

An almost contact structure on M^{2n+1} is a G-structure by the 1-jets of local contact transformations of $(J^1(n,1),0)$. It can be indicated by a pair $([\alpha],[\omega])$ of conformal classes of a 1-form α and a 2-form ω on M^{2n+1} which satisfies $[\alpha]\wedge[\omega]^n > 0$. A contact form α determines an almost contact structure just by putting $\omega = d\alpha$. Then the almost contact structure is said to be exact. Another typical example of an almost contact structure is a leafwise almost symplectic structure of a codimension one foliation. Recently, motivated by efforts of Verjovsky, Mitsumatsu constructed a leafwise symplectic spinnable foliation on S^5. Here we say that a foliation is spinnable if it is associated to an open-book. Following Mitsumatsu's construction, I constructed a family of almost contact structures which starts with the exact one and ends with a leafwise symplectic structure through ones with $\ker[\alpha]$ contact. Note that an almost contact structure on a 3-manifold is nothing but fixing a transverse direction to a co-oriented and oriented plane field. Eliashberg and Thurston considered a path of plane fields connecting a 3-dimensional contact structure to a codimension one foliation. Here they required that the path stays in the closure $\overline{\mathcal{C}}$ of the space $\mathcal{C}$ of contact structures. Slightly changing the original definition, we call a plane field in $\overline{\mathcal{C}} \cup \mathcal{I}$ a confoliation where $\mathcal{I}$ denotes the space of completely integrable plane fields. Here is a natural generalization of this notion to higer dimension: Let $\mathcal{C}$ be the subspace of the space of almost contact structures $([\alpha],[\omega])$ consisting of those which $\ker[\alpha]$ are contact structures, and $\mathcal{I}$ the subspace consisting of leafwise (almost) symplectic foliations. Then each element of $\overline{\mathcal{C}} \cup \mathcal{I}$ is called a confoliation (Note that Altschuler-Wu defined another notion with the same name.) The above result is the first phenomenon in higher dimensional confoliation theory.

Problem 5. Find other phenomena in higher dimensional confoliation theory.

Problem 6. Generalize the Novikov closed leaf theorem to leafwise symplectic case. Note that the recent work of Meigniez implies that the Novikov theorem never hold for higher dimensional smooth foliations. Also note that Ibort and Martínez Torres studied almost contact structures with closed representatives $\omega \in [\omega]$, which are called 2-calibrated structures, and showed that 2-calibrated foliations can be obtained by fattening 3-dimensional taut foliations.

A.6 Murphy, Emmy

I am particularly interested in flexibility phenomena in high dimensional contact topology. A result of this type is one I gave in my thesis, available at

http://arxiv.org/abs/1201.2245

Briefly; though Legendrian embeddings exhibit a lot of rigidity which can be detected with pseudo-holomorphic curves, my thesis defines a class of Legendrian embeddings, called loose Legendrians, that are classified (existence and uniqueness) by their formal invariants.

I expect other flexibility results could be found and better understood during the timespan of the conference. A long outstanding question is high dimensional overtwistedness: the problem of finding a flexible class of contact structures on closed manifolds. I am very interested in this problem for its own sake, and have some new ideas. Though loose Legendrians exist in every contact manifold, I conjecture in my thesis that a Legendrian in an overtwisted manifold is loose if and only if it has overtwisted complement. I believe this result should immediately follow from the basic flexibility properties of overtwisted contact structures, once these are established.

Another interesting idea in this direction relates loose Legendrian embeddings with the contact non/squeezing phenomenon. Specifically, it is known that $\mathrm{S}^1 invariant domains in \mathrm{S}^1 \times R^{2n}_{std}$ admit a discrete capacity, which obstructs contact squeezings only for domains larger than a certain minimal size. It seems likely that this non-squeezing phenomenon can be detected by non-loose Legendrian embedddings in these domains. Conversely, the squeezability of contact domains below the threshold size might be interpreted as a flexibility result for Legendrians contained in these domains.

