An introduction to the cohomology and modular representation theory of the symmetric group.

David J.Hemmer

University of Toledo ⇒ University at Buffalo, SUNY

June, 2007

Outline

1. Some notation: partitions and tableaux.
2. Brief overview of complex representation theory of Σ_d.
3. Changing the field- branching and decomposition numbers.
4. Relations with the general linear group.
5. Open problems in cohomology.
6. Filtrations and cohomology.

Partitions

Many of the modules we consider will be indexed by partitions.

Definition

$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ is a partition of d, denoted $\lambda \vdash d$, if $\lambda_i \geq \lambda_{i+1} > 0$ and $\sum \lambda_i = d$.

Partitions

Many of the modules we consider will be indexed by partitions.

Definition

$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_s)$ is a partition of d, denoted $\lambda \vdash d$, if $\lambda_i \geq \lambda_{i+1} > 0$ and $\sum \lambda_i = d$.

Example

$\lambda = (3, 3, 2, 2, 1, 1, 1) = (3^2, 2^2, 1^3) \vdash 13.$

Partitions

Many of the modules we consider will be indexed by partitions.

Definition

$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ is a partition of d, denoted $\lambda \vdash d$, if $\lambda_i \geq \lambda_{i+1} > 0$ and $\sum \lambda_i = d$.

Example

$\lambda = (3,3,2,2,1,1,1) = (3^2, 2^2, 1^3) \vdash 13.$

The transpose λ' is the partition of d obtained from transposing the Young diagram. For example:

$$\lambda = (3,1) = \begin{matrix} X & X & X \\ X & & \end{matrix} \qquad \lambda' = (2, 1^2) = \begin{matrix} X & X \\ X & \\ X & \end{matrix}$$

Young tableaux and tabloids

Definition

A λ- tableau t is an assignment of the numbers $\{1, 2, \ldots, d\}$ to the boxes in the Young diagram for λ.

For example:

$$t = \begin{array}{ccc} 1 & 4 & 2 \\ 6 & 3 & \\ 5 & & \end{array}$$

is a $(3, 2, 1)$-tableau.

Young tableaux and tabloids

Definition

A λ- tableau t is an assignment of the numbers $\{1, 2, \ldots, d\}$ to the boxes in the Young diagram for λ.

For example:

$$t = \begin{matrix} 1 & 4 & 2 \\ 6 & 3 & \\ 5 & & \end{matrix}$$

is a $(3, 2, 1)$-tableau.

Definition

A tabloid $\{t\}$ is an equivalence class of tableau under row equivalence.

To each partition λ of d there is a corresponding Young subgroup

$$\Sigma_\lambda \cong \Sigma_{\lambda_1} \times \cdots \times \Sigma_{\lambda_s}.$$

Over any field one can define the permutation module

$$M^\lambda := \mathrm{Ind}_{\Sigma_\lambda}^{\Sigma_d} k.$$

To each partition λ of d there is a corresponding Young subgroup

$$\Sigma_\lambda \cong \Sigma_{\lambda_1} \times \cdots \times \Sigma_{\lambda_s}.$$

Over any field one can define the permutation module

$$M^\lambda := \mathrm{Ind}_{\Sigma_\lambda}^{\Sigma_d} k.$$

Notice that the set of λ-tabloids is a basis for the module M^λ. For example:

$$M^{(2,2)} = \langle \{ \begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix} \}, \{ \begin{matrix} 1 & 3 \\ 2 & 4 \end{matrix} \}, \{ \begin{matrix} 1 & 4 \\ 2 & 3 \end{matrix} \}, \{ \begin{matrix} 2 & 3 \\ 1 & 4 \end{matrix} \}, \{ \begin{matrix} 2 & 4 \\ 1 & 3 \end{matrix} \}, \{ \begin{matrix} 3 & 4 \\ 1 & 2 \end{matrix} \} \rangle$$

Specht modules

To each $\lambda \vdash d$ there is a Specht module $S^\lambda \subseteq M^\lambda$ with the following properties:

- The dimension of S^λ is the number of standard Young tableaux of shape λ.

Specht modules

To each $\lambda \vdash d$ there is a Specht module $S^\lambda \subseteq M^\lambda$ with the following properties:

- The dimension of S^λ is the number of standard Young tableaux of shape λ.
- S^λ is defined over $\mathbb{Z}$ in a easily described combinatorial way.

Specht modules

To each $\lambda \vdash d$ there is a Specht module $S^\lambda \subseteq M^\lambda$ with the following properties:

- The dimension of S^λ is the number of standard Young tableaux of shape λ.
- S^λ is defined over $\mathbb{Z}$ in a easily described combinatorial way.
- $S^\lambda \otimes \mathrm{sgn} \cong (S^{\lambda'})^*$

Specht modules

To each $\lambda \vdash d$ there is a Specht module $S^\lambda \subseteq M^\lambda$ with the following properties:

- The dimension of S^λ is the number of standard Young tableaux of shape λ.
- S^λ is defined over $\mathbb{Z}$ in a easily described combinatorial way.
- $S^\lambda \otimes \text{sgn} \cong (S^{\lambda'})^*$
- Over $\mathbb{C}$ the set $\{S^\lambda \mid \lambda \vdash d\}$ is a complete set of irreducible Σ_d modules.

Specht modules

To each $\lambda \vdash d$ there is a Specht module $S^\lambda \subseteq M^\lambda$ with the following properties:

- The dimension of S^λ is the number of standard Young tableaux of shape λ.
- S^λ is defined over $\mathbb{Z}$ in a easily described combinatorial way.
- $S^\lambda \otimes \text{sgn} \cong (S^{\lambda'})^*$
- Over $\mathbb{C}$ the set $\{S^\lambda \mid \lambda \vdash d\}$ is a complete set of irreducible Σ_d modules.
- M^λ / S^λ has a filtration by Specht modules S^μ with $\mu \rhd \lambda$ and known multiplicities.

Specht modules

To each $\lambda \vdash d$ there is a Specht module $S^\lambda \subseteq M^\lambda$ with the following properties:

- The dimension of S^λ is the number of standard Young tableaux of shape λ.
- S^λ is defined over $\mathbb{Z}$ in a easily described combinatorial way.
- $S^\lambda \otimes \mathrm{sgn} \cong (S^{\lambda'})^*$
- Over $\mathbb{C}$ the set $\{S^\lambda \mid \lambda \vdash d\}$ is a complete set of irreducible Σ_d modules.
- M^λ / S^λ has a filtration by Specht modules S^μ with $\mu \rhd \lambda$ and known multiplicities.

Complex representation theory of the symmetric group.

The representation theory of $\mathbb{C}\Sigma_d$ is very well understood:

- Easy methods exist to determine the character values. In particular a nice closed form formula for the dimension of the irreducibles.

Complex representation theory of the symmetric group.

The representation theory of $\mathbb{C}\Sigma_d$ is very well understood:

- Easy methods exist to determine the character values. In particular a nice closed form formula for the dimension of the irreducibles.
- Induction and restriction of Specht modules is well understood via the classical branching theorems and Littlewood-Richardson rule, which are easily described combinatorially in terms of Young diagrams. Induction and restriction of Specht modules are multiplicity free.

