Irreducible Modular Representations of Finite and Algebraic Groups

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June 8, 2007

Abstract

In these notes we outline some aspects of the modular representation theories of finite groups of Lie type in defining and cross-characteristics, with particular interest paid to how these theories relate to the modular representation theory of algebraic groups and the (characteristic 0) representation theory of Lie algebras and quantum groups. We begin by summarizing some classical results on the representation theory of complex semisimple Lie algebras and Lie groups, and then compare the classical theory to the representation theory of algebraic groups, discussing some of the issues encountered in moving to fields of positive characteristic and discussing some of the progress that has been in resolving these issues. We then discuss how the study of maximal subgroups leads to the study of linear representations in cross-characteristic, and conclude with a discussion of how the theory of quantum enveloping algebras (quantum groups) helps us to understand this situation.

1 Characteristic Zero Lie Theory

In order to establish our notation and to provide a reference to which we will later refer, we begin by summarizing some well known results on the structure and representation theory of complex semisimple Lie algebras and Lie groups. Readers who are already familiar with this material may skip ahead to Section 2. For further reference consult [6], [16].

1.1 Structure of Complex Semisimple Lie Algebras

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. (Readers not familiar with these notions may want to look ahead to Example 1.1.1.) It is a fact that $\mathfrak{h}$ is diagonalizable for any finite-dimensional representation of $\mathfrak{g}$. Specializing to the adjoint

*This document is based on two lectures delivered by Leonard Scott on April 25, 2007 and May 2, 2007, in the University of Virginia Algebra Seminar and in Professor Scott's course MATH 852 Representation Theory. Some additional material has been added, especially from [16] and [37] for the first part of the notes, and from [8], [20] and other references listed herein for the second part of the notes.

representation, $\mathfrak{g}$ decomposes as a direct sum of weight spaces for the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigsqcup_{\alpha\in\Phi} \mathfrak{g}_\alpha$$

Here $\mathfrak{g}_\alpha$ denotes the weight space of weight $\alpha \in \mathfrak{h}^*$. (Note that $\mathfrak{g}_0 = \mathfrak{h}$.) We call the nonzero weights $\Phi \subset \mathfrak{h}^*$ the roots of $\mathfrak{g}$. The $\mathbb{R}$-span $E = \mathbb{R}\Phi$ of the roots in $\mathfrak{h}^*$ is an ℓ-dimensional ($\ell = \dim_\mathbb{C} \mathfrak{h}^*$) euclidean space, in which Φ is a root system and on which we have a non-degenerate symmetric bilinear form, denoted $\langle\cdot,\cdot\rangle$, which is invariant under the Weyl group W of Φ.

Fix a base $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ of the root system Φ. We refer to the elements of Π as the simple roots. Then every element of Φ can be written as an integral linear combination of the simple roots with all coefficients of like sign, and $\Phi = \Phi^+ \bigsqcup \Phi^-$, where Φ^+ is the collection of positive roots (i.e., those roots which can be written as a non-negative integral combination of the α_i), and where $\Phi^- = -\Phi^+$ is the collection of negative roots.

If $\alpha \in \Phi$ and $\alpha = \sum_{i=1}^{\ell} c_i\alpha_i$, define the height of α by $\operatorname{ht}\alpha = \sum_{i=1}^{\ell} c_i$. We have a partial order $\leq$ on Λ (in fact, on all of $\mathfrak{h}^*$) defined by $\mu \leq \lambda$ if $\lambda - \mu$ is a non-negative integral combination of positive roots. So $\Phi^+ = \{\alpha \in \Phi : \alpha > 0\} = \{\alpha \in \Phi : \operatorname{ht}\alpha > 0\}$, and $\Phi^- = \{\alpha \in \Phi : \alpha < 0\} = \{\alpha \in \Phi : \operatorname{ht}\alpha < 0\}$. (Note that $\Pi \bigsqcup (-\Pi) = \{\alpha \in \Phi : \operatorname{ht}\alpha = 1\}$.)

To each root $\alpha \in \Phi$ we have its associated coroot $\alpha^\vee = 2\alpha/\langle\alpha, \alpha\rangle$. The set of all coroots $\Phi^\vee = \{\alpha^\vee : \alpha \in \Phi\}$ forms a root system in E, called the dual or coroot system of Φ. Evidently $\langle\alpha, \alpha^\vee\rangle = 2$ for all $\alpha \in \Phi$. The set $\Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_\ell^\vee\}$ is a base for the coroot system $\Phi^\vee$.

Let C denote the $\ell \times \ell$ Cartan matrix $C = (c_{ij}) = (\langle\alpha_i^\vee, \alpha_j\rangle)$. The matrix C is symmetrizable, that is, there exists an $\ell \times \ell$ diagonal matrix $D = \operatorname{diag}(d_1, \ldots, d_\ell)$ with entries in $\mathbb{Z}^+$ such that DC is symmetric.

For $\alpha \in \Phi$, define the reflection $s_\alpha \in GL(E)$ by $s_\alpha(x) = x - \langle x, \alpha^\vee\rangle\alpha$ for all $x \in E$. The Weyl group W of Φ is the finite reflection group generated by the s_α. In fact, W is a finite Coxeter group generated by the simple reflections $s_i := s_{\alpha_i}$ ($\alpha_i \in \Pi$).