Consider the space of all Legendrian embeddings of a closed manifold L. We are typically interested in connected components of this space with the C^∞ topology. Instead, equip this space with the C^0 topology; in this topology the set of loose Legendrians is dense. I also expect it is C^0 open. I do not have any specific methods to attack this problem currently however.

There are a number of other interesting open questions concerning loose Legendrians, for instance their relations with Lagrangian cobordisms, and even basic existence questions for non-loose Legendrians. I'll stop listing potential questions for brevity, but many of these questions can be answered with known invariants (Legendrian contact homology), and interesting examples could likely be constructed in the span of a week with no new techniques being introduced.

A.7 Ng, Lenny

I am interested in all of the main topics to be addressed by the workshop, but particularly in the study of Legendrian submanifolds and relative Symplectic Field Theory in high dimensions. As part of an ongoing joint project on knot contact homology with John Etnyre as well as Tobias Ekholm and Michael Sullivan, we have developed techniques to count the holomorphic curves with Legendrian boundary that arise in Legendrian contact homology in arbitrary dimension. These techniques rely on gradient flow trees and are computable for a large family of Legendrian 2-tori in five dimensions. I would like to understand how to extend these tools to other circumstances, including even higher dimensions. Because of this, I am very interested to learn about the state of the art in contact topology in dimensions 5 and above.

A.8 Niederkrueger, Klaus

The main interest of my research has been finding qualitative properties of contact manifolds in dimension five and higher. Typical examples are properties that have been modeled and adapted according to the three dimensional case, and include overtwistedness, different degrees of fillability, Giroux torsion etc.

Significant progress has been made in this area during the last years, and many examples of manifolds have been constructed that display these properties, showing in particular that these properties really exist. Unfortunately, none of these contact structures have been found naturally, but all had to be constructed with the explicit aim in mind of finding a given property.

This is the reason why I feel that the whole theory is only in the prototype stage, where every part needs still to be carefully assembled by hand, and to move the theory to the next step, one would need to develop tools to effectively analyze given contact structures.

I will be more explicit: In my opinion, a useful theory should be able to provide the tools to understand at least the following three families of contact manifolds:

Giroux has defined negative stabilization of higher dimensional open books. There is no doubt that structures obtained this way correspond in some abstract sense to the property of being overtwisted. Are these structures also PS-overtwisted? I have worked for many years on this question, and every one of my attempts shows "almost" that they are PS-overtwisted. What is the situation here? Is the definition of PS-overtwisted too strong, or do we simply miss the correct tools to check that these manifolds are actually PS-overtwisted?

Contact structures which are S^1-invariant provide in dimension three, one of the most important sources of examples to test new theories on. While very special, they are still general enough to display many different properties that can often be directly read off from their orbit spaces. In higher dimensions, there are invariant contact structures where one can often do the same (Mori, Massot-N-Wendl) and which have lead to a tentative definition of Giroux torsion. But again, examples had to be carefully manufactured by hand, and it would be important to develop this theory further to apply it to "real world examples" of S^1-invariant contact structures.

Finally, and while already included in the previous example, Bourgeois showed that given a contact manifold M with an open book decomposition, one can obtain a contact structure on $M \times T^2$. In fact, these manifolds are contact fibrations, and historically they provided the building blocks to construct the first PS-overtwisted contact structures (Presas).

Unfortunately, the question remains open if these manifolds are sometimes already PS-overtwisted from the beginning. More precisely, the question would be: Start with a contact manifold M that has property X, does the manifold $M \times T^2$ share the same property or not. For example it is easy to show (again by hand) that if M is weakly fillable then $M \times T^2$ will be weakly fillable; if M is subcritically Stein fillable then M will be Stein fillable (for the canonical open book), but on the negative side, we do not have any information.

A.9 Pancholi, Dishant

My principal interests are in understanding the meaning of overtwistedness in higher dimensions and constructions of contact structures on almost-contact manifolds. There are many interesting questions related to these topics, for example:

1) Relationship between so called PS-overtwistedness and open-books.