Complex representation theory of the symmetric group.

The representation theory of $\mathbb{C}\Sigma_d$ is very well understood:

- Easy methods exist to determine the character values. In particular a nice closed form formula for the dimension of the irreducibles.
- Induction and restriction of Specht modules is well understood via the classical branching theorems and Littlewood-Richardson rule, which are easily described combinatorially in terms of Young diagrams. Induction and restriction of Specht modules are multiplicity free.
- Other nice bases of the Specht modules exist. For example the seminormal basis which is well-adapted to the branching behavior.

Complex representation theory of the symmetric group.

The representation theory of $\mathbb{C}\Sigma_d$ is very well understood:

- Easy methods exist to determine the character values. In particular a nice closed form formula for the dimension of the irreducibles.
- Induction and restriction of Specht modules is well understood via the classical branching theorems and Littlewood-Richardson rule, which are easily described combinatorially in terms of Young diagrams. Induction and restriction of Specht modules are multiplicity free.
- Other nice bases of the Specht modules exist. For example the seminormal basis which is well-adapted to the branching behavior.
- The theory is closely related to algebraic combinatorics and symmetric functions.

Some open problems in the representation theory of $\mathbb{C}\Sigma_d$

Problem

Find a general formula for the constituents of $S^\lambda \otimes S^\mu$, i.e. decompose the tensor product of two irreducible characters.

Some open problems in the representation theory of $\mathbb{C}\Sigma_d$

Problem

Find a general formula for the constituents of $S^\lambda \otimes S^\mu$, i.e. decompose the tensor product of two irreducible characters.

Problem

Foulkes conjecture (1950). Suppose $a < b$ and consider the Young subgroups Σ_{a^b} and Σ_{b^a} inside Σ_{ab}. Consider the permutation characters on the cosets of their respective normalizers. Then the multiplicity of each character in the smaller module is $\leq$ the multiplicity in the bigger module.

Some open problems in the representation theory of $\mathbb{C}\Sigma_d$

Problem

Find a general formula for the constituents of $S^\lambda \otimes S^\mu$, i.e. decompose the tensor product of two irreducible characters.

Problem

Foulkes conjecture (1950). Suppose $a < b$ and consider the Young subgroups Σ_{a^b} and Σ_{b^a} inside Σ_{ab}. Consider the permutation characters on the cosets of their respective normalizers. Then the multiplicity of each character in the smaller module is $\leq$ the multiplicity in the bigger module.

Problem

Find fast Fourier transforms for symmetric groups. For example find a sparse factorization of the change of basis matrix from the usual basis of $\mathbb{C}\Sigma_d$ to the basis of matrix coefficients.

Modular representation theory of Σ_d

Definition

$\lambda \vdash d$ is *p*-regular if no part repeats p or more times. $\lambda \vdash d$ is *p*-restricted λ' is p-regular.

Modular representation theory of Σ_d

Definition

$\lambda \vdash d$ is p-regular if no part repeats p or more times. $\lambda \vdash d$ is p-restricted λ' is p-regular.

Theorem (James)

For λ p-regular let $D^\lambda = S^\lambda / \operatorname{rad} S^\lambda$. Then D^λ is irreducible and the various D^λ give a complete set of nonisomorphic irreducibles.

Definition

$\lambda \vdash d$ is *p-regular* if no part repeats p or more times. $\lambda \vdash d$ is *p-restricted* λ' is p-regular.

Theorem (James)

For λ p-regular let $D^\lambda = S^\lambda / \operatorname{rad} S^\lambda$. Then D^λ is irreducible and the various D^λ give a complete set of nonisomorphic irreducibles.

Theorem

For λ p-restricted let $D_\lambda = D^{\lambda'} \otimes \operatorname{sgn}$. Then $D_\lambda = \operatorname{soc} S^\lambda$

Modular representation theory of Σ_d

Definition

$\lambda \vdash d$ is *p-regular* if no part repeats p or more times. $\lambda \vdash d$ is *p-restricted* λ' is p-regular.

Theorem (James)

For λ p-regular let $D^\lambda = S^\lambda / \operatorname{rad} S^\lambda$. Then D^λ is irreducible and the various D^λ give a complete set of nonisomorphic irreducibles.

Theorem

For λ p-restricted let $D_\lambda = D^{\lambda'} \otimes \operatorname{sgn}$. Then $D_\lambda = \operatorname{soc} S^\lambda$

Problem

Determine the dimension of D_λ.

Decomposition matrices

Thus we come to our first major open problem:

Problem

What is the multiplicity $[S^\lambda : D_\mu]$ *of* D_μ *in* S^λ*?*

Decomposition matrices

Thus we come to our first major open problem:

Problem

What is the multiplicity $[S^\lambda : D_\mu]$ *of* D_μ *in* S^λ*?*

What is known:

- $[S^\lambda : D_\mu] = 0$ unless $\lambda \trianglerighteq \mu$ and $[S^\lambda : D_\lambda] = 1$, i.e. the decomposition matrix has a nice triangular structure.

Decomposition matrices

Thus we come to our first major open problem:

Problem

What is the multiplicity $[S^\lambda : D_\mu]$ *of* D_μ *in* S^λ*?*

What is known:

- $[S^\lambda : D_\mu] = 0$ unless $\lambda \trianglerighteq \mu$ and $[S^\lambda : D_\lambda] = 1$, i.e. the decomposition matrix has a nice triangular structure.
- There are some specific results, such as when the partition is a "hook" or has 2 or 3 parts. There are results for blocks of small defect, most recently for defect 3 and $p > 3$ due to Matt Fayers.

Decomposition matrices

Thus we come to our first major open problem:

Problem

What is the multiplicity $[S^\lambda : D_\mu]$ of D_μ in S^λ?

What is known:

- $[S^\lambda : D_\mu] = 0$ unless $\lambda \trianglerighteq \mu$ and $[S^\lambda : D_\lambda] = 1$, i.e. the decomposition matrix has a nice triangular structure.
- There are some specific results, such as when the partition is a "hook" or has 2 or 3 parts. There are results for blocks of small defect, most recently for defect 3 and $p > 3$ due to Matt Fayers.
- The blocks of $k\Sigma_d$ are known by "Nakayama's conjecture".

Decomposition matrices

Thus we come to our first major open problem:

Problem

What is the multiplicity $[S^\lambda : D_\mu]$ of D_μ in S^λ?

What is known:

- $[S^\lambda : D_\mu] = 0$ unless $\lambda \trianglerighteq \mu$ and $[S^\lambda : D_\lambda] = 1$, i.e. the decomposition matrix has a nice triangular structure.
- There are some specific results, such as when the partition is a "hook" or has 2 or 3 parts. There are results for blocks of small defect, most recently for defect 3 and $p > 3$ due to Matt Fayers.
- The blocks of $k\Sigma_d$ are known by "Nakayama's conjecture".
- There is a conjecture of Gordon James that would, in theory, allow one to compute the decomposition matrix of Σ_d for $d < p^2$.

Problem

For λ not p-restricted, the socle of the Specht module is not even known.