If $\lambda \in E$ is a vector such that $\langle\lambda, \alpha^\vee\rangle \in \mathbb{Z}$ for all $\alpha \in \Phi$, we call λ an integral weight. Let $\varpi_i \in E$ denote the unique vector such that $\langle\varpi_i, \alpha_j^\vee\rangle = \delta_{ij}$ (Kronecker delta) for all simple roots $\alpha_j \in \Pi$. We call ϖ_i the fundamental dominant (integral) weight corresponding to the simple root α_i. The free abelian group Λ generated by the fundamental dominant weights $\varpi_1, \ldots, \varpi_\ell$ is called the weight lattice. As a set, $\Lambda = \{\lambda \in E : \langle\lambda, \alpha^\vee\rangle \in \mathbb{Z},\ \forall\, \alpha \in \Phi\}$, the set of all integral weights. Note that $\alpha \in \Lambda$ for all $\alpha \in \Phi$. We call the subgroup Λ_r of Λ generated by the roots $\alpha \in \Phi$ the root lattice. It is a subgroup of finite index in Λ. If $\mu \in \Lambda$ and $\langle\mu, \alpha^\vee\rangle \geq 0$ for all $\alpha \in \Phi$, we call μ a dominant (integral) weight, and we denote the collection of all dominant weights by Λ^+. We have $\mu \in \Lambda^+$ if and only if μ is equal to a non-negative integral combination of the fundamental dominant weights.

Associated to the choice Π of a base for Φ, we have the triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) is the subalgebra generated by all positive (resp. negative) root spaces. We often denote the subalgebra $\mathfrak{h} \oplus \mathfrak{n}^+$ by $\mathfrak{b}^+$ (resp. $\mathfrak{b}^- = \mathfrak{n}^- \oplus \mathfrak{h}$). Denote the universal enveloping algebras of $\mathfrak{g}$, $\mathfrak{n}^+$, $\mathfrak{h}$, etc. by $\mathcal{U}(\mathfrak{g})$, $\mathcal{U}(\mathfrak{n}^+)$, $\mathcal{U}(\mathfrak{h})$, etc. By the PBW Basis Theorem, the natural multiplication maps define isomorphisms of vector spaces $\mathcal{U}(\mathfrak{g}) \cong \mathcal{U}(\mathfrak{n}^-) \otimes_\mathbb{C} \mathcal{U}(\mathfrak{h}) \otimes_\mathbb{C} \mathcal{U}(\mathfrak{n}^+) \cong \mathcal{U}(\mathfrak{n}^-) \otimes_\mathbb{C} \mathcal{U}(\mathfrak{b}^+)$.

Example 1.1.1. Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. Let $\mathfrak{h}$ denote the subalgebra of trace zero diagaonal matrices. Then $\mathfrak{h}$ is a Cartan subalgebra in $\mathfrak{g}$. For $1 \leq i \leq n$ let $\epsilon_i \in \mathfrak{h}^*$ be defined by $\epsilon_i(\text{diag}(a_1, \dots, a_n)) = a_i$. Then $\Phi = \{\epsilon_i - \epsilon_j : i \neq j\}$. For $1 \leq i \leq n-1$ let $\alpha_i = \epsilon_i - \epsilon_{i+1}$. Then $\Pi = \{\alpha_1, \dots, \alpha_{n-1}\}$ is a base for Φ, and with respect to this base we have $\Phi^+ = \{\epsilon_i - \epsilon_j : i < j\}$. For $1 \leq i \leq n-1$ we have $\varpi_i = \epsilon_1 + \cdots + \epsilon_i$. For $i \neq j$ we have $\mathfrak{g}_{\epsilon_i - \epsilon_j} = \mathbb{C}E_{ij}$, where $E_{ij} \in \mathfrak{g}$ denotes the matrix having 1 in the (i, j) position and zeros elsewhere. Then $\mathfrak{n}^+$ is the subalgebra of strictly upper triangular matrcies, and $\mathfrak{b}^+$ is the subalgebra of trace zero upper triangular matrices.

1.2 Irreducible $\mathfrak{g}$-modules

Fix a base $\Pi \subset \Phi$, and hence a corresponding triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{b}^+$.

Let V be a $\mathfrak{g}$-module, and suppose that there exists a weight vector $v^+ \in V_\lambda$ such that $\mathfrak{b}^+.v^+ = 0$. Then we call v^+ a maximal vector of weight λ in V. Suppose furthermore that V is generated as a $\mathfrak{g}$-module by v^+. Then all other weights μ of V satisfy $\mu \leq \lambda$, and we call λ the highest weight of V. Suppose finally that V is an irreducible $\mathfrak{g}$-module. Then the line $\mathbb{C}v^+ \subset V$ is uniquely determined by the fact that v^+ is a maximal vector, and any other irreducible $\mathfrak{g}$-module generated by a maximal vector of highest weight λ is necessarily isomorphic to V.

As a specific example of the above setup, given $\lambda \in \mathfrak{h}^*$ define a $\mathfrak{g}$-module of highest weight λ as follows. Let $\mathbb{C}_\lambda$ be the one-dimensional $\mathbb{C}$-vector space with basis element denoted by λ. Define a one-dimensional representation of $\mathfrak{b}^+$ on $\mathbb{C}_\lambda$ by $\mathfrak{b}^+.\lambda = 0$ and $h.\lambda = \lambda(h)\lambda$ for all $h \in \mathfrak{h}$. Set $V(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b}^+)} \mathbb{C}_\lambda$, the induced module for $\mathbb{C}_\lambda$ from $\mathfrak{b}^+$ to $\mathfrak{g}$. Then $V(\lambda)$ is a $\mathfrak{g}$-module of highest weight λ, called the Verma module (or standard module) of highest weight λ. As a vector space, $V(\lambda) = \mathcal{U}(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathbb{C}_\lambda$. Every $\mathfrak{g}$-module of highest weight λ is a homomorphic image of $V(\lambda)$. The Verma module $V(\lambda)$ has a unique maximal submodule, and its irreducible head is denoted $L(\lambda)$. The necessary and sufficient condition for $L(\lambda)$ to be finite-dimensional is for λ to be a dominant integral weight.

