New connections between dynamical systems and PDE's

The American Institute of Mathematics

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Chapter A: Open problems

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1. (Albert Fathi) Let f be a smooth vector field on a compact manifold M and let $\phi_f^t : M \to M$ be its flow. There is a standard way to include the flow ϕ_f^t in the flow of a positive definite Lagrangian system. Take a Riemannian metric on M and let $\| \ \|$ be the corresponding norm. Define a Lagrangian $L : TM \to \mathbf{R}$ by

$$L(x, v) = \frac{1}{2}\|v - f(x)\|^2.$$

Then the zero section $M \subset TM$ of the tangent bundle is an invariant manifold for the Lagrangian flow. The Aubry set $A_0 \subset TM$ corresponding to the zero cohomology class $0 \in H^1(M, \mathbf{R})$ is contained in M.

Problem: give a characterization of A_0 in terms of the dynamics of the flow ϕ_f^t. In particular, does A_0 contain the chain recurrent set of the flow ϕ_f^t?

2. Let $H : T^*M \to \mathbf{R}$,

$$H(x, p) = \frac{1}{2}\|p\|^2 - f(x) \cdot p$$

be the Hamiltonian corresponding to L. Consider the Hamilton–Jacobi equation

$$H(x, Du(x)) = \frac{1}{2}\|Du(x)\|^2 - f(x) \cdot Du(x) = 0.$$

Question: Under what conditions is $u = 0$ the unique viscosity solution?

3. (John Mather) Let M be a compact manifold and $L : TM \times \mathbf{T} \to \mathbf{R}$ a C^∞ Lagrangian satisfying the usual hypotheses of the Mather theory – convexity, superlinearity and completeness. Let $A_c \subset M$ be the Aubry set corresponding to the cohomology class $c \in H^1(M, \mathbf{R})$. Define a pseudo metric ρ_c on A_c as follows. Modify the Lagrangian by subtracting a closed 1-form from the cohomology class c and adding a constant $\alpha(c)$ so that for the new L_c we have

$$\inf \left\{ \int L_c \, d\mu \mid \mu \text{ is an invariant probablility measure on } TM \times \mathbf{T} \right\} = 0.$$

Then set

$$h_c^n(x, y) = \inf \left\{ \int_0^n L_c(\gamma(t), \dot\gamma(t), t) \, dt \mid \gamma : [0, n] \to M \text{ connects } x \text{ and } y \right\}$$

and

$$\rho_c(x, y) = \liminf_{n\to\infty} h_c^n(x, y) + \liminf_{n\to\infty} h_c^n(y, x).$$

Define an equivalence relation on A_c by $x \sim y$ iff $\rho_c(x, y) = 0$. Let $\bar A_c$ be the corresponding quotient space.

Question: Is it true that $\bar A_c$ is totally disconnected? Does $\bar A_c$ have zero Hausdorff dimension?

4. (Albert Fathi) Is it true that for generic $L \in C^\infty$ there exists only a one-parameter family of viscosity solutions of the corresponding Hamilton–Jacobi equation?

5. (Patrick Bernard) Consider an analytic positive definite Lagrangian $L : \mathbf{T}^n \times \mathbf{R}^n \times \mathbf{T} \to \mathbf{R}$, $L = L(x, v, t)$, satisfying Mather's conditions. Let
$$\alpha_L(c) = -\inf_{\mu} \int (L(x, v, t) - c \cdot v)\, d\mu$$
be Mather's function $\alpha_L : \mathbf{R}^n \to \mathbf{R}$.

   Question: Is it true that if α_L is analytic with positive definite second derivative, then the Lagrangian system is completely integrable?

6. (Gonzalo Contreras) Is it true that under the same condition on α_L, each Aubry set A_c covers the whole configuration space $\mathbf{T}^n$?

7. (John Mather) Let $f : \mathbf{T} \times \mathbf{R} \to \mathbf{T} \times \mathbf{R}$ be an analytic twist map. Suppose that $\mathbf{T} \times \mathbf{R}$ is foliated by invariant curves Γ_c (which are Aubry-Mather sets). Are these curves Γ_c necessarily analytic?

8. (Vadim Kaloshin) Consider a billiard inside a region bounded by a closed convex curve $\Gamma \subset \mathbf{R}^2$. Suppose that a neighborhood of Γ is foliated by caustics for the billiard. Is it true that then Γ is an ellipse and hence the caustics are analytic curves? The billiard in Γ is described by a twist map f, and caustics correspond to invariant curves of f. Hence this questions is a particular case of the previous one.