Only in the last two years has it been known which rows of the decomposition matrix have a single nonzero entry:

Theorem (Fayers 2005)

The Specht module S^{λ} remains irreducible in characteristic p if and only if λ satisfies the combinatorial conditions conjectured by James and Mathas.

The cases where λ is p-restricted or p-regular were already known.

Branching Problems

Recall that the Branching rule gives a complete description of $\operatorname{Res} S^\lambda$ and $\operatorname{Ind} S^\lambda$ in characteristic zero.

Branching Problems

Recall that the Branching rule gives a complete description of $\operatorname{Res} S^\lambda$ and $\operatorname{Ind} S^\lambda$ in characteristic zero.

Problem

Describe the module $U := \operatorname{Res}_{\Sigma_{d-1}} D_\lambda$.

At a minimum we would like to know the composition factors of U.

Kleshchev's branching theorems

In a series of papers in the 1990's Kleshchev proved, among other things, that:

- Each indecomposable summand of $U := \text{Res}_{\Sigma_{d-1}} D_\lambda$ is self-dual with (known) simple socle and lies in a distinct block. Similarly for Ind D_λ.

Kleshchev's branching theorems

In a series of papers in the 1990's Kleshchev proved, among other things, that:

- Each indecomposable summand of $U := \text{Res}_{\Sigma_{d-1}} D_\lambda$ is self-dual with (known) simple socle and lies in a distinct block. Similarly for Ind D_λ.
- Solving the branching problem of determining $[U : D_\mu]$ is equivalent to solving the decomposition number problem .

Kleshchev's branching theorems

In a series of papers in the 1990's Kleshchev proved, among other things, that:

- Each indecomposable summand of $U := \mathrm{Res}_{\Sigma_{d-1}} D_\lambda$ is self-dual with (known) simple socle and lies in a distinct block. Similarly for Ind D_λ.
- Solving the branching problem of determining $[U : D_\mu]$ is equivalent to solving the decomposition number problem .
- The endomorphism algebra of U is known.

Kleshchev's branching theorems

In a series of papers in the 1990's Kleshchev proved, among other things, that:

- Each indecomposable summand of $U := \text{Res}_{\Sigma_{d-1}} D_\lambda$ is self-dual with (known) simple socle and lies in a distinct block. Similarly for Ind D_λ.
- Solving the branching problem of determining $[U : D_\mu]$ is equivalent to solving the decomposition number problem .
- The endomorphism algebra of U is known.
- Some of the multiplicities in U are known but not even a complete description of *which* simple modules appear in U known.

Kleshchev's branching theorems

In a series of papers in the 1990's Kleshchev proved, among other things, that:

- Each indecomposable summand of $U := \mathrm{Res}_{\Sigma_{d-1}} D_\lambda$ is self-dual with (known) simple socle and lies in a distinct block. Similarly for Ind D_λ.
- Solving the branching problem of determining $[U : D_\mu]$ is equivalent to solving the decomposition number problem .
- The endomorphism algebra of U is known.
- Some of the multiplicities in U are known but not even a complete description of *which* simple modules appear in U known.

Much about U remains a mystery. James showed the composition multiplicities in U can be arbitrarily large! The proofs all use the connections with representation theory of the general linear group.

More recently (2007) Ellers and Murray proved:

Theorem

Suppose $p > 2$. Let $U = \mathrm{Res}_{\Sigma_{d-1}} S^{\lambda}$. Then each indecomposable summand of U is in a distinct block. Similarly for the induced module.

In characteristic 2 Specht modules need not be indecomposable and this theorem fails.

Some modules for the general linear group.

Let $G = GL_n(k)$ and let $\lambda \vdash d$. There are $GL_n(k)$-modules:

Induced modules	$H^0(\lambda)$
Weyl modules	$V(\lambda) \cong H^0(\lambda)^\tau$
Irreducible modules	$L(\lambda) \cong \mathrm{soc}(H^0(\lambda))$

The $H^0(\lambda)$ are induced from natural one-dimensional representations of the subgroup of upper triangular matrices. These modules behave much like Specht, dual Specht and irreducible $k\Sigma_d$-modules.

Some modules for the general linear group.

Let $G = GL_n(k)$ and let $\lambda \vdash d$. There are $GL_n(k)$-modules:

Induced modules	$H^0(\lambda)$
Weyl modules	$V(\lambda) \cong H^0(\lambda)^\tau$
Irreducible modules	$L(\lambda) \cong \mathrm{soc}(H^0(\lambda))$

The $H^0(\lambda)$ are induced from natural one-dimensional representations of the subgroup of upper triangular matrices. These modules behave much like Specht, dual Specht and irreducible $k\Sigma_d$-modules.

Technicality: The relevant category is polynomial representations of $GL_n(k)$ of homogeneous degree d.

Nice cohomological properties

These modules have many nice cohomological properties. Key is the following:

Theorem

Suppose $\mu \not\triangleright \lambda$. *Then:*

$$\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \cong \mathrm{Hom}_G(\mathrm{rad}(V(\lambda), L(\mu)).$$

In particular

$$\mathrm{Ext}^1_G(L(\mu), L(\mu)) = 0.$$

We say there are no *self-extensions* for simple G modules.

Relating the representation theory of Σ_d and $GL_n(k)$

Let $E \cong k^n$ be column vectors of length n, so E is naturally a $GL_n(k)$ module and thus so is the tensor space $E^{\otimes d}$.

Σ_d acts on $E^{\otimes d}$ by permuting the coordinates. The two actions commute!

Relating the representation theory of Σ_d and $GL_n(k)$

Let $E \cong k^n$ be column vectors of length n, so E is naturally a $GL_n(k)$ module and thus so is the tensor space $E^{\otimes d}$.

Σ_d acts on $E^{\otimes d}$ by permuting the coordinates. The two actions commute!

When $n \geq d$ this commuting action allows one to define the Schur functor $\mathcal{F} : \operatorname{mod} GL_n(k) \to \operatorname{mod} \Sigma_d$ by

$$\mathcal{F}(U) \cong \operatorname{Hom}_{GL_n(k)}(E^{\otimes d}, U)$$

and its adjoint $\mathcal{G} : \operatorname{mod} \Sigma_d \to \operatorname{mod} GL_n(k)$ given by

$$\mathcal{G}(N) \cong \operatorname{Hom}_{\Sigma_d}(E^{\otimes d}, N).$$

Properties of the Schur and adjoint Schur functors

- $\mathcal{F}$: mod $GL_n(k) \longrightarrow$ mod Σ_d is exact.
- $\mathcal{G}$: mod $\Sigma_d \longrightarrow$ mod $GL_n(k)$. is only left exact.

Properties of the Schur and adjoint Schur functors

- $\mathcal{F} : \text{mod}\, GL_n(k) \to \text{mod}\, \Sigma_d$ is exact.
- $\mathcal{G} : \text{mod}\, \Sigma_d \to \text{mod}\, GL_n(k)$. is only left exact.
- $\mathcal{F}(\mathcal{G}(N)) \cong N$.