Consider an arbitrary finite-dimensional irreducible $\mathfrak{g}$-module V. Being a finite direct sum of weight spaces for $\mathfrak{h}$, V must contain a weight space V_λ with $V_{\lambda+\alpha} = 0$ for all $\alpha \in \Phi^+$ (i.e., λ is maximal among all weights μ of V with respect to the partial order $\leq$ on $\mathfrak{h}^*$). Then any $v \in V_\lambda$ is necessarily a maximal vector of weight λ in V. By irreducibility, V is generated as a $\mathfrak{g}$-module by v, hence is a homomorphic image of the Verma module $V(\lambda)$. But $L(\lambda)$ is the unique irreducible quotient of $V(\lambda)$, so we conclude that $V \cong L(\lambda)$ and $\lambda \in \Lambda^+$. Thus, the finite-dimensional irreducible $\mathfrak{g}$-modules are parametrized (up to isomorphism) by their highest weights, and we have a bijection between elements of Λ^+ and the finite-dimensional irreducible $\mathfrak{g}$-modules given by $\lambda \leftrightarrow L(\lambda)$.

For future reference, we mention the following theorem:

Theorem 1.2.1 (Weyl's Complete Reducibility Theorem)**.** Let V be a finite-dimensional $\mathfrak{g}$-module. Then V decomposes as a direct sum of irreducible $\mathfrak{g}$-modules. In particular, $\text{Ext}^1_{\mathfrak{g}}(L(\lambda), L(\mu)) = 0$ for $\lambda \neq \mu \in \Lambda^+$.

1.3 Character and Dimension Formulae

Given a (not necessarily finite-dimensional) $\mathfrak{g}$-module V such that V is a direct sum of finite-dimensional $\mathfrak{h}$-weight spaces V_λ ($\lambda \in \mathfrak{h}^*$), we define the formal character of V by

$$\operatorname{ch} V = \sum_{\lambda \in \mathfrak{h}^*} (\dim V_\lambda) e^\lambda$$

Here the e^λ are formal symbols satisfying $e^\lambda e^\mu = e^{\lambda+\mu}$ for all $\lambda, \mu \in \mathfrak{h}^*$. By Weyl's Complete Reducibility Theorem 1.2.1, every finite-dimensional $\mathfrak{g}$-module decomposes as a direct sum of irreducible $\mathfrak{g}$-submodules. So the problem of determining all characters of finite-dimensional $\mathfrak{g}$-modules reduces to the problem of computing the $\operatorname{ch} L(\lambda)$ for $\lambda \in \Lambda^+$.

We begin by computing the formal character of the Verma module $V(\lambda)$. The Verma module $V(\lambda)$ has a finite composition series with composition factors of the form $L(w \cdot \lambda)$ for $w \in W$, where $w \cdot \lambda := w(\lambda + \rho) - \rho$ and $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha = \sum_{i=1}^{l} \varpi_i$ is the Weyl weight. Moreover, $L(w \cdot \lambda)$ occurs as a composition factor of $V(\lambda)$ only if $w \cdot \lambda \leq \lambda$, and $L(\lambda)$ occurs as a composition factor of $V(\lambda)$ with multiplicity one.

Now $\operatorname{ch} V(\lambda) = \sum_{w \in W} a_w \operatorname{ch} L(w \cdot \lambda)$ for some non-negative integers a_w with $a_1 = 1$. A similar equation holds for each $\operatorname{ch} V(w \cdot \lambda)$ with $w \cdot \lambda \leq \lambda$. We thus obtain a system of equations describing each $\operatorname{ch} V(w \cdot \lambda)$ ($w \in W$) in terms of the characters $\operatorname{ch} L(y \cdot \lambda)$ ($y \in W$). Writing the coefficients of this system of equations with respect to a suitable ordering of the set $\{w \cdot \lambda : w \in W, w \cdot \lambda \leq \lambda\}$, we obtain an upper triangular matrix over $\mathbb{Z}$ having all diagonal entries equal to one. Inverting this matrix we obtain an equation of the form

$$\operatorname{ch} L(\lambda) = \sum_{w \in W} b_w \operatorname{ch} V(w \cdot \lambda) \tag{1}$$

for some $b_w \in \mathbb{Z}$. Following [16], we can look at the action of the Weyl group[1] on both sides of this equation and deduce the following famous result of Weyl:

Theorem 1.3.1 (Weyl's Character Formula)**.** Let $\lambda \in \Lambda^+$. For $w \in W$ let $l(w)$ denote the length of w as a word in the generators $s_1, \ldots, s_l$ of W. Then

$$\operatorname{ch} L(\lambda) = \sum_{w \in W} (-1)^{l(w)} \operatorname{ch} V(w \cdot \lambda) \tag{2}$$

From the vector space isomorphism $V(w \cdot \lambda) = \mathcal{U}(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathbb{C}_{w \cdot \lambda}$ we have that, for each $\mu \in \mathfrak{h}^*$, $\dim V(w \cdot \lambda)_\mu$ is equal to the number of ways that $w \cdot \lambda - \mu$ can be written as a non-negative integral sum of positive roots. From this we deduce the formula

$$\operatorname{ch} V(w \cdot \lambda) = \frac{e^{w(\lambda+\rho)}}{\sum_{y \in W} (-1)^{l(y)} e^{y\rho}}$$

and hence the following alternate formulation of Weyl's character formula:

[1]Actually, to this point λ could be any element of Λ. But when λ is dominant, so that $L(\lambda)$ is finite-dimensional, the Weyl group fixes char $L(\lambda)$ is its action $w(e^\mu) = e^{w(\mu)}$, $\mu \in \Lambda$, $w \in W$.