2) Questions related to the possible extension of the result about existence of contact structure on 5-fold to higher dimensions.

A.10 Presas Mata, Francisco

I am particularly interested in understanding the existence problem of contact structures in higher dimensions. My last contribution is a preprint in which, under fairly general hypothesis, we prove that any 5-dimensional almost contact structure does admit a contact structure. I could explain the ideas behind the proof of that result during the conference.

Other topics in which I am interested:

a) Restrictions to fillability in contact geometry.

b) Orderability of the contactomorphism group and its relations with non-fillability.

c) Homotopy classification of contact structures (wild topic, at least some ideas in dimension 5).

A.11 Sandon, Sheila

My main interests for this workshop are in contact rigidity phenomena such as contact non-squeezing, the existence of bi-invariant metrics and quasimorphisms on the contactomorphism group, orderability of contact manifolds and Morse estimates for translated points of contactomorphisms. Although their symplectic analogues have played since the 80's a crucial role in the development of symplectic topology, these phenomena are still quite mysterious and not so well understood in the contact case.

The notion of orderability of contact manifolds was introduced by Eliashberg and Polterovich in 2000. A contact isotopy is said to be *positive* if it moves every point in a direction positively transverse to the contact distribution. In some cases (for the so-called *orderable* contact manifolds) this notion induces a bi-invariant partial order on the universal cover of the contactomorphism group. Orderability turns out to be sensitive to the topology of the underlying manifold (for example $\mathbb{R}P^{2n-1}$ is orderable while its universal cover S^{2n-1} is not) and, as discovered by Eliashberg, Kim and Polterovich in 2006, related to a contact non-squeezing phenomenon for domains. Up to now contact non-squeezing has been observed only for domains in $\mathbb{R}^{2n} \times S^1$. More precisely, it has been proved that if k is an integer and R_1, R_2 two real numbers with $R_2 < k < R_1$ then the prequantization of the ball of radius R_1 cannot be squeezed into the prequantization of the ball of radius R_2. The special role played by the integers is strictly related to the 1-periodicity of the Reeb flow of $\mathbb{R}^{2n} \times S^1$. It is not known, and I think an interesting problem to be discussed in the workshop, whether in more general contact manifolds the existence of a 1-periodic Reeb flow, or more generally the existence of closed Reeb orbits, should imply some contact non-squeezing phenomena for domains.

The original motivation of Eliashberg and Polterovich for introducing the notion of orderability was that if (M, ξ) is an orderable contact manifold then we can apply a general procedure, that works for any partially ordered group, to associate to the universal cover of the contactomorphism group of (M, ξ) a partially ordered metric space $\big(Z(M, \xi), \delta\big)$. This is done by first defining, in terms of the *relative growth*, a pseudo-distance δ on the group of elements of the universal cover that are generated by a positive contact isotopy, and then by quotienting this group by the equivalence classes of elements which are at zero distance from each other. This construction has been the first geometric structure to be associated to the contactomorphism group. After this, integer-valued bi-invariant metrics on the contactomorphism group itself have been discovered in the case of $\mathbb{R}^{2n} \times S^1$ (by myself in 2009) and $T^*B \times S^1$ (by Zapolsky in 2012). There is currently work in progress by Fraser and Polterovich to define integer-valued bi-invariant metrics on the contactomorphism group of a class of circle bundles, including $T^*B \times S^1$. In joint work with Vincent Colin we recently found a way to define an integer-valued bi-invariant metric on the universal cover of the contactomorphism group of any contact manifold. This bi-invariant metric has a very simple geometric definition and is always non-degenerate. However in some cases it can be bounded, and hence in some sense equivalent to the trivial metric. This is the case for example for $\mathbb{R}^{2n+1}$. On the other hand we can prove that our metric is unbounded at least in the cases of $\mathbb{R}^{2n} \times S^1$ and $\mathbb{R}P^{2n-1}$. Again, as for the non-squeezing phenomenon, this fact relies heavily on the periodicity of the Reeb flow. We would like very much to have some feedback about this metric from the participants of the workshop. Interestingly enough our method does not work to show that the metric is unbounded also in the case of S^{2n-1}, and we suspect that this might be related to the fact that S^{2n-1} is non-orderable while $\mathbb{R}P^{2n-1}$ is orderable. It is in fact conjectured by Polterovich that any bi-invariant metric on the contactomorphism group of S^{2n-1} might be equivalent to the trivial metric. If this conjecture was confirmed, it would then show that the existence of a 1-periodic Reeb flow would not be enough to guarantee the existence of a bi-invariant metric on the contactomorphism group. A similar phenomenon has been observed also in the context of quasimorphisms. In 1990 Givental defined a quasimorphism on the universal cover of the contactomorphism group of $\mathbb{R}P^{2n-1}$ (the *non-linear Maslov index*, which is by now the only known example of a quasimorphism on a contactomorphism group). The construction relies also on the topology of $\mathbb{R}P^{2n-1}$ and does not work on its universal cover S^{2n-1}. I find this interplay between contact rigidity phenomena and the topology of the underlying manifold quite mysterious, and I would be happy to hear some discussion about this during the workshop.