Properties of the Schur and adjoint Schur functors

- $\mathcal{F} : \text{mod}\, GL_n(k) \to \text{mod}\, \Sigma_d$ is exact.
- $\mathcal{G} : \text{mod}\, \Sigma_d \to \text{mod}\, GL_n(k)$. is only left exact.
- $\mathcal{F}(\mathcal{G}(N)) \cong N$.
- Over $\mathbb{C}$ these are equivalences of categories. (Schur)

Properties of the Schur and adjoint Schur functors

- $\mathcal{F} : \operatorname{mod} GL_n(k) \to \operatorname{mod} \Sigma_d$ is exact.
- $\mathcal{G} : \operatorname{mod} \Sigma_d \to \operatorname{mod} GL_n(k)$. is only left exact.
- $\mathcal{F}(\mathcal{G}(N)) \cong N$.
- Over $\mathbb{C}$ these are equivalences of categories. (Schur)
- There is a Grothendieck spectral sequence taking cohomology (i.e. extensions) for $GL_n(k)$ and converging to cohomology for Σ_d.

Properties of the Schur and adjoint Schur functors

- $\mathcal{F} : \text{mod}\, GL_n(k) \to \text{mod}\, \Sigma_d$ is exact.
- $\mathcal{G} : \text{mod}\, \Sigma_d \to \text{mod}\, GL_n(k)$. is only left exact.
- $\mathcal{F}(\mathcal{G}(N)) \cong N$.
- Over $\mathbb{C}$ these are equivalences of categories. (Schur)
- There is a Grothendieck spectral sequence taking cohomology (i.e. extensions) for $GL_n(k)$ and converging to cohomology for Σ_d.
- And on actual modules...

$$H^0(\lambda) \xrightarrow{\mathcal{F}} S^\lambda \xrightarrow{\mathcal{G}} ???$$

$$V(\lambda) \xrightarrow{\mathcal{F}} (S_\lambda) \xrightarrow{\mathcal{G}\ p>3} V(\lambda)$$

$$L(\lambda) \xrightarrow{\mathcal{F}} D_\lambda \text{ or } 0 \xrightarrow{\mathcal{G}} ???$$

Extensions between simple $k\Sigma_d$ modules.

It is natural to ask what nice cohomological properties from GL_n hold for the symmetric group.

Problem

Determine the nonsplit extensions between irreducible modules for a group G, i.e. calculate $\mathrm{Ext}^1_G(S, T)$ for irreducible modules S, T. This is known as determining the Ext^1-quiver. Very few results are known for $G = \Sigma_d$.

Extensions between simple $k\Sigma_d$ modules.

It is natural to ask what nice cohomological properties from GL_n hold for the symmetric group.

Problem

Determine the nonsplit extensions between irreducible modules for a group G, i.e. calculate $\mathrm{Ext}^1_G(S, T)$ for irreducible modules S, T. This is known as determining the Ext^1-quiver. Very few results are known for $G = \Sigma_d$.

Conjecture

(Kleshchev-Martin) Let D_λ be an irreducible Σ_d-module and $p > 2$. Then:

$$\mathrm{Ext}^1_{\Sigma_d}(D_\lambda, D_\lambda) = 0.$$

If only the corresponding result about extensions between simple modules all living at the top of Specht modules were true, the Kleshchev-Martin conjecture would follow. It is definitely false in characteristic two. However:

If only the corresponding result about extensions between simple modules all living at the top of Specht modules were true, the Kleshchev-Martin conjecture would follow. It is definitely false in characteristic two. However:

Theorem (Kleshchev-Sheth 1999)

Let $p > 2$ and suppose $\lambda \not\triangleright \mu$. If λ and μ have at most $p-1$ parts then:

$$\mathrm{Ext}^1_{\Sigma_d}(D^\lambda, D^\mu) \cong \mathrm{Hom}_{\Sigma_d}(\mathrm{rad}(S^\lambda), D^\mu).$$

If only the corresponding result about extensions between simple modules all living at the top of Specht modules were true, the Kleshchev-Martin conjecture would follow. It is definitely false in characteristic two. However:

Theorem (Kleshchev-Sheth 1999)

Let $p > 2$ and suppose $\lambda \not\trianglerighteq \mu$. If λ and μ have at most $p-1$ parts then:

$$\mathrm{Ext}^1_{\Sigma_d}(D^\lambda, D^\mu) \cong \mathrm{Hom}_{\Sigma_d}(\mathrm{rad}(S^\lambda), D^\mu).$$

There are no known counterexamples for partitions with more than p parts.

Even the much weaker statement below is not known:

Problem

Suppose $\mathrm{Ext}^1_{\Sigma_d}(D_\lambda, D_\mu) \neq 0$. *Must* λ *and* μ *be comparable in the dominance order?*

Even the much weaker statement below is not known:

Problem

Suppose $\mathrm{Ext}^1_{\Sigma_d}(D_\lambda, D_\mu) \neq 0$*. Must* λ *and* μ *be comparable in the dominance order?*

What is known about the Ext-quiver:

- Small defect blocks.

Even the much weaker statement below is not known:

Problem

Suppose $\mathrm{Ext}^1_{\Sigma_d}(D_\lambda, D_\mu) \neq 0$*. Must* λ *and* μ *be comparable in the dominance order?*

What is known about the Ext-quiver:

- Small defect blocks.
- Completely splittable modules. (my thesis)

Even the much weaker statement below is not known:

Problem

Suppose $\mathrm{Ext}^1_{\Sigma_d}(D_\lambda, D_\mu) \neq 0$*. Must* λ *and* μ *be comparable in the dominance order?*

What is known about the Ext-quiver:

- Small defect blocks.
- Completely splittable modules. (my thesis)
- 2-part and 2-column partitions.

Even the much weaker statement below is not known:

Problem

Suppose $\mathrm{Ext}^1_{\Sigma_d}(D_\lambda, D_\mu) \neq 0$*. Must* λ *and* μ *be comparable in the dominance order?*

What is known about the Ext-quiver:

- Small defect blocks.
- Completely splittable modules. (my thesis)
- 2-part and 2-column partitions.
- Certain λ and μ "far apart" in some blocks of abelian defect.

Even the much weaker statement below is not known:

Problem

Suppose $\mathrm{Ext}^1_{\Sigma_d}(D_\lambda, D_\mu) \neq 0$. *Must* λ *and* μ *be comparable in the dominance order?*

What is known about the Ext-quiver:

- Small defect blocks.
- Completely splittable modules. (my thesis)
- 2-part and 2-column partitions.
- Certain λ and μ "far apart" in some blocks of abelian defect.

Conjecture

If $d < p^2$ *then the Ext-quiver is bipartite.*

More cohomology problems

Also natural to consider are the extension groups $\mathrm{Ext}^i_{\Sigma_d}(k, M)$ for small i and natural choices of M. Again open problems abound!

More cohomology problems

Also natural to consider are the extension groups $\text{Ext}^i_{\Sigma_d}(k, M)$ for small i and natural choices of M. Again open problems abound!

Theorem (BKM)

$\text{Ext}^i_{\Sigma_d}(k, (S^\lambda)^*) = 0$ *for* $1 \leq i \leq p - 3$.

More cohomology problems

Also natural to consider are the extension groups $\mathrm{Ext}^i_{\Sigma_d}(k, M)$ for small i and natural choices of M. Again open problems abound!