Theorem 1.3.2 (Weyl's Character Formula, Alternate Formulation)**.** Let $\lambda \in \Lambda^+$. For $w \in W$ let $l(w)$ denote the length of w as a word in the generators $s_1, \ldots, s_l$ of W. Then

$$\operatorname{ch} L(\lambda) = \frac{\sum_{w \in W} (-1)^{l(w)} e^{w(\lambda+\rho)}}{\sum_{w \in W} (-1)^{l(w)} e^{w\rho}} \tag{3}$$

A further consequence of Weyl's character formula is the following formula for the dimensions of the irreducible $\mathfrak{g}$-modules.

Theorem 1.3.3 (Weyl's dimension formula)**.** Let $\lambda \in \Lambda^+$. Then

$$\dim L(\lambda) = \frac{\prod_{\alpha \in \Phi^+} \langle \lambda + \rho, \alpha \rangle}{\prod_{\alpha \in \Phi^+} \langle \rho, \alpha \rangle} \tag{4}$$

So far we have concentrated on the representation theory of semisimple complex Lie algebras and have ignored the corresponding Lie groups. But since one can pass from the Lie algebra to the Lie group through the process of exponentiation, the representation theory of semisimple complex Lie groups exactly parallels that of the Lie algebras.

2 Algebraic Groups in Positive Characteristic

This section is based on [32].

It is well-known that the finite simple groups fall into three classes: the simple groups associated to finite groups of Lie type (loosely also called groups of Lie type, or simple groups of Lie type), the alternating groups, and the 26 sporadic finite simple groups, with the simple groups of Lie type taking up the bulk of the simple groups in some sense. (Tits has suggested that the alternating groups may be considered as groups of Lie type over the field of one element, in which case the simple groups of Lie type take up all but 26 of the known finite simple groups.[2]) We begin our study of the irreducible representations of the finite groups of Lie type with the irreducible representations of semisimple algebraic groups over fields of positive characteristic.

The Lie algebra $\mathfrak{g}$ of a semisimple algebraic group G over an algebraically closed field of characteristic zero carries much information about the structure of G, and in this case one can deduce results on the representation theory of G from results on the representation theory of the semisimple Lie algebra $\mathfrak{g}$. But the situation becomes more complicated when one passes from algebraically closed fields of characteristic zero to fields of arbitrary characteristic.

While the representation theory of semisimple algebraic groups in positive characteristic largely parallels that of the complex semisimple Lie algebras elucidated above in Section 1—the finite-dimensional irreducible modules are still parametrized by dominant integral weights—we lack complete information regarding the structure of the irreducible representations. Indeed, the problem of determining the formal characters and dimensions of the irreducible modules, and the progress that has been made towards this end, will be our central focus in the sections that follow.

[2]This is more than a joke, as it turns out. A theorem of Gordon James guarantees that the irreducible modular representations of the symmetric group of degree r are determined by those of irreducible modular representations of the degree r general linear group, via Schur-Weyl duality.

2.1 Notational Conventions

Let p be a prime number and let $k = \overline{\mathbb{F}}_p$. An (affine) algebraic group G over k is an affine algebraic variety $G \subset k^n$ (for some n) with a compatible group structure in the sense that multiplication $G \times G \to G$ and inversion $G \to G$ are morphisms of algebraic varieties. Denote the coordinate algebra of G by $k[G]$.

More generally, we call G an affine k-group scheme if G is a representable functor from the category of commutative k-algebras to the category of groups. (For this more general notion k need only be a commutative ring, though we will always assume that k is at least a field.) Thus there exists a commutative k-algebra $k[G]$ such that $G(A) = \mathrm{Hom}_{k-alg}(k[G], A)$ for all commutative k-algebras A. The fact that G is a group forces $k[G]$ to be a Hopf algebra. We say that G is an algebraic affine k-group scheme if $k[G]$ is finitely generated over k. Given a k-algebra A and $f, g \in G(A) = \mathrm{Hom}_{k-alg}(k[G], A)$, the product $fg \in G(A)$ is defined by $fg = \mu \circ (f \otimes g) \circ \Delta$, where here $\mu : A \otimes A \to A$ denotes multiplication in A and $\Delta : k[G] \to k[G] \otimes k[G]$ is the comultiplication in $k[G]$. When $k[G]$ is an affine algebraic k-group scheme that is reduced (i.e., when $k[G]$ has no nonzero nilpotent elements) and when we specialize to the case $k = \overline{\mathbb{F}}_p$ and $A = k$, then $G(A)$ is an affine algebraic group over k (in the classical sense of an affine algebraic variety) having coordinate algebra $k[G]$. We will feel free to consider our algebraic groups both in the classical sense as affine algebraic varieties, and in the functorial sense as affine algebraic k-group schemes.