9. (John Mather) Does there exist a C^3 twist map $f : \mathbf{T} \times \mathbf{R} \to \mathbf{T} \times \mathbf{R}$ which has an invariant curve Γ with irrational rotation number and such that $f|_\Gamma$ is not conjugate to a rotation? By the Denjoy example there exist C^1 maps of a circle with irrational rotation number not conjugate to a rotation. Can this happen for invariant curves of twist maps?

10. (Patrick Bernard) The existence of KAM invariant tori of a Hamiltonian system on $\mathbf{T}^n \times \mathbf{R}^n$ is equivalent to the existence of regular solutions of the Hamilton–Jacobi equation $H(x, c + Du(x)) = \alpha(c)$. Hence KAM theory can be regarded as regularity result for viscosity solutions.

    Question: Is it possible to obtain proofs of KAM results using this connection?

11. (Albert Fathi) For a Hamiltonian $H(x, p)$ on $\mathbf{T}^n \times \mathbf{R}^n$, which is superlinear and strictly convex in p, we have a fairly good description both in dynamical terms (Mather theory) and in PDE terms (viscosity solutions) of the Aubry set. It would be nice to understand the situation where H is still superlinear but not necessarily convex:

    - Does there exist an Aubry set from the PDE point of view (for example as a uniqueness set for viscosity solutions)?
    - Does there exist an Aubry set from the point of view of dynamics, i.e. a canonical graph invariant under the Hamiltonian flow?
    - Is there a relationship between the two sets if they exist?

12. Let c_0 be the infimum of the c's such that the Hamilton-Jacobi equation $H(x, Du(x)) = c$ with superlinear H admits a global viscosity subsolution $u : \mathbf{T}^n \to \mathbf{R}$. Let SS be the set of global viscosity subsolutions of $H(x, Du(x)) = c_0$. Define a function S on $\mathbf{T}^n \times \mathbf{T}^n$ by
$$S(x, y) = \sup\{u(x) - u(y) \mid u \in SS\}.$$

Then for fixed x, the function $S(x,\cdot)$ is a viscosity subsolution of $H(x, Du(x)) = c_0$ on $\mathbf{T}^n$, and a viscosity solution on $\mathbf{T}^n \setminus \{0\}$. When H is convex in p, the Aubry set A_0 is the set of $x \in \mathbf{T}^n$ such that $S(x,\cdot)$ is a global viscosity solution on the whole $\mathbf{T}^n$.

Question: If H is not necessarily convex, does there exist $x \in \mathbf{T}^n$ such that $S(x,\cdot)$ is a viscosity solution on the whole $\mathbf{T}^n$?

13. (Walter Craig) Consider a positive definite Lagrangian $L : TM \times \mathbf{T} \to \mathbf{R}$. Let $\alpha : H^1(M, \mathbf{R}) \to \mathbf{R}$ be Mather's α-function.

    Problem: Relate the regularity of the Mather set M_c and the arithmetic properties of the frequency set $\omega = D\alpha(c) \subset H_1(M, \mathbf{R})$.

14. (Walter Craig) This question concerns the minimax Birkhoff orbits in an area preserving twist map of an annulus. It is well known that

    - There exist at least two periodic orbits of a twist map with given rational rotation number $\alpha = p/q$, namely a minimal orbit and a minimax orbit.
    - Taking a limit $\alpha_j \to \omega \in \mathbf{R} \setminus \mathbf{Q}$ of rational rotation numbers, the minimal Birkhoff orbits converge to the Mather set Π_ω.
    - If Π_ω is an invariant circle, then the minimax orbits also converge to Π_ω.

    Question: What is the fate of the minimax orbits for the case when the Mather set is a Cantor set?

15. (Gonzalo Contreras) Let M be a compact manifold, $L : TM \to \mathbf{R}$ a convex superlinear Lagrangian, and $\tilde{L} : T\tilde{M} \to \mathbf{R}$ – lift of L to the universal cover $\tilde{M}$ of M. Let c_u be the Mane critical value for $\tilde{L}$.

    Question: Is it true that

    $$c_u = \inf\{h \in \mathbf{R} \mid \exists x \in \tilde{M} \forall y \in M \exists \text{trajectory with energy } h \text{ joining } x \text{ to } y\}.$$

16. Let M be a compact manifold, $L : TM \to \mathbf{R}$ a convex superlinear Lagrangian. The only example known of an energy level without periodic orbits or singularities is $\{H = c_u\}$ with c_u a Mane critical value for the lift of the Lagrangian to $\tilde{M}$.