Theorem (BKM)

$\mathrm{Ext}^i_{\Sigma_d}(k, (S^\lambda)^*) = 0$ *for* $1 \leq i \leq p - 3$.

The situation for cohomology of Specht modules is more interesting.

- In his book, Gordon James determined $\mathrm{Hom}_{\Sigma_d}(k, S^\lambda)$.

More cohomology problems

Also natural to consider are the extension groups $\mathrm{Ext}^i_{\Sigma_d}(k, M)$ for small i and natural choices of M. Again open problems abound!

Theorem (BKM)

$\mathrm{Ext}^i_{\Sigma_d}(k, (S^\lambda)^*) = 0$ *for* $1 \leq i \leq p - 3$.

The situation for cohomology of Specht modules is more interesting.

- In his book, Gordon James determined $\mathrm{Hom}_{\Sigma_d}(k, S^\lambda)$.
- I have a conjecture for when $\mathrm{Ext}^1_{\Sigma_d}(k, S^\lambda)$ is nonzero.

More cohomology problems

Also natural to consider are the extension groups $\mathrm{Ext}^i_{\Sigma_d}(k, M)$ for small i and natural choices of M. Again open problems abound!

Theorem (BKM)

$\mathrm{Ext}^i_{\Sigma_d}(k, (S^\lambda)^*) = 0$ *for* $1 \leq i \leq p-3$.

The situation for cohomology of Specht modules is more interesting.

- In his book, Gordon James determined $\mathrm{Hom}_{\Sigma_d}(k, S^\lambda)$.
- I have a conjecture for when $\mathrm{Ext}^1_{\Sigma_d}(k, S^\lambda)$ is nonzero.

However partitions with all parts divisible by p are trouble. For example

Problem

Can anyone compute

$$\mathrm{Ext}^1_{\Sigma_{15}}(k, S^{(5,5,5)})$$

in characteristic 5?

Even more cohomology problems

Problem

Compute the complexity or even the support varieties for Specht modules.

This problem was considered for a time by the UGA VIGRE group.

Even more cohomology problems

Problem

Compute the complexity or even the support varieties for Specht modules.

This problem was considered for a time by the UGA VIGRE group.

Problem

Is it possible to have $\mathrm{Ext}^i_{\Sigma_d}(k, D_\lambda) = 0 \forall i$*? where* D_λ *is in the principal block.*

Conjecturally no. (for any finite group!)

Good filtrations

Definition

A $GL_n(k)$-module U has a good filtration if there is a chain $0 = U_0 \subset U_1 \subset \cdots \subset U_s = U$ of submodules with $U_i/U_{i-1} \cong H^0(\mu^i)$.

Good filtrations

Definition

A $GL_n(k)$-module U has a good filtration if there is a chain $0 = U_0 \subset U_1 \subset \cdots \subset U_s = U$ of submodules with $U_i/U_{i-1} \cong H^0(\mu^i)$.

Theorem

(Donkin) A $GL_n(k)$-module U has a good filtration iff

$$\mathrm{Ext}^1_{GL_n(k)}(V(\mu), U) = 0 \;\forall\; \mu \vdash d.$$

Moreover the multiplicity of $H^0(\mu)$ in such a filtration is independent of the choice of filtration and equal to

$$\dim \mathrm{Hom}_{GL_n(k)}(V(\mu), U).$$

Specht filtrations for symmetric group modules

It was long thought that a theory of Specht filtrations analogous to that of good filtrations was hopeless. For example: [1]

The genesis of this remarkable result is interesting: James [1983] proved that the character of the Young module is precisely the one predicted by 4.6.4. Then Donkin [1987] exhibited a filtration to realise the numerical result. Another comment worth making is to the effect that in general the filtration multiplicity of a KG-module with a Specht series is not

well defined. Take $r = 2$ and $p = 2$ and let $M = Y^{(1^2)} = K\Sigma_2$. Then there are three short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & S^{(2)} & \to & M & \to & S^{(2)} & \to & 0 \\ 0 & \to & S^{(2)} & \to & M & \to & S^{(1^2)} & \to & 0 \; . \\ 0 & \to & S^{(1^2)} & \to & M & \to & S^{(1^2)} & \to & 0 \end{array}$$

So $S^{(1^2)}$ occurs with multiplicities 0,1 and 2 in M, respectively.

[1] "Schur algebras and representation theory," Stuart Martin, 1993.

Two and three are odd primes

Theorem

For $p > 3$, $\mathrm{Ext}^1_{\Sigma_d}(\mathrm{sgn}, k) = 0$.

Two and three are odd primes

Theorem

For $p > 3$, $\mathrm{Ext}^1_{\Sigma_d}(\mathrm{sgn}, k) = 0$.

Proof.

Suppose not, then there is a homomorphism

$$\psi : \Sigma_d \to \begin{pmatrix} 1 & * \\ 0 & \pm 1 \end{pmatrix}.$$

The image of ψ is a group of order divisible by p. Thus the kernel is a normal subgroup of Σ_d of index divisible by p. This can not happen for $p > 3$.

□

Theorem

(Hemmer-Nakano, 2004)
Let $p > 3$ *and let* $M \in \text{mod-}k\Sigma_d$. *Then* M *has a Specht filtration if and only if :*

$$\mathrm{Ext}^1_{GL_n(k)}(\mathcal{G}(M^*), H^0(\lambda)) = 0 \text{ for all } \lambda.$$

Theorem

(Hemmer-Nakano, 2004)
Let $p > 3$ *and let* $M \in mod{-}k\Sigma_d$. *Then* M *has a Specht filtration if and only if :*

$$\mathrm{Ext}^1_{GL_n(k)}(\mathcal{G}(M^*), H^0(\lambda)) = 0 \ \textit{for all } \lambda.$$

If M *has a Specht filtration, then the multiplicities are independent of the choice of filtration, and are given by:*

$$[M : S^\lambda] = \dim_k \mathrm{Hom}_{GL_n(k)}(\mathcal{G}(M^*), H^0(\lambda)).$$

Theorem

(Hemmer-Nakano, 2004)
Let $p > 3$ and let $M \in \text{mod-}k\Sigma_d$. Then M has a Specht filtration if and only if :

$$\mathrm{Ext}^1_{GL_n(k)}(\mathcal{G}(M^*), H^0(\lambda)) = 0 \text{ for all } \lambda.$$

If M has a Specht filtration, then the multiplicities are independent of the choice of filtration, and are given by:

$$[M : S^\lambda] = \dim_k \mathrm{Hom}_{GL_n(k)}(\mathcal{G}(M^*), H^0(\lambda)).$$

- Thus in characteristic > 3 one may hope for a theory of Specht filtrations analogous to the theory of good filtrations for $GL_n(k)$.