We say that an affine algebraic group G is defined over a subfield k_0 of k provided that there exists a Hopf algebra A_0 over k_0 such that the natural map $k \otimes_{k_0} A_0 \to k[G]$ is Hopf algebra isomorphism. (We identify A_0 with an algebra of k_0-valued functions on G.) In what follows we will generally assume G to be defined over some finite subfield $\mathbb{F}_q$ of $k = \overline{\mathbb{F}}_p$ ($q = p^r$ a prime power).

Let G be a semisimple algebraic group over $k = \overline{\mathbb{F}}_p$. Fix a maximal torus T in G, a Borel subgroup B containing T and the opposite Borel subgroup B^+. Let $U^+ = R_u(B^+)$ the unipotent radical of B, and let $U = R_u(B)$. Denote the character group of T by $X = X(T)$. The torus T acts on the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ through the adjoint map Ad, and $\mathfrak{g}$ decomposes as a direct sum of weight spaces $\mathfrak{g}_\alpha$ ($\alpha \in X$) for T. The nonzero weights form a root system Φ in $E = X \otimes_{\mathbb{Z}} \mathbb{R}$, and the choice of Borel subgroup B^+ determines a positive system Φ^+ in Φ and hence a base $\Pi \subset \Phi^+$ for Φ.

Identify the co-character group $Y = Y(T) = \mathrm{Hom}(\mathbb{G}_m, T)$ with the group $\mathrm{Hom}_{\mathbb{Z}}(X(T), \mathbb{Z})$ via the pairing $\langle \cdot, \cdot \rangle : X(T) \times Y(T) \to \mathbb{Z}$, where $t^{\langle \lambda, \mu \rangle} = \lambda \circ \mu(t)$ for all $t \in T$, $\lambda \in X(T)$, $\mu \in Y(T)$. (Note that $\mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$.) Now identify $E^* = Y \otimes_{\mathbb{Z}} \mathbb{R}$ with E via the pairing $\langle \cdot, \cdot \rangle$. Then for each root $\alpha \in \Phi$ we have the associated coroot $\alpha^\vee \in Y$.

The abstract weights in E (i.e., the $\mu \in X \otimes_{\mathbb{Z}} \mathbb{R}$ satisfying $\langle \mu, \alpha^\vee \rangle \in \mathbb{Z}$ for all $\alpha \in \Phi$) span a lattice Λ containing the root lattice Λ_r (subgroup generated by all $\alpha \in \Phi$) as a subgroup of finite index. In fact, we have $\Lambda_r \subseteq X(T) \subseteq \Lambda$. If $X(T) = \Lambda$ we say that G is simply connected.

We say that a character $\lambda \in X$ is a dominant weight if $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all $\alpha \in \Phi^+$, or equivalently if λ can be written as a non-negative integral combination of the fundamental dominant weights $\varpi_i \in \Lambda$. Denote the collection of dominant weights in X by $X^+ = X(T)^+$. We call $\lambda \in X(T)^+$ an r-th restricted dominant weight if $0 \leq \langle \lambda, \alpha^\vee \rangle < p^r$ for all $\alpha \in \Pi$. Denote the collection of r-th restricted dominant weights by $X_r(T)$. We may refer to the elements of $X_1(T)$ simply as the restricted dominant weights.

Example 2.1.1. Let $G = GL_n$. Then $k[G] = k[(\det)^{-1}, X_{ij} : 1 \leq i,j \leq n]$, where $\det \in k[X_{ij} : 1 \leq i,j \leq n]$ is the determinant function (a polynomial in the variables X_{ij}). Identifying $GL_n(A)$ with the set of all $n \times n$ invertible matrices with entries in the given k-algebra A, we may take B^+ to be the subgroup of all upper triangular matrices, U^+ to be the subgroup of all strictly upper triangular matrices, and T to be the subgroup of all diagonal matrices. Multiplication in G is given by ordinary matrix multiplication. Comultiplication in $k[G]$ is given by $\Delta(X_{ij}) = \sum_{k=1}^{n} X_{ik} \otimes X_{kj}$. The group $G = GL_n$ has unipotent radical equal to the collection of scalar matrices, hence is reductive but not semisimple.

Example 2.1.2. Let $G = SL_n$. Since SL_n is the subgroup of GL_n defined by the vanishing of the polynomial $\det -1 \in k[GL_n]$, SL_n is a closed subgroup (closed subfunctor) of GL_n. We have $k[G] = k[(1-\det), X_{ij} : 1 \leq i,j \leq n]$, and we may take B^+, U^+, T to be the subgroups of upper triangular, strictly upper triangular and diagonal matrices, respectively. The group $G = SL_n$ is semisimple. If $\epsilon_i = X_{ii}|_T$ is the i-th coordinate function on T, then the characters $\alpha_i := \epsilon_i - \epsilon_{i+1}$ $(1 \leq i \leq n-1)$ form a basis for $X(T)$ and a base for Φ. (So SL_n is simply connected.) We have $\varpi_i = \epsilon_1 + \cdots + \epsilon_i$.