    Question: Is there an example with other energy levels without periodic orbits or singularities?

17. Let M be a compact manifold, $H : TM \to \mathbf{R}$ a convex superlinear Hamiltonian. The energy level $\Sigma_h = \{H = h\}$ is of contact type if there exists a 1-form λ on Σ_h such that $d\lambda$ is the symplectic 2-form on T^*M and $\lambda(X_H) > 0$, where X_H is the Hamiltonian vector field. Let c_u be the critical value on $\tilde{M}$ and let

    $$e_0 = \min\{h \in \mathbf{R} \mid \pi(\Sigma_h) = M\}.$$

    Question: Is it true that if $M \neq \mathbf{T}^2$, then for all $e_0 < h < c_u$ the energy level Σ_h is not of contact type?

18. Let M be a non-compact manifold and $L : TM \to \mathbf{R}$ a convex superlinear Lagrangian satisfying appropriate completeness conditions at infinity. There is a compactification

of M by adding an "extended Aubry set" whose points correspond to "Busemann viscosity solutions" of the Hamilton–Jacobi equation (Calc. Var. 13 (2001), 427–458).

Problem: Understand the geometry of this compactification.

19. Let $L : \mathbf{T}^n \times \mathbf{R}^n \to \mathbf{R}$ be an autonomous convex superlinear Lagrangian and H the corresponding Hamiltonian. Let $c_0 = \alpha(0)$ be the Mane critical value.

    Question: Is it true that for any $h < c_0$ the set

    $$\{(x,p) \in \mathbf{R}^{2n} \mid H(x,p) < h\}$$

    has finite symplectic capacity?

20. Let $H : \mathbf{T}^n \times \mathbf{R}^n \to \mathbf{R}$ be an autonomous convex superlinear Hamiltonian. Suppose that the Aubry set A_0 satisfies $\pi(A_0) = \mathbf{T}^n$ and there exists a unique (up to a constant) viscosity solution of the Hamilton–Jacobi equation $H(x, Du(x)) = \alpha(0)$. For $c \in \mathbf{R}^n$, let A_c be the Aubry set corresponding to the Hamiltonian $H_c(x,p) = H(x, p - c)$. Let v_c be a viscosity solution of $H_c(x, Dv_c(x)) = \alpha(c)$.

    Question: is it true that

    $$\lim_{|c|\to 0} \|Dv_c - Du\|_{\mathrm{Lip}\,(\pi(A_c))} = 0.$$

21. Mane proved that for a generic positive definite Lagrangian $L : TM \times \mathbf{T} \to \mathbf{R}$ and generic $c \in H^1(M, \mathbf{R})$, the minimizing measure in the Mather set M_c is unique.

    Problem: For generic L and c is the unique minimizing measure in M_c supported in a periodic orbit?

    Conjecture (Jeff Xia): For a generic L and *all* $c \in H^1(M, \mathbf{R})$, the number of ergodic invariant measures in M_c is finite.

22. (Kostia Khanin) Consider a convex superlinear positive definite Lagrangian on $\mathbf{T}^n \times \mathbf{R}^n$ with white noise perturbation:

    $$L(x,v,t) = L_0(x,v) + \sum_{i=1}^{N} F_i(x)\dot{w}_i(t),$$

    where $w_i(t)$ are independent Brownian motions. Suppose that the map $F = (F_1, \ldots, F_n) : \mathbf{T}^n \to \mathbf{R}^N$ is an embedding. Then with probability 1 there exists a unique global minimizer $\gamma : \mathbf{R} \to M$.

    Conjecture: With probability 1, the minimizer γ is a hyperbolic trajectory of the Lagrangian flow.

    This is proved (E–Khanin–Mazel–Sinai) for $n = 1$.

23. (Arnold diffusion) Consider a C^r, $3 \le r \le \omega$, Hamiltonian $H_\varepsilon : \mathbf{T}^n \times \mathbf{R}^n \times \mathbf{T} \to \mathbf{R}$ of the form

    $$H_\varepsilon(x,p,t) = H_0(p) + \varepsilon H_1(x,p,t).$$

    Suppose that H_0 is positive definite and superlinear.

Conjecture: Suppose that $n \geq 2$. Then for given open sets $U_0, U_1 \subset \mathbf{R}^n$ and typical in the C^r topology H_1, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, there exists a trajectory $(x(t), p(t))$ of the Hamiltonian system such that $p(t_0) \in U_0$ and $p(t_1) \in U_1$.