Theorem

(Hemmer-Nakano, 2004)
Let $p > 3$ and let $M \in \text{mod}-k\Sigma_d$. Then M has a Specht filtration if and only if :

$$\mathrm{Ext}^1_{GL_n(k)}(\mathcal{G}(M^*), H^0(\lambda)) = 0 \ \textit{for all } \lambda.$$

If M has a Specht filtration, then the multiplicities are independent of the choice of filtration, and are given by:

$$[M : S^\lambda] = \dim_k \mathrm{Hom}_{GL_n(k)}(\mathcal{G}(M^*), H^0(\lambda)).$$

- Thus in characteristic > 3 one may hope for a theory of Specht filtrations analogous to the theory of good filtrations for $GL_n(k)$.
- We could not find a criterion or a multiplicity formula stated in terms of symmetric group cohomology.

Recently we proved two cohomological criteria for a symmetric group module to have a Specht filtration:

Theorem

(H, 2007) Let $M \in$ mod–Σ_d.

- *If* $\mathrm{Ext}^1_{\Sigma_d}(M, S_\lambda) = 0 \ \forall \lambda \vdash d$ *then M has a Specht filtration. The multiplicity of S^μ in any filtration is* $\dim_k \mathrm{Hom}_{\Sigma_d}(M, S_\mu)$
- *If* $\mathrm{Ext}^1_{\Sigma_d}(S_\lambda, M) = 0 \ \forall \lambda \vdash d$ *then M has a Specht filtration. The multiplicity of S^μ in any filtration is* $\dim_k \mathrm{Hom}_{\Sigma_d}(S_\mu, M)$.

Recently we proved two cohomological criteria for a symmetric group module to have a Specht filtration:

Theorem

(H, 2007) Let $M \in mod\text{-}\Sigma_d$.

- *If $\mathrm{Ext}^1_{\Sigma_d}(M, S_\lambda) = 0 \ \forall \lambda \vdash d$ then M has a Specht filtration. The multiplicity of S^μ in any filtration is $\dim_k \mathrm{Hom}_{\Sigma_d}(M, S_\mu)$*
- *If $\mathrm{Ext}^1_{\Sigma_d}(S_\lambda, M) = 0 \ \forall \lambda \vdash d$ then M has a Specht filtration. The multiplicity of S^μ in any filtration is $\dim_k \mathrm{Hom}_{\Sigma_d}(S_\mu, M)$.*

- The two criteria detect different modules.
- Two corresponding criteria for dual Specht filtrations.
- Neither is necessary.
- Multiplicity formula generalizes special case for M projective.

More good filtration theory

Recall $GL_n(k)$ has induced modules $H^0(\lambda)$ and their duals, the Weyl modules $V(\lambda)$.

Theorem

(Donkin) $H^0(\lambda) \otimes H^0(\mu)$ has a good filtration. Thus the tensor product of any two modules with a good filtration has a good filtration.

More good filtration theory

Recall $GL_n(k)$ has induced modules $H^0(\lambda)$ and their duals, the Weyl modules $V(\lambda)$.

Theorem

(Donkin) $H^0(\lambda) \otimes H^0(\mu)$ has a good filtration. Thus the tensor product of any two modules with a good filtration has a good filtration.

Problem: Classify indecomposable $GL_n(k)$ modules which have both a good and Weyl filtration.

More good filtration theory

Recall $GL_n(k)$ has induced modules $H^0(\lambda)$ and their duals, the Weyl modules $V(\lambda)$.

Theorem

(Donkin) $H^0(\lambda) \otimes H^0(\mu)$ has a good filtration. Thus the tensor product of any two modules with a good filtration has a good filtration.

Problem: Classify indecomposable $GL_n(k)$ modules which have both a good and Weyl filtration.

Theorem

(Ringel) To each $\lambda \vdash d$ there is an indecomposable module $T(\lambda)$, called a tilting module, which has both a good and Weyl filtration. Moreover $T(\lambda)$ is self-dual and these are the only examples.

Modules with Specht and dual Specht filtrations

Problem

Classify indecomposable Σ_d modules which have both a Specht and dual Specht filtration, i.e. find "tilting modules" for Σ_d.

Modules with Specht and dual Specht filtrations

Problem

Classify indecomposable Σ_d modules which have both a Specht and dual Specht filtration, i.e. find "tilting modules" for Σ_d.

Warning: For $p > 3$ we know the adjoint Schur functor $\mathcal{G}$ maps dual Specht filtrations to Weyl filtrations but does not respect Specht module filtrations.

Modules with Specht and dual Specht filtrations

Problem

Classify indecomposable Σ_d modules which have both a Specht and dual Specht filtration, i.e. find "tilting modules" for Σ_d.

Warning: For $p > 3$ we know the adjoint Schur functor $\mathcal{G}$ maps dual Specht filtrations to Weyl filtrations but does not respect Specht module filtrations.

Fact

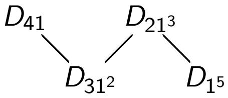
$$S^{\lambda} \otimes \mathrm{sgn} \cong S_{\lambda'}.$$

Thus tensoring with sgn *takes modules with a Specht filtration to modules with dual Specht filtrations and vice versa. In particular the class of "tilting modules" for Σ_d is closed under tensoring with* sgn.

A horrifying example?

Let $p = d = 5$ and let $U \cong \Omega^2(D_{21^3})$ where Ω is the Heller translate. Then

$U \cong$



A horrifying example?

Let $p = d = 5$ and let $U \cong \Omega^2(D_{21^3})$ where Ω is the Heller translate. Then

$$U \cong \begin{array}{ccccccc} D_{41} & & & & D_{21^3} & & \\ & \diagdown & & \diagup & & \diagdown & \\ & & D_{31^2} & & & & D_{1^5} \end{array}$$

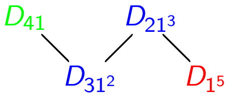
The Specht modules look like:

$$S^5 \cong D_{41}, \quad S^{41} \cong \begin{array}{c} D_{31^2} \\ D_{41} \end{array}, \quad S^{31^2} \cong \begin{array}{c} D_{21^3} \\ D_{31^2} \end{array}, \quad S^{21^3} \cong \begin{array}{c} D_{1^5} \\ D_{21^3} \end{array}, \quad S^{1^5} \cong D_{1^5},$$

A horrifying example?

Let $p = d = 5$ and let $U \cong \Omega^2(D_{21^3})$ where Ω is the Heller translate. Then

$U \cong$



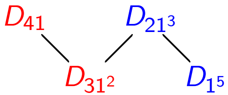
The Specht modules look like:

$$S^5 \cong D_{41}, \quad S^{41} \cong \begin{array}{c} D_{31^2} \\ D_{41} \end{array}, \quad S^{31^2} \cong \begin{array}{c} D_{21^3} \\ D_{31^2} \end{array}, \quad S^{21^3} \cong \begin{array}{c} D_{1^5} \\ D_{21^3} \end{array}, \quad S^{1^5} \cong D_{1^5},$$

So U has a Specht filtration with subquotients S^{1^5}, S^{31^2}, and S^5

A horrifying example?

Let $p = d = 5$ and let $U \cong \Omega^2(D_{21^3})$ where Ω is the Heller translate. Then

$U \cong$ 

The Specht modules look like:

$$S^5 \cong D_{41}, \quad S^{41} \cong \begin{array}{c} D_{31^2} \\ D_{41} \end{array}, \quad S^{31^2} \cong \begin{array}{c} D_{21^3} \\ D_{31^2} \end{array}, \quad S^{21^3} \cong \begin{array}{c} D_{1^5} \\ D_{21^3} \end{array}, \quad S^{1^5} \cong D_{1^5},$$

So U has a Specht filtration with subquotients S^{1^5}, S^{31^2}, and S^5

And U has a dual Specht filtration with subquotients S_{41} and S_{21^3}.

