ℓ-modular Representations of Finite Reductive Groups

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G is a finite group.

Frobenius created the theory of characters of G. He defined induction from a subgroup H of G, taking characters of H to characters of G. He then computed the character table of $PSL(2, p)$ in 1896. The character of a representation of G over an algebraically closed field of characteristic 0 is an "ordinary" character. The set of ordinary characters of G is denoted by $\mathrm{Irr}(G)$.

Richard Brauer developed the modular representation theory of finite groups, starting in the thirties.

G	a finite group
p	a prime integer
K	a sufficiently large field of characteristic 0
$\mathcal{O}$	a complete discrete valuation ring with quotient field K
k	residue field of $\mathcal{O}$, char k=p

A representation of G over K is equivalent to a representation over $\mathcal{O}$, and can then be reduced mod p to get a modular representation of G over k.

Brauer defined the character of a modular representation: a complex-valued function on the p-regular elements of G. Then we can compare ordinary and p-modular (Brauer) characters.

The decomposition map $d : K_0(KG) \to K_0(kG)$, where K_0 denotes the Grothendieck group expresses an ordinary character in terms of Brauer characters.

The decomposition matrix D (over $\mathbf{Z}$) is the transition matrix between ordinary and Brauer characters.

Consider the algebras KG, $\mathcal{O}G$, kG.

$$\mathcal{O}G = B_1 \oplus B_2 \oplus \ldots \oplus B_n$$

where the B_i are "block algebras", indecomposable ideals of $\mathcal{O}G$. We have a corresponding decomposition of kG.

Leads to:

- a partition of the ordinary characters, or KG-modules, into blocks
- a partition of the Brauer characters, or kG-modules, into blocks
- a partition of the decomposition matrix into blocks

Example: $G = S_n$. If $\chi \in \mathrm{Irr}(G)$ then $\chi = \chi_\lambda$ where λ is a partition of n. Then there is a Young diagram corresponding to λ and p-hooks, p-cores are defined. Then:

Theorem (Brauer-Nakayama) χ_λ, χ_μ are in the same p-block if and only if λ, μ have the same p-core.

An invariant of a block B of G: The defect group, a p- subgroup of G, unique up to G-conjugacy

D is minimal with respect to: Every B-module is a direct summand of an induced module from D

The "Brauer correspondence" gives:

There is a bijection between blocks of G of defect group D and blocks of $N_G(D)$ of defect group D

Some main problems of modular representation theory:

- Describe the irreducible modular representations, e.g. their degrees
- Describe the blocks
- Find the decomposition matrix D
- Global to local: Describe information on the block B by "local information", i.e. from blocks of subgroups of the form $N_G(P)$, P a p-group

$\mathbf{G}$	connected reductive group over $\mathbf{F}_q$, $\mathbf{F} = \overline{\mathbf{F}}_q$
q	a power p^n of the prime p
F	Frobenius endomorphism, $F : \mathbf{G} \to \mathbf{G}$
$G = \mathbf{G}^F$	finite reductive group
$\mathbf{T}$	torus, closed subgroup $\simeq \mathbf{F}^\times \times \mathbf{F}^\times \times \cdots \times \mathbf{F}^\times$
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Let **P** be an F-stable parabolic subgroup of **G** and **L** an F-stable Levi subgroup of **P** so that $L \leqslant P \leqslant G$.

Harish-Chandra induction is the following map:

$$R_L^G : K_0(KL) \to K_0(KG).$$

If $\psi \in \mathrm{Irr}(L)$ then $R_L^G(\psi) = \mathrm{Ind}_P^G(\tilde{\psi})$ where $\tilde{\psi}$ is the character of P obtained by inflating ψ to P.

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$\chi \in \mathrm{Irr}(G)$ is cuspidal if $\langle \chi, R_L^G(\psi) \rangle = 0$ for any $L \leqslant P < G$ where P is a proper parabolic subgroup of G. The pair (L, θ) a cuspidal pair if $\theta \in \mathrm{Irr}(L)$ is cuspidal.

