Exact crossing numbers

The American Institute of Mathematics

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A.1 Aichholzer, Oswin

My main interest in this workshop is on the crossing number of the complete graph K_n, where I'm especially interested in the following two variations:

1) The rectilinear crossing number: Here the vertices of K_n are points in the Euclidean plane, and the edges are straight segments connecting these points. This version is also called the geometric crossing number. In recent years, several observations have been made on the structure of point sets such that the resulting crossing number is minimized. Also the exact value of the rectilinear crossing number for up to 30 points (with one exception) has been computed. Still, up to now there is no closed formula and actually not even a conjecture on the exact value of the rectilinear crossing number for arbitrary n.

2) The crossing number of good drawings of the complete graph K_n: This graphs are also called simple complete topological graph. Vertices are distinct points and edges are non-self-intersecting continuous curves connecting two points; edges do not pass through vertices and any pair of edges intersects at most once, either in a proper crossing or a common end point. The Harary-Hill Conjecture from 1958 gives a closed formulare for the (conjectured) crossing number of these graphs. In the last two years several classes have been shown to fulfill the Harary-Hill Conjecture: 2-page book drawings, monotone-drawings, cylindrical drawings, x-bounded drawings. Also interesting connections to other structures, like k-sets or rotation systems, have been identified, and seem to be very promising to otain further progress.

A.2 Fulek, Radoslav

I am particularly interested in the crossing number and its variants of the complete graph K_n. Let $cr(K_n)$ denote the crossing number of K_n. In my recent joint paper with Martin Balko and Jan Kynčl we stated a conjecture that would imply the value of $cr(K_n)$ conjectured by Harary and Hill for every natural number n.

Let $E_k(D)$ denote the number of generalized k-edges in a drawing D of a complete graph. Intuitively, a k-edge has precisely k vertices on one of its sides. Our conjecture states that for every drawing D of a complete graph on n vertices and $0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 2$ we have $E_{\leq\leq\leq k}(D) \geq 3\binom{k+4}{4}$, where $E_{\leq\leq\leq k}(D) = \sum_{i=0}^{k} E_{\leq\leq i}(D)$, $E_{\leq\leq k}(D) = \sum_{i=0}^{k} E_{\leq i}(D)$ and $E_{\leq k}(D) = \sum_{i=0}^{k} E_i(D)$.

We know that the conjecture holds for x-monotone, shellable, and some other closely related classes of drawings, where in an x-monotone drawing we require that every edge is intersected by every vertical line at most once, and shellable drawings generalize slightly x-monotone drawings. It would be interesting to enlarge the class of drawings of K_n for which our conjecture holds.

A.3 Hogben, Leslie

I hope to learn more about recent developments for the exact crossing number and rectilinear exact crossing number, including Zarankiewicz's Conjecture and the Harary-Hill Conjecture.

I have a background in combinatorial matrix theory, and in particular have worked with the Colin de Verdière number $\mu(G)$, which characterizes planarity. I am interested in exploring whether μ might be helpful in the study of crossing numbers. It seems doubtful that there is any direct relationship, since $\mu(K_n) = n-1$ but for $3 \leq p \leq q$, $\mu(K_{p,q}) = p+1$. But the techniques involved might be useful.

A.4 McQuillan, Dan

I am very interested in crossing numbers of complete graphs and complete bipartite graphs. My 3 most recent papers on this subject are all joint work with R.B. Richter, and one also with S. Pan. A key strategy in these works involves a "strong induction" approach; instead of considering drawings of K_{n-1} in an optimal drawing of K_n, we instead consider drawings of K_m in an optimal K_n for m much smaller than n. The ideas were strong enough to give a complete conceptual proof that $cr(K_9) = 36$.

A.5 Mohar, Bojan

At the workshop I would like to discuss crossing numbers of graphs of bounded genus. One particular question is to find precise dependence of the crossing number of projective planar graphs in terms of the dual edge-width and its variations.

A.6 Mutzel, Petra

I am interested in algorithmical questions concerning crossing number computations. E.g., there are approximation algorithms for the crossing number of almost planar graphs and apex graphs with bounded degree (e.g., Cabello, Mohar 2008, Gutwenger, Mutzel, Weiskircher 2005, Chimani, Gutwenger, Mutzel, Wolf 2009, and Chimani, Hlineny, Mutzel 2012) . What happens if a constant number of high degree vertices is present? Current algorithms will not guarantee good approximations; however, I think that it is possible to find new algorithms to handle these problems. A second line of research could be the following. Recently, partial planarity problems have been studied where subsets of the edges $F \subseteq E$ (and sometimes also their embedding) are given, which ask for embeddings of the whole graph $G = (V, E)$ fulfilling certain properties, e.g., F must be crossing free or the embedding of F must be fixed. Up to my knowledge, so far no crossing number theory or algorithms for these cases have been studied.