Example 2.1.3. Let p be a prime number, $q = p^r$ a power of p, and $G = GU_n(q)$ the general unitary group. Then $G = \left\{ U \in GL_n(\mathbb{F}_{q^2}) : U\overline{U}^t = 1 \right\}$, where $\overline{U} \in GL_n(\mathbb{F}_{q^2})$ denotes the matrix obtained from U by raising each entry to the q-th power and U^t denotes the matrix transpose of U. The special unitary group $SU_n(q)$ is the subgroup of $GU_n(q)$ of elements with determinant equal to one. Evidently $GU_n(q)$ and $SU_n(q)$ are both closed subgroups of $GL_n(\mathbb{F}_{q^2})$.

For further reference on general properties of affine algebraic groups, consult the standard references [5], [17], [36]. For further reference on algebraic group schemes and their representations, consult [20].

2.2 Chevalley Groups and other Finite Groups of Lie Type

The Chevalley groups over k (also called split semisimple groups of Lie type over k) are certain concrete constructions of semisimple algebraic groups from representations of complex semisimple Lie algebras. We briefly sketch their construction here; for further reference consult [37]. To begin, let $\mathfrak{g}_{\mathbb{C}}$ be a complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$, and let V be a faithful finite-dimensional $\mathfrak{g}_{\mathbb{C}}$-module. Let $L_1 = \Lambda$ the weight lattice of $\mathfrak{g}_{\mathbb{C}}$, (cf. Section 1.1), let $L_0 = \Lambda_r$ the root lattice of $\mathfrak{g}_{\mathbb{C}}$, and let L_V denote the sublattice of L_0 generated by all weights of $\mathfrak{h}_{\mathbb{C}}$ on V. Then $L_0 \subset L_V \subset L_1$.

There exists a basis $\{X_\alpha, H_i : \alpha \in \Phi, 1 \leq i \leq l\}$ for $\mathfrak{g}_{\mathbb{C}}$ with $X_\alpha \in \mathfrak{g}_\alpha$, $H_i \in \mathfrak{h}_{\mathbb{C}}$, called a Chevalley basis, such that all of the structure constants of $\mathfrak{g}_{\mathbb{C}}$ relative to the Chevalley basis are integers. Let $\mathcal{U}_{\mathbb{Z}}$ denote the subalgebra of $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ generated by all $X_\alpha^{(n)} := X_\alpha^n/n!$ $(\alpha \in \Phi, n \in \mathbb{N})$. (This subalgebra is known as the Kostant $\mathbb{Z}$-form of $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$.) Then there exists a lattice $V_{\mathbb{Z}}$ in V invariant under $\mathcal{U}_{\mathbb{Z}}$. Now given $t \in k$ and $\alpha \in \Phi$, we define an automorphism of $V^k := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$ as follows. Because V is finite-dimensional and since $X_\alpha V_\lambda \subset V_{\lambda+\alpha}$ for all weights λ of V, multiplication by X_α is a locally nilpotent endormorphism of V. Then the map $\exp tX_\alpha : V^k \to V^k$ defined by $(\exp tX_\alpha)(v \otimes a) = \sum_{n=0}^{\infty} X_\alpha^{(n)}.v \otimes t^n a$ is a well-defined automorphism of V^k.

Let G be the group of automorphisms of V^k generated by all $\exp tX_\alpha$ $(t \in k, \alpha \in \Phi)$. We call G a Chevalley group. In fact, G is a semisimple algebraic group with $\mathrm{Lie}(G) \cong \mathfrak{g} := \mathfrak{g}_\mathbb{Z} \otimes_\mathbb{Z} k$, where $\mathfrak{g}_\mathbb{Z}$ is the lattice in $\mathfrak{g}_\mathbb{C}$ preserving the $\mathbb{Z}$-form $V_\mathbb{Z}$. The lattice L_V is realized as the character group of a maximal torus T of G, and the lattices L_0 and L_1 are realized respectively as the root and weight lattices of G with respect to T. Moreover, every semisimple algebraic group G' over k can be constructed in this fashion by the choice of an appropriate faithful finite-dimensional $\mathfrak{g}_\mathbb{C}$-module V' for some complex semisimple Lie algebra $\mathfrak{g}_\mathbb{C}$.

If $L_V = L_1$ we say that G is a universal Chevalley group. The reason for this terminology is that if G' is another Chevalley group over k constructed from a faithful finite-dimensional $\mathfrak{g}_\mathbb{C}$-module V', then there exists a surjective homomorphism $\varphi : G \to G'$ with $\ker \varphi \subseteq Z(G)$ the center of G. So if G is the universal Chevalley group constructed from $\mathfrak{g}_\mathbb{C}$, then G is a covering group for all other Chevalley groups constructed from $\mathfrak{g}_\mathbb{C}$. Since $L_1 \cong \Lambda$ and $L_V \cong X(T)$ for some maximal torus T of G, we see that G is universal if and only if it is simply connected.

2.3 Frobenius Morphisms

Let G be an affine algebraic group defined over $\mathbb{F}_p$. Identify $k[G]$ with $k \otimes_{\mathbb{F}_p} A_0$, and define the Frobenius comorphism $F^* : k[G] \to k[G]$ by $F^*(a \otimes f) = a \otimes f^p$. This map is readily seen to be a Hopf algebra map, and hence is the comorphism of an algebraic group morphism $F = F_G : G \to G$ called the Frobenius morphism. We call the r-th power F^r of F the r-th Frobenius morphism.