The last problem that I would like to mention is the existence of translated points for contactomorphisms. A point q in a contact manifold M is called a *translated point* of ϕ (with respect to a fixed contact form) if $\phi(q)$ belongs to the same Reeb orbit as q, and the contact form is preserved at q. Translated points play a crucial role in the proof of the non-squeezing theorem in $\mathbb{R}^{2n} \times S^1$ and in the constructions of the bi-invariant metric in $\mathbb{R}^{2n} \times S^1$ and of the non-linear Maslov index in $\mathbb{R}P^{2n-1}$. I suspect that they might play an important role also in the study of contact rigidity phenomena in more general contact manifolds. Translated points can be regarded as the natural contact analogue of fixed points of Hamiltonian symplectomorphisms, because an analogue of the Arnold conjecture seems to be true for translated points of contactomorphisms which are contact isotopic to the identity. More precisely, it can be conjectured that on a closed contact manifold the number of translated points for a contactomorphism isotopic to the identity is always greater or equal than the minimal number of critical points of a function on the manifold. This has been proved for S^{2n-1} and $\mathbb{R}P^{2n-1}$, and for any contact manifold if the contactomorphism is $\mathcal{C}^0$-small. I think that it would be interesting to develop a Floer homology theory for translated points, by imitating the original construction of Floer for fixed points of Hamiltonian symplectomorphisms. I think that such Floer homology theory might turn out to be an important tool to study contact rigidity phenomena (but I would like to hear the opinion of more experienced participants in the workshop regarding the doability of this project).

A.12 Traynor, Lisa

I am quite interested in studying Legendrian submanifolds in 1-jet bundles using the technique of generating families. Much of my work has focused on defining and studying generating family homology groups for a Legendrian submanifold; these homology groups parallel the linearized contact homology groups defined through the theory of pseudo-holomorphic curves. Recently, Sabloff and I have found obstructions to the existence of Lagrangian cobordisms between Legendrian submanifolds; here the Lagrangians and the Legendrians have compatible generating families. Bourgeois, Sabloff and I are currently investigating how one can use the technique of generating families to construct Lagrangian cobordisms between Legendrian submanifolds. Through the construction of cobordisms, we have been able to address some fundamental geography and botany questions about higher dimensional Legendrian submanifolds. For example, we now understand which polynomials can be realized as the generating family polynomial of an n-dimensional Legendrian submanifold (with a generating family), and we can show that when n is greater than 1, there are infinitely many Legendrians (having generating families) with the same classical invariants.

A.13 Vertesi, Vera

I am mostly interested in the relation between the two different generalizations of overtwistedness: -plastikstufe -and open books that are negative stabilizations