A.7 Nicholson, Emlee

My interest in the Exact Crossing Numbers workshop stems from several factors. The first is that I am interested in gaining expertise on crossing numbers and the techniques used in their arguments. My previous research has been of the extremal variety, investigating the minimum degree sum of a graph and some of its implications. The topic of crossing numbers appeals to me in part because of its interesting history and clear applications. Primarily though, I enjoy structural and counting type arguments. Secondly, I am interested in meeting new people with similar interests and forming new collaborative relationships. Finally, I am interested in jump starting a new project and getting in position to spend the summer and beyond making progress on an interesting problem. I am open to a variety of specific issues for discussion at the workshop: expanding the known body of graphs for which the Harary-Hill conjecture holds to include a new class, looking at previously unconsidered constructions of K_n, or considering the possible implications of the crossing number to other properties of the graph. I am still reading the literature and learning more about the existing body of work on crossing numbers. I look forward to working with you!

A.8 Ramos, Pedro

Harary-Hill (also attributed to Guy) Conjecture (1962-63) states that the crossing number of the complete graph is at least

$$Z(n) := \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$$

Up to now, two families of drawings achieving this lower bound have been found: cylindrical drawings and 2-page drawings. Recently, it has been shown that for those families of drawings, the known lower bound is tight.

In [1] the class of *t-shellable* drawings is introduced, and it is shown that the already known families of drawings achieving the conjectured optimal number of crossings are t-shellable (for $t \geq n/2$. Furthermore, it is also shown that, if $t \geq n/2$, a t-shellable drawing has at least $Z(n)$ crossings. This suggest two main questions:

A. find families of optimal drawings that are not t-shellable
B. extend the lower bound to a more general class of drawings

[1] http://arxiv.org/abs/1309.3665

A.9 Richter, Bruce

Determining the crossing numbers of the complete graph K_n and complete bipartite graph $K_{m,n}$ are long-standing open problems. In recent years, I have been involved with progress for the former.

With Pan, we used a computer to determine the crossing number of K_{11} (and K_{12} comes for free). With McQuillan and Pan, we developed some theory and then used a computer to show that the crossing number of K_{13} is not 217 (the counting lower bound from K_{12}).

Most interestingly, with McQuillan, we have a truly "by hand" proof that the crossing number of K_9 is 36. The methods are radically different from the existing proofs, which test all possible drawings arising from an optimal K_8. We have hopes to continue this line of work to get K_{11} "by hand". It is my belief that, if these kinds of methods can the Harary-Hill Conjecture up to about K_{15}, then we will have enough information to see how to prove the entire conjecture.

There are many other interesting open questions in the area.

A. What is the crossing number of the n-dimensional cube? Faria et al recently confirmed the Erdös-Guy Conjecture that it is at most

$$4^n\frac{5}{32} - \lfloor\frac{n^2+1}{2}\rfloor 2^{n-2}.$$

B. A long-time favourite of mine: what is the crossing number of the Cartesian Product $C_m \square C_n$? For $m \leq n$, this is known to be at most $(m-2)n$, with equality for $m \leq 7$ and, in a wonderful paper by Glebsky and Salazar, for $n \geq m(m+1)$.
Is the crossing number of $C_8 \square C_8$ really 48? Why is $C_m \square C_m$ so hard?

A.10 Salazar, Gelasio

I expect to learn from the experts the latest developments on Hill's and Zarankiewicz's Conjectures, as well as share our latest results (jointly obtained with other participants). With so much talent and expertise put together, we should be able to come up with an ambitious, yet doable research program to tackle particularly interesting cases of these conjectures.

A.11 Sparks, Athena

I am currently preparing my master's thesis on the crossing number of a cylindrical drawing of a complete bipartite graph, $K_{m,n}$. I have derived a formula for evaluating the crossing number of $K_{m,n}$ and an algorithm for creating a drawing that achieves that number of crossings. First, a labeling scheme is defined on drawings of $K_{m,n}$, and then a formula is constructed for a lower bound on the number of crossings given by that drawing. The formula for the crossing number has been derived by minimizing this multivariable function and showing that a drawing exists that achieves that minimum.