(The comorphism F^* defined above is called the *geometric* Frobenius endomorphism of $k[G]$. There is a second Frobenius endomorphism of $k[G]$, called the *arithmetic* Frobenius endomorphism, defined by $a \otimes f \mapsto a^p \otimes f$. There are also more general notions of a Frobenius morphism: Let X be an affine algebraic variety over $k = \overline{\mathbb{F}}_p$ with coordinate algebra A. If $F^* : A \to A$ is an algebra homomorphism such that F^* is injective, $F^*(A) = A^q$ for some p-th power $q = p^r$, and if for each $f \in A$ there exists $m \geq 1$ such that $(F^*)^m(f) = f^{q^m}$, then we call the coordesponding morphism of varieties $F : X \to X$ a generalized Frobenius morphism on X, cf. [13] Chapter 4.)

Associated to the r-th Frobenius morphism F^r are two subgroups of G. The kernel of F^r is a normal subgroup called the r-th Frobenius kernel of G, and is denoted by G_r. In the language of affine group schemes, G_r is an infinitesimal group scheme. We have $G_r(K) = \mathrm{Hom}_{k-alg}(k[G_r], K) = \{e\}$, the trivial group, for any field extension K of k. So Frobenius kernels are always trivial when we consider them in the classical sense as affine algebraic varieties, but they may no longer be trivial when we consider G as an affine group scheme and permit k-algebras A that are not fields.

The fixed points G^{F^r} of G under the r-th Frobenius morphism form a finite subgroup of G, denoted $G(q)$ or $G(\mathbb{F}_q)$ (where $q = p^r$). (If we consider G as an affine algebraic variety over $k = \overline{\mathbb{F}}_p$, then $G(q)$ consists of those points in $G \subset k^n$ all of whose coordinates lie in $\mathbb{F}_q$.) If G is a Chevalley group, then we call $G(q)$ a finite Chevalley group (also called a finite group of Lie type). Other finite groups of Lie type are obtained from a Chevalley group G through various "twistings" of the Frobenius morphism or by taking the fixed point subgroup of G under a generalized Frobenius morphism.

The table on page 6 of [18] lists the orders of the finite Chevalley groups $G(q)$ when G is a universal Chevalley group.

Example 2.3.1. Let $G = \mathbb{G}_a$, the additive group, considered as an affine group scheme. Then $k[G] = k[X]$, a polynomial ring in one indeterminate X, and $G(A) = (A, +)$ the additive group of A, for each k-algebras A. The Frobenius morphism satisfies $F(t) = t^p$ for all $t \in G(A)$, so

$$G^{F^r}(A) = \{t \in A : t^{p^r} = t\} \qquad \text{and} \qquad G_r(A) = \{t \in G(A) : t^{p^r} = 0\}$$

Example 2.3.2. Let $G = \mathbb{G}_m$, the multiplicative group, considered as an affine group scheme. Then $k[G] = k[X, X^{-1}]$, a Laurent polynomial ring in one indeterminate X, and $G(A) = (A^\times, \times)$ the multiplicative group of units in A, for each k-algebra A. The Frobenius morphism satisfies $F(t) = t^p$ for all $t \in G(A)$, so

$$G^{F^r}(A) = \{t \in A^\times : t^{p^r} = t\} \qquad \text{and} \qquad G_r(A) = \{t \in A^\times : t^{p^r} = 1\}$$

2.4 Representations of Algebraic Groups

Let G be the universal Chevalley group constructed from $\mathfrak{g}_\mathbb{C}$ as in Section 2.2.

While the Lie algebra of an algebraic group over a field of positive characteristic carries less information concerning the structure and representations of the group than it does in characteristic zero, Chevalley showed that the high weight theory of complex Lie algebras and Lie groups does carry over to semisimple algebraic groups in the sense that the irreducible modules for a semisimple algebraic group G are still parameterized by dominant highest weights. So, loosely speaking, we have the same "number" of irreducible modules in positive characteristic as in characteristic zero.

One approach to constructing the irreducible G-module $L(\lambda)$ parametrized by a given dominant weight $\lambda \in X^+$ is to construct $L(\lambda)$ as a certain submodule of the coordinate algebra $k[G]$ (which is a G-module via the left regular representation). To begin, for a given $\lambda \in X^+$ let λ denote the one-dimensional B-module of T-weight λ. Set

$$\nabla(\lambda) := \operatorname{ind}_B^G \lambda = \{f \in k[G] : f(gb) = \lambda(b^{-1})f(g) \,\forall\, g \in G, \,\forall\, b \in B\}$$

the induced module of λ from B to G. (Another common notation for $\operatorname{ind}_B^G \lambda$ is $H^0(\lambda)$.) Then $\nabla(\lambda)$ is a (finite-dimensional) G-submodule of $k[G]$ of highest weight λ. One can show that $L(\lambda) := \operatorname{soc}_G \nabla(\lambda)$ is an irreducible G-module of highest weight $\lambda \in X^+$. Now let $\Delta(\lambda) = \nabla(-w_0\lambda)^*$. One can show that $\Delta(\lambda)$ is generated by a B^+-stable line of highest weight λ and that any other such G-module is a homomorphic image of $\Delta(\lambda)$. Moreover, $\Delta(\lambda)$ has a unique maximal submodule, and $\Delta(\lambda)/\operatorname{rad}_G \Delta(\lambda) \cong L(\lambda)$. We call $\Delta(\lambda)$ the Weyl module of highest weight λ. Evidently the finite-dimensional Weyl module $\Delta(\lambda)$ assumes a position in the representation theory of G similar to that held by the infinite-dimensional Verma module $V(\lambda)$ in the representation theory of $\mathfrak{g}_\mathbb{C}$ (cf. Section 1.2). (Another common notation for $\Delta(\lambda)$ is $V(\lambda)$, further emphasizing its similarity to the Verma modules of $\mathfrak{g}_\mathbb{C}$.)