Properties of U

Theorem

(H, 2007) Let $U \cong \Omega^2(D_{21^3})$ *be as above. Then:*

- $U \otimes U$ *has neither a Specht nor dual Specht filtration.*

Properties of U

Theorem

(H, 2007) Let $U \cong \Omega^2(D_{21^3})$ be as above. Then:

- *$U \otimes U$ has neither a Specht nor dual Specht filtration.*
- $\text{Ext}^1_{\Sigma_5}(U, U^*) \cong k$.

Properties of U

Theorem

(H, 2007) Let $U \cong \Omega^2(D_{21^3})$ be as above. Then:

- *$U \otimes U$ has neither a Specht nor dual Specht filtration.*
- $\mathrm{Ext}^1_{\Sigma_5}(U, U^*) \cong k$.
- *U does not lift to characteristic zero.*

Properties of U

Theorem

(H, 2007) Let $U \cong \Omega^2(D_{21^3})$ be as above. Then:

- *$U \otimes U$ has neither a Specht nor dual Specht filtration.*
- $\mathrm{Ext}^1_{\Sigma_5}(U, U^*) \cong k$.
- *U does not lift to characteristic zero.*

This suggests we either give up on a theory of "tilting" modules or try to restrict the class of modules which we consider.

Some permutation modules for Σ_d

For $\lambda \vdash d$ there is a Young subgroup $\Sigma_\lambda \leq \Sigma_d$. For example

$$\Sigma_{(3,2,1)} = \Sigma_{\{1,2,3\}} \times \Sigma_{\{4,5\}} \times \Sigma_{\{6\}} \leq \Sigma_6.$$

Definition

The permutation module

$$M^\lambda \cong \mathrm{Ind}_{\Sigma_\lambda}^{\Sigma_d} k.$$

Some permutation modules for Σ_d

For $\lambda \vdash d$ there is a Young subgroup $\Sigma_\lambda \leq \Sigma_d$. For example

$$\Sigma_{(3,2,1)} = \Sigma_{\{1,2,3\}} \times \Sigma_{\{4,5\}} \times \Sigma_{\{6\}} \leq \Sigma_6.$$

Definition

The permutation module

$$M^\lambda \cong \mathrm{Ind}_{\Sigma_\lambda}^{\Sigma_d} k.$$

Theorem

(James, Donkin) The indecomposable summands of the M^λ's are indexed by partitions of d and called Young modules Y^μ. The Young modules are all self-dual and have Specht (and hence dual Specht) filtrations.

A complete example, Σ_6, $p = 3$

$$M^{(3,2,1)} \cong Y^{(3,2,1)} \oplus Y^{(4,2)} \oplus Y^{(4,2)} \oplus Y^{(5,1)}$$

A complete example, Σ_6, $p = 3$

$$M^{(3,2,1)} \cong Y^{(3,2,1)} \oplus Y^{(4,2)} \oplus Y^{(4,2)} \oplus Y^{(5,1)}$$

$$M^{(3,2,1)} \cong \begin{matrix} S^{(5,1)} \\ S^{(4,1^2)} \\ S^{(3,3)} \\ S^{(3,2,1)} \end{matrix} \oplus S^{(4,2)} \oplus S^{(4,2)} \oplus \begin{matrix} S^{(6)} \\ S^{(5,1)} \end{matrix}$$

A complete example, Σ_6, $p = 3$

$$M^{(3,2,1)} \cong Y^{(3,2,1)} \oplus Y^{(4,2)} \oplus Y^{(4,2)} \oplus Y^{(5,1)}$$

$$M^{(3,2,1)} \cong \begin{array}{c} S^{(5,1)} \\ S^{(4,1^2)} \\ S^{(3,3)} \\ S^{(3,2,1)} \end{array} \oplus S^{(4,2)} \oplus S^{(4,2)} \oplus \begin{array}{c} S^{(6)} \\ S^{(5,1)} \end{array}$$

$$M^{(3,2,1)} \cong \begin{array}{ccc} & 321 & \\ 2^3 & 1^6 & 31^3 \\ 321 & 21^4 & 321 \\ 2^3 & 1^6 & 31^3 \\ & 321 & \end{array} \oplus 42 \oplus 42 \oplus \begin{array}{c} 2^3 \\ 321 \\ 2^3 \end{array}$$

A complete example, Σ_6, $p = 3$

$$M^{(3,2,1)} \cong Y^{(3,2,1)} \oplus Y^{(4,2)} \oplus Y^{(4,2)} \oplus Y^{(5,1)}$$

$$M^{(3,2,1)} \cong \begin{matrix} S^{(5,1)} \\ S^{(4,1^2)} \\ S^{(3,3)} \\ S^{(3,2,1)} \end{matrix} \oplus S^{(4,2)} \oplus S^{(4,2)} \oplus \begin{matrix} S^{(6)} \\ S^{(5,1)} \end{matrix}$$

$$M^{(3,2,1)} \cong \begin{matrix} & 321 & \\ 2^3 & 1^6 & 31^3 \\ 321 & 21^4 & 321 \\ 2^3 & 1^6 & 31^3 \\ & 321 & \end{matrix} \oplus 42 \oplus 42 \oplus \begin{matrix} 2^3 \\ 321 \\ 2^3 \end{matrix}$$

Perhaps Young modules are the tilting modules for Σ_d?

But there are more! Remember that tensoring with sgn preserves this set so the twisted Young modules $Y^\lambda \otimes \text{sgn}$ also belong to this set, and these modules are not always isomorphic to Young modules!

But there are more! Remember that tensoring with sgn preserves this set so the twisted Young modules $Y^\lambda \otimes \mathrm{sgn}$ also belong to this set, and these modules are not always isomorphic to Young modules!

Definition

Suppose $\lambda \vdash a$ and $\mu \vdash d-a$. The signed permutation module is:

$$M^{(\lambda|\mu)} \cong \mathrm{Ind}_{\Sigma_\lambda \times \Sigma_\mu}^{\Sigma_d} k \boxtimes \mathrm{sgn}\,.$$

Summands of these are called signed Young modules, and are simultaneous generalizations of Young and twisted Young modules.

But there are more! Remember that tensoring with sgn preserves this set so the twisted Young modules $Y^\lambda \otimes \mathrm{sgn}$ also belong to this set, and these modules are not always isomorphic to Young modules!

Definition

Suppose $\lambda \vdash a$ and $\mu \vdash d - a$. The signed permutation module is:

$$M^{(\lambda|\mu)} \cong \mathrm{Ind}_{\Sigma_\lambda \times \Sigma_\mu}^{\Sigma_d} k \boxtimes \mathrm{sgn}\,.$$

Summands of these are called signed Young modules, and are simultaneous generalizations of Young and twisted Young modules.

Fact

Signed Young modules are self-dual with both Specht and dual Specht filtrations.