A.12 Szekely, Laszlo

A graph is biplanar, if its edge set can be written as the union of the edge sets of two planar graphs. An excellent survey on them is L.W. Beineke [Biplanar graphs: a survey, *Computers Math. Applic.* **34**(11) (1997) 1–8]. The biplanar crossing number of a graph G, $cr_2(G)$, was introduced in electrical engineering, A. Owens [On the biplanar crossing number, *IEEE Trans. Circuit Theory* **18** (1971) 277–280]. The definition of the biplanar crossing number is the minimum of the sum of the (ordinary) crossing numbers of two graphs obtained by a bipartition of the edge set of the graph. One defines k-planar crossing numbers analogously.

É. Czabarka, O. Sýkora, L.A. Székely and I. Vrťo [Biplanar crossing numbers I: a survey of results and problems, in: *More Sets, Graphs, and Numbers*, eds. E. Győri, G.O.H. Katona, L. Lovász, Springer 2006] investigated, among others, biplanar crossing numbers of complete bipartite graphs. We obtained that

$$cr_2(K_{5,q}) = \left\lfloor \frac{q}{12} \right\rfloor \left(q - 6 \left\lfloor \frac{q}{12} \right\rfloor - 6 \right)$$

and conjectured

$$cr_2(K_{6,q}) = 2 \left\lfloor \frac{q}{8} \right\rfloor \left(q - 4 \left\lfloor \frac{q}{8} \right\rfloor - 4 \right),$$

supporting this conjecture with two drawings with this number of crossings. Even more, for both problems above, if q is even, the stated number of crossings can be achieved such that $K_{5,q}$ or $K_{6,q}$ is bipartitioned into two isomorphic graphs to be drawn on the two planes. We did not dare to state a conjecture for $cr_2(K_{7,q})$. In this paper we posed a problem whether $cr_2(P_n \times C_n \times C_n \times C_n)$ is zero. To our surprise, Joshua Lambert in his Ph.D. thesis proved that this is the case, even if we allow different lengths for the path and each and every cycle.

F. Shahrokhi, O. Sýkora, L.A. Székely and I. Vrťo [On k-planar crossing numbers, *Discrete Appl. Math.* **155** (2007) 1106–1115] determined exact k-planar crossing numbers of some complete bipartite graphs. We showed for $k \geq 2$, $q \geq 1$ that

$$cr_k(K_{2k+1,q}) = \left\lfloor \frac{q}{2k(2k-1)} \right\rfloor \left(q - k(2k-1) \left\lfloor \frac{q}{2k(2k-1)} \right\rfloor - 1 \right),$$

and for $k \geq 2$, $q \geq 1$ that

$$cr_k(K_{2k+2,q}) \leq 2\left\lfloor \frac{q}{2k^2} \right\rfloor \left(q - k^2 \left\lfloor \frac{q}{2k^2} \right\rfloor - k^2) \right),$$

and equality holds for $1 \leq q \leq 4k^2$.

A.13 Toth, Csaba

Is the intersection graph of the (open or closed) edges in a straight-line of drawing of K_n recognizable. Kratochvil and Matousek (JCTB, 1994) show that the recognition of the intersection graph of line segments is complete for the existential theory of the reals (see also Schaefer (GD, 2009)).

What is the smallest planar point set S_n such that every n-vertex graph G admits a straight-line embedding where the vertices are mapped into S_n and attaining the rectilinear crossing number of G? Bienstock (DCG, 1991) proved that a rectangular section of the integer lattice into which K_n can be embedded with straight-line edges and realizing the rectilinear crossing number of K_n requires exponential area, but S_n may not be a section of the integer lattice.

A.14 Toth, Geza

One of my most important research topics is the theory of representations, drawings of graphs. In particular, topological graphs, geometric graphs, and crossing numbers of graphs. I have some results about the best possible constant in the Crossing Lemma, crossing number of random graphs, relationships between different versions of the crossing number, and some others. So, this workshop, about exact crossing numbers, is of great interest for me.

A.15 Vogtenhuber, Birgit

My main interest in this workshop is to learn more about and to do research on the crossing number of the complete graph and the complete bipartite graph.

For the complete bipartite graph, I am especially interested in the question about the rectilinear crossing number and variations of it. As an example, one could consider drawings where the two partitions of the vertices are drawn such that they are linearly separated.

Concerning the crossing number of the complete graph, I am most interested in connections with good drawings (also called simple complete topological graphs) and rotation systems.