$\mathrm{Irr}(G)$ partitioned into Harish-Chandra families: A family is the set of constituents of $R_L^G(\theta)$ where (L, θ) is cuspidal.

Now let ℓ be a prime not dividing q.

Suppose $\mathbf{L}$ is an F-stable Levi subgroup, not necessarily in an F-stable parabolic $\mathbf{P}$ of $\mathbf{G}$.

- The Deligne-Lusztig linear operator:

$$R_{\mathbf{L}}^{\mathbf{G}} : K_0(\overline{\mathbf{Q}}_l L) \to K_0(\overline{\mathbf{Q}}_l G).$$

- Every χ in $\mathrm{Irr}(G)$ is in $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ for some $(\mathbf{T}, \theta)$, where $\mathbf{T}$ is an F-stable maximal torus and $\theta \in \mathrm{Irr}(T)$.
- The unipotent characters of G are the irreducible characters χ in $R_{\mathbf{T}}^{\mathbf{G}}(1)$ as $\mathbf{T}$ runs over F-stable maximal tori of $\mathbf{G}$.

If $L \leqslant P \leqslant G$, where $\mathbf{P}$ is a F-stable parabolic subgroup, $R_{\mathbf{L}}^{\mathbf{G}}$ is just Harish-Chandra induction.

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If $L \leqslant P \leqslant G$, where $\mathbf{P}$ is a F-stable parabolic subgroup, $R_{\mathbf{L}}^{\mathbf{G}}$ is just Harish-Chandra induction.

Example: $G = GL(n, q)$. If L is the subgroup of diagonal matrices contained in the (Borel) subgroup of upper triangular matrices, we can do Harish-Chandra induction. But if L is a torus (Coxeter torus) of order $q^n - 1$, we must do Deligne-Lusztig induction to obtain generalized characters from characters of L.

G is a finite reductive group, ℓ a prime not dividing q.

Problem: Describe the ℓ-blocks of G.

Let $G = GL(n, q)$, e the order of q mod ℓ. The unipotent characters of G are indexed by partitions of n. Then:

Theorem (Fong-Srinivasan, 1982) χ_λ, χ_μ are in the same ℓ-block if and only if λ, μ have the same e-core.

As before, G is a finite reductive group, e the order of q mod ℓ

SURPRISE: Brauer Theory and Lusztig Theory are compatible!

$\phi_e(q)$ is the e-th cyclotomic polynomial. The order of G is the product of a power of q and certain cyclotomic polynomials. A torus T of G is a ϕ_e-torus if T has order a power of $\phi_e(q)$.

The centralizer in G of a ϕ_e-torus is an e-split Levi subgroup of G.

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Example. In GL_n e-split Levi subgroups L are isomorphic to $\prod_i GL(m_i, q^e) \times GL(r, q)$.

An e-cuspidal pair (L, θ) is defined as in the Harish-Chandra case, using only e-split Levi subgroups. Thus $\chi \in \mathrm{Irr}(G)$ is e-cuspidal if $\langle \chi, R_L^G(\psi) \rangle = 0$ for any e-split Levi subgroup L.

The unipotent characters of G are divided into e-Harish-Chandra families, as in the usual Harish-Chandra case of $e = 1$.

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Definition. A unipotent block of G is a block which contains unipotent characters.

THEOREM (Cabanes-Enguehard) Let B be a unipotent block of G, ℓ odd. Then the unipotent characters in B are precisely the constituents of $R_L^G(\lambda)$ where the pair (L, λ) is e-cuspidal.

Thus the unipotent blocks of G are parametrized by e-cuspidal pairs (L, λ) up to G-conjugacy. The subgroup $N_G(L)$ here plays the role of a "local subgroup".