This conjecture seems very hard in the C^ω category, when the change of the action is exponentially slow in ε by the Nekhoroshev theorem. The case of finite r should be easier.

A precise definition of "typical" needs to be established. Mather proved this conjecture for $n = 2$ and for a cusp residual set of perturbations εH_1 in the C^∞ topology. This doesn't prove the above conjecture: for given H_1 the set of admissible ε does not cover an interval $(0, \varepsilon_0)$.

Question: Does Mather's theorem holds for an εH_1 in a cusp residual set of trigonometric polynomials of high order N? Exponentially small perturbations of coefficients are allowed, so the transcendental problem of exponentially small separatrice splitting is avoided.

24. Consider an apriori unstable Hamiltonian of the form

$$H(\theta, I, x, y) = H_0(I) + F(x, y) + \varepsilon H_1(\theta, I, x, y),$$

on $\mathbf{T}^n \times \mathbf{R}^n \times \mathbf{R}^2$, where H_0 is convex and superlinear, the Hamiltonian $F : \mathbf{R}^2 \to \mathbf{R}$ has a separatrix loop, and H_1 is a generic perturbation. The large gap problem of Arnold diffusion was overpassed recently for such systems by variational methods of Mather (Xia), by geometrical methods using secondary KAM-tori (de la Llave, Delshams, Seara), and by the method of separatrix map (Treschev).

Problem: Understand the relation between the variational method and the method of separatrix map. They seem similar in spirit.

25. (Paul Rabinowitz) Is there a reasonable PDE analogue of Arnold's diffusion?

It seems that a version of finite dimensional transition chains is not the right mechanism for PDE. There are results of Kuksin which prove "diffusion" for PDE with respect to high order Sobolev norms.

The same problem for infinite lattices like Fermi–Pasta–Ulam system.

26. (Victor Bangert) Let $L : \mathbf{T}^n \times \mathbf{T} \times \mathbf{R}^n \to \mathbf{R}$, $L = L(x, u, p)$ be a smooth multi-dimensional Lagrangian satisfying the usual convexity assumptions in $p \in \mathbf{R}^n$. For example,

$$L(x, u, p) = \frac{1}{2}|p|^2 + F(x, u),$$

where the potential $F : \mathbf{T}^{n+1} \to \mathbf{R}$ is periodic in all variables.

A function $u : \mathbf{R}^n \to \mathbf{R}$ is called minimal if it minimizes the action functional

$$\int L(x, u(x), Du(x))\, dx$$

for all variations with compact support. The set of minimizers u "without self-intersections" is very well understood. Here $u : \mathbf{R}^n \to \mathbf{R}$ is said to be "without self-intersections" if the projection to $\mathbf{T}^{n+1}$ of graph$(u) \subset \mathbf{R}^{n+1}$ is a hypersurface without self-intersections. In particular, for every minimizer u without self-intersections there exists a "rotation vector" $c \in (\mathbf{R}^n)^*$ such that

$$|u(x) - u(0) - c \cdot x|$$

is bounded by a constant that only depends on L. Conversely, for every $c \in (\mathbf{R}^n)^*$ there exists a minimizer u without self-intersections such that $|u(x) - c \cdot x|$ is bounded.

Question: Is there an analytical condition implying that the minimizers satisfying this condition have no self-intersections.

More concrete question: Suppose $u : \mathbf{R}^n \to \mathbf{R}$ is minimal and $|Du(x)|$ is bounded. Is it true that the graph of u in $\mathbf{T}^{n+1}$ has no self-intersections. For partial results see V. Bangert, Ann. Inst. Henri Poincaré – Analyse non linéaire 6(1989), 95–138, in particular Sect. 8.

27. (Victor Bangert) Let M^n be a compact Riemannian manifold and let C_qM be the set of closed q-currents on M:

$$C_qM = \{T \in (\Omega^q(M))^* : T(d\alpha) = 0 \text{ for all } \alpha \in \Omega^{q-1}(M)\}.$$

Every q-current T defines a homology class $[T] \in H_q(M, \mathbf{R})$ and a mass

$$M(T) = \sup\{T(\omega) : \omega \in \Omega^q(M),\ \|\omega\|_\infty = 1\}.$$

Every homology class $h \in H_q(M, \mathbf{R})$ has a representative with minimal mass.

For $q = 1$ the supports of minimal currents consist of minimal geodesics. For $q = n-1$ any minimizer T is given by a measured lamination by minimizing hypersurfaces (possibly with singularities).