Signed Young modules

Tilting modules correspond bijectively to projective modules for the Schur algebra. Signed Young modules are in natural bijective correspondence with projective modules for the Schur superalgebra.

I once believed that the signed Young modules would be exactly the set of indecomposable, *self-dual* modules for Σ_d which had both Specht and dual Specht filtrations. However...

Signed Young modules

Tilting modules correspond bijectively to projective modules for the Schur algebra. Signed Young modules are in natural bijective correspondence with projective modules for the Schur superalgebra.

I once believed that the signed Young modules would be exactly the set of indecomposable, *self-dual* modules for Σ_d which had both Specht and dual Specht filtrations. However...

Theorem

(Paget, Wildon 2006) Let $H \cong (\Sigma_p \times \Sigma_p) \rtimes \Sigma_2 \leq \Sigma_{2p}$. *Then* $\mathrm{Ind}_H^{\Sigma_{2p}} k$ *has a summand which is indecomposable, self-dual and has both Specht and dual Specht filtration. It is not isomorphic to a signed Young module.*

Conjectural tilting modules for Σ_d.

Definition

An indecomposable module has trivial source if it is a direct summand of a permutation module on the cosets of some subgroup $H \leq G$.

Conjectural tilting modules for Σ_d.

Definition

An indecomposable module has trivial source if it is a direct summand of a permutation module on the cosets of some subgroup $H \leq G$.

Conjecture: The indecomposable self-dual modules for Σ_d which have both Specht and dual Specht filtration have trivial source.

Conjectural tilting modules for Σ_d.

Definition

An indecomposable module has trivial source if it is a direct summand of a permutation module on the cosets of some subgroup $H \leq G$.

Conjecture: The indecomposable self-dual modules for Σ_d which have both Specht and dual Specht filtration have trivial source.

Remarks:

- This includes all signed Young modules and the modules of Paget/Wildon.

Conjectural tilting modules for Σ_d.

Definition

An indecomposable module has trivial source if it is a direct summand of a permutation module on the cosets of some subgroup $H \leq G$.

Conjecture: The indecomposable self-dual modules for Σ_d which have both Specht and dual Specht filtration have trivial source.

Remarks:

- This includes all signed Young modules and the modules of Paget/Wildon.
- This would immediately imply there are only finitely many.

Conjectural tilting modules for Σ_d.

Definition

An indecomposable module has trivial source if it is a direct summand of a permutation module on the cosets of some subgroup $H \leq G$.

Conjecture: The indecomposable self-dual modules for Σ_d which have both Specht and dual Specht filtration have trivial source.

Remarks:

- This includes all signed Young modules and the modules of Paget/Wildon.
- This would immediately imply there are only finitely many.
- It would also imply they all lift to characteristic zero.

Irreducible Specht modules

The irreducibles D_λ are all self-dual. Thus if a Specht module S^λ is irreducible then it is self-dual and hence (trivially) has both a Specht and dual Specht filtration.

Irreducible Specht modules

The irreducibles D_λ are all self-dual. Thus if a Specht module S^λ is irreducible then it is self-dual and hence (trivially) has both a Specht and dual Specht filtration.

Theorem

(Fayers, 2005) There is a nice combinatorial description for precisely which Specht modules S^λ remain irreducible upon reduction to characteristic p.

Irreducible Specht modules

The irreducibles D_λ are all self-dual. Thus if a Specht module S^λ is irreducible then it is self-dual and hence (trivially) has both a Specht and dual Specht filtration.

Theorem

(Fayers, 2005) There is a nice combinatorial description for precisely which Specht modules S^λ remain irreducible upon reduction to characteristic p.

Theorem

(H, 2006) A Specht module S^λ is isomorphic to a signed Young module if and only if it is irreducible. In particular, irreducible Specht modules have trivial source.

Even more problems

- Suppose M and N are indecomposable self-dual modules with Specht filtrations. Is $M \otimes N$ a direct sum of such modules?

Even more problems

- Suppose M and N are indecomposable self-dual modules with Specht filtrations. Is $M \otimes N$ a direct sum of such modules?
- Must indecomposable, trivial source modules have either Specht or dual Specht filtrations? More generally, must Σ_d modules which are reductions of some characteristic zero module have Specht or dual Specht filtrations?

Even more problems

- Suppose M and N are indecomposable self-dual modules with Specht filtrations. Is $M \otimes N$ a direct sum of such modules?
- Must indecomposable, trivial source modules have either Specht or dual Specht filtrations? More generally, must Σ_d modules which are reductions of some characteristic zero module have Specht or dual Specht filtrations?
- Suppose M and N are indecomposable self-dual modules with Specht filtrations and suppose $p > 3$. Is $\mathrm{Ext}^i_{\Sigma_d}(M, N) = 0$ for $1 \leq i \leq p - 3$?

Even more problems

- Suppose M and N are indecomposable self-dual modules with Specht filtrations. Is $M \otimes N$ a direct sum of such modules?
- Must indecomposable, trivial source modules have either Specht or dual Specht filtrations? More generally, must Σ_d modules which are reductions of some characteristic zero module have Specht or dual Specht filtrations?
- Suppose M and N are indecomposable self-dual modules with Specht filtrations and suppose $p > 3$. Is $\mathrm{Ext}^i_{\Sigma_d}(M, N) = 0$ for $1 \leq i \leq p - 3$?
- Suppose M and N are indecomposable self-dual modules with Specht filtrations and suppose $p > 3$. Do M and N lift to characteristic zero?

Even more problems

- Suppose M and N are indecomposable self-dual modules with Specht filtrations. Is $M \otimes N$ a direct sum of such modules?
- Must indecomposable, trivial source modules have either Specht or dual Specht filtrations? More generally, must Σ_d modules which are reductions of some characteristic zero module have Specht or dual Specht filtrations?
- Suppose M and N are indecomposable self-dual modules with Specht filtrations and suppose $p > 3$. Is $\mathrm{Ext}^i_{\Sigma_d}(M, N) = 0$ for $1 \leq i \leq p - 3$?
- Suppose M and N are indecomposable self-dual modules with Specht filtrations and suppose $p > 3$. Do M and N lift to characteristic zero?
- Suppose U is indecomposable with both Specht and dual Specht filtrations. Must $\mathrm{Ext}^1_{\Sigma_d}(U, U) = 0$?

Even more problems

- Suppose M and N are indecomposable self-dual modules with Specht filtrations. Is $M \otimes N$ a direct sum of such modules?
- Must indecomposable, trivial source modules have either Specht or dual Specht filtrations? More generally, must Σ_d modules which are reductions of some characteristic zero module have Specht or dual Specht filtrations?
- Suppose M and N are indecomposable self-dual modules with Specht filtrations and suppose $p > 3$. Is $\mathrm{Ext}^i_{\Sigma_d}(M, N) = 0$ for $1 \leq i \leq p - 3$?
- Suppose M and N are indecomposable self-dual modules with Specht filtrations and suppose $p > 3$. Do M and N lift to characteristic zero?
- Suppose U is indecomposable with both Specht and dual Specht filtrations. Must $\mathrm{Ext}^1_{\Sigma_d}(U, U) = 0$?

Thank you for your attention!