Decomposition Numbers:

Much less is known. A main example is $GL(n, q)$, $l >> 0$ where one knows how to compute decomposition numbers in principle using the q-Schur algebra. See the notes of L. Scott.

Local to Global: Conjectures

G a finite group: Conjectures at different levels:

Characters	Perfect Isometries
Characters	Isotypies
kG-modules	Alperin Weight Conjecture
Derived Categories	Broué's Abelian Group Conjecture

An example: $GL(3,2)$

Character table for $GL(3,2)$.

order of element	1	2	3	4	7	7
class size	1	21	56	42	24	24
χ_1	1	1	1	1	1	1
χ_2	6	2	0	0	-1	-1
χ_3	7	-1	1	-1	0	0
χ_4	8	0	-1	0	1	1
χ_5	3	-1	0	1	$\frac{-1+i\sqrt{7}}{2}$	$\frac{-1-i\sqrt{7}}{2}$
χ_6	3	-1	0	1	$\frac{-1-i\sqrt{7}}{2}$	$\frac{-1+i\sqrt{7}}{2}$

Next look at the character table of $N(P_7)$ where P_7 is a Sylow 7-subgroup.

order of element	1	3	3	7	7
class size	1	7	7	3	3
ψ_1	1	1	1	1	1
ψ_2	1	ζ	ζ^2	1	1
ψ_3	1	ζ^2	ζ	1	1
ψ_4	3	0	0	$\frac{-1+i\sqrt{7}}{2}$	$\frac{-1-i\sqrt{7}}{2}$
ψ_5	3	0	0	$\frac{-1-i\sqrt{7}}{2}$	$\frac{-1+i\sqrt{7}}{2}$

Here ζ is a primitive 3rd root of unity.

The map

$$I_7 : \begin{Bmatrix} \chi_1 \\ -\chi_2 \\ \chi_4 \\ \chi_5 \\ \chi_6 \end{Bmatrix} \rightarrow \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \chi_4 \\ \chi_5 \end{Bmatrix}$$

preserves the character degrees mod 7 and preserves the values of the characters on 7-elements. Then I_7 is a simple example of an isotypy.

Block B of G, block b of H (e.g. $H = N_G(D)$), D defect group of B:

- A perfect isometry is a bijection between $K_0(B)$ and $K_0(b)$, preserving certain invariants of B and b.
- An isotypy is a collection of compatible perfect isometries
- Alperin's Weight Conjecture gives the number of simple kG-modules in terms of local data

If A is an $\mathcal{O} - algebra$, $\mathcal{D}^b(A)$ is the bounded derived category of $mod - A$, a triangulated category.

- Objects: Complexes of finitely generated projective $\mathcal{O}$-modules, bounded on the right, exact almost everywhere.
- Morphisms: Chain maps up to homotopy

Abelian Defect Group Conjecture: B a block of G with the abelian defect group D, b the Brauer correspondent of B in $N_G(D)$. Then $\mathcal{D}^b(B)$ and $\mathcal{D}^b(b)$ are equivalent as triangulated categories.

If the (ADG) conjecture is true for B and b, then there is a perfect isometry and B and b share various invariants, such as the number of characters in the blocks. This weaker property for unipotent blocks of a finite reductive group was proved by Broué, Malle and Michel. See Chuang and Rickard [LMS Lecture Notes 332] for cases where the conjecture has been proved. Many cases have been proved by constructing a "tilting complex" X such that

$$- \otimes X : \mathcal{D}^b(B) \to \mathcal{D}^b(b)$$

is an equivalence.

Chuang and Rouquier proved the (ADG) conjecture in the case $G = S_n$ or $G = GL(n, q)$, unipotent block, by "Categorification":

Replace the action of a group on a vector space by the action of functors on the Grothendieck group of a suitable abelian category.

For S_n, the Grothendieck group is $\oplus_{n \geqslant 0} K_0(mod - kS_n)$.

"Geometrization" and "Categorification" appear to be new directions in Representation Theory.