Problem: What can one say about minimal currents for $1 < q < n - 1$.

28. (Franz Auer) Conjecture: If $T \in C_{n-1}M$ is a minimizing closed current, then the Hausdorff dimension of the corresponding singularity set is at most $n - 7$.

29. Define a stable norm on $H_q(M, \mathbf{R})$ by

$$\|h\| = \inf\{M(T) : [T] = h\}.$$

This norm is an analog of Mather's β-function).

Problem: Study convexity and differentiability properties of the unit ball

$$B = \{h \in H_q(M, \mathbf{R}) : \|h\| \le 1\}.$$

30. (Victor Bangert) Let M^{2n} be a manifold with an almost complex structure J. A pseudo-holomorphic line is a map $f : \mathbf{R}^2 \to M$ satisfying the PDE: $f_y = Jf_x$. Moser proved that for an almost complex structure J on $\mathbf{T}^{2n}$ which is close to a standard complex structure J_0, and any sufficiently irrational vector $v \in \mathbf{R}^{2n}$, there exists a foliation of $\mathbf{T}^{2n}$ by pseudo-holomorphic curves which is conjugate to a linear foliation by real 2-planes containing the vector v.

Problem: Develop a global (non-perturbative) theory of such foliations or laminations.

The theory is expected to be particularly rich for the case $\mathbf{T}^4$ due to the positivity of the intersection number of pseudoholomorphic curves. This is an analog of the intersection number for geodesics on $\mathbf{T}^2$ – the basis for Hedlund's results on minimal geodesics.

31. (Craig Evans) Consider the Hamiltonian $H(x,p) = \frac{1}{2}|p|^2 + V(x)$, where the potential V is $\mathbf{T}^n$ periodic. The corresponding Hamiltonian operator in quantum mechanics is then $-\frac{h^2}{2}\Delta + V(x)$.

    Much exciting research in "semiclassical analysis" concerns studying the limit as $h \to 0$ of solutions $u(h)$ of the eigenvalue problem

    $$\frac{h^2}{2}\Delta u(h) + V(x)u(h) = E(h)u(h) \qquad (*)$$

    and finding connections with classical Hamiltonian dynamics.

    Problem: Do Mather sets play any role here? Or, conversely, can we somehow "quantize" Mather sets? This would presumably mean to build quasimodes (ie approximate solutions of (*)) corresponding to Mather's sets and to prove good error bounds. Would some sort of Diophantine condition be useful here?

32. (Takis Souganidis) Homogenization problem for random Hamilton–Jacobi equation

    $$F(D^2u, Du, u, x/\varepsilon, x, \omega) = 0$$

    with non-convex stationary ergodic in x/ε Hamiltonian F.

33. (Elena Kosygina) Let Ω be a probability space with probability measure P ergodic under a shift transformation group $\tau_x : \Omega \to \Omega$, $x \in \mathbf{R}^n$. Consider the stochastic Hamilton–Jacobi equation

    $$u_t + H(D_x u, \tau_x \omega) = 0, \qquad (x,t) \in \mathbf{R}^n \times (0,\infty),$$

    where $H(p,\omega)$ is convex in p and satisfies some regularity assumptions, for example

    $$H(p,\omega) = \frac{1}{2}|p|^2 - V(\omega).$$

    A homogenization result for this problem was obtained by Lions and Souganidis. For the Hamiltonian above homogenization is equivalent to a large deviation result for Brownian motion among random obstacles (Snitzman).

    Problem: Derive a large deviation result for general stationary ergodic setting without any independence or mixing assumptions on P under the shifts.

34. (Diogo Gomes) Investigate possible extensions of the techniques used in the Aubry–Mather theory and Monge–Kantorovich problems to study linear programming problems in infinite dimensions. An example of such extensions are the stochastic Mather measures which can be used to analyze second order Hamilton–Jacobi equations.

35. (Diogo Gomes) For a small perturbation of a completely integrable Hamiltonian system a viscosity solution corresponding to a non-resonant unperturbed torus can be uniformly approximated by formal expansions. Similar expansions can be constructed for the density of the Mather measures. However, it is not known how well Mather measures themselves are approximated by these formal expansions.

36. (Massimiliano Berti) For a nonlinear wave equation

    $$u_{tt} - u_{xx} = f(u), \qquad f(0) = f'(0) = 0$$

with Dirichlet boundary conditions and general nonlinearity there exist a large number of small amplitude periodic orbit with fixed period. Could one find small amplitude periodic orbits of large period and quasi-periodic solutions for the wave equation?