A second approach to the construction of the irreducible G-modules makes more clear the connection between the simple modules $L(\lambda)_{\mathbb{C}}$ for $\mathfrak{g}_{\mathbb{C}}$ and the simple modules $L(\lambda)$ for G. Given $\lambda \in \Lambda^+$, let $L(\lambda)_{\mathbb{C}}$ denote the simple $\mathfrak{g}_{\mathbb{C}}$-module of highest weight λ. Then by the construction in Section 2.2, there exists a lattice $L(\lambda)_{\mathbb{Z}} \subset L(\lambda)_{\mathbb{C}}$ such that the group G acts naturally on $L(\lambda)_{\mathbb{Z}} \otimes k$. This G-module is no longer simple in general, but it does have an irreducible head. Indeed, $L(\lambda)_{\mathbb{Z}} \otimes k \cong \Delta(\lambda)$ the Weyl module[3] of highest weight λ, so its head is isomorphic to the simple G-module $L(\lambda)$.

Before we address the structure of the simple modules for the finite Chevalley group $G(q)$, we state the following theorem of Steinberg:

Theorem 2.4.1 (Steinberg's Tensor Product Theorem)**.** Let $\lambda \in X(T)^+$ and write $\lambda = \sum_{i=0}^{m} p^i \lambda_i$ with $\lambda_i \in X_1(T)$. Then $L(\lambda) \cong L(\lambda_0) \otimes L(\lambda_1)^{[1]} \otimes \cdots \otimes L(\lambda_m)^{[m]}$, where $L(\lambda_j)^{[j]}$ denotes the G-module obtained by composing the structure map for $L(\lambda_j)$ with the j-th Frobenius morphism.

In principle then, the structures of the irreducible G-modules $L(\lambda)$ are completely determined by those $L(\lambda)$ with restricted weights $\lambda \in X_1(T)$ and by the Frobenius morphism $F : G \to G$.

Now, each simple G-module $L(\lambda)$ remains simple on restriction to the finite Chevalley group $G(q)$, cf. [18] Section 2.12. Steinberg showed that in fact every irreducible $G(q)$-module can be obtained in this manner. (His result also holds for any finite group G^F of Lie type, F as discussed above.)

Theorem 2.4.2 (Steinberg, 1963)**.** Let L be an irreducible module over k for the finite group $G(q)$. Then L is the restriction from G of an irreducible G-module.

On the other hand, distinct irreducible G-modules may no longer be non-isomorphic on restriction to $G(q)$. Indeed, let $\lambda \in X_1(T)$. Then by the Tensor Product Theorem, $L(p^r\lambda) \cong L(\lambda)^{[r]}$. But $G(q) = G^{F^r}$ is the fixed point subgroup of G under the r-th Frobenius morphism, so $G(q)$ doesn't "see" the twist on $L(\lambda)$ and we have $L(\lambda) \cong L(p^r\lambda)$ as $G(q)$-modules. To parametrize the simple $G(q)$-modules, we must then restrict our attention to some subset of the dominant weights. Steinberg showed that the necessary dominant weights are precisely the r-th restricted dominant weights $\lambda \in X_r(\lambda)$. (He also gives a precise description of the weights needed in the general finite group of Lie type case. We stick to the Chevalley groups here and below for simplicity.) By the Tensor Product Theorem, one may even restrict attention to the restricted weights $\lambda \in X_1(T)$.

An important step in understanding the irreducible G-modules $L(\lambda)$ with $\lambda \in X_r(T)$, and hence their restrictions to the finite group $G(q)$, would be to know their dimensions and the dimensions of their weight spaces.

Problem 1. Give a "Weyl character formula" (cf. Theorem 1.3.1) for the irreducible G-modules $L(\lambda)$, $\lambda \in X_r(T)$ an r-th restricted dominant weight.

[3]This fact is quite nontrivial. It is a consequence of 'Kempf's Vanishing Theorem' for line bundles on G/B; equivalently, a statement regarding the vanishing of higher derived functors $H^i(\lambda) = R^i\mathrm{Ind}_B^G(\lambda), i > 0$ of certain induction functors. See [20] for an exposition.

A second problem, and one that only becomes interesting for finite-dimensional representations in the case of fields of positive characteristic (cf. Theorem 1.2.1), is to understand the ways in which the irreducible G-modules $L(\lambda)$ can "fit together." More precisely, we want to understand the morphisms between the irreducible G-modules, and hence (recursively) the structure of G-modules admitting a composition series.

Problem 2. Determine $\mathrm{Ext}^1_G(L(\lambda), L(\mu))$, $\mathrm{Ext}^1_{G(q)}(L(\lambda), L(\mu))$ for $\lambda, \mu \in X_r(T)$.

Both problems may (and should) be formulated for general finite groups G^F of Lie type, with a suitable modification of $X_r(T)$. Also, one may pose problems analogous to Problem 2 for Ext^2 or higher Ext groups. The emphasis on Ext^1 here is partly motivated by the special role of H^1 in the next section.

Analogous to the situation of Section 1.3 Equation 1, we can write an equation of the form

$$\mathrm{ch}\, L(\lambda) = \sum_{\substack{w \in W_p \\ w\cdot\lambda \in X^+}} b_w \,\mathrm{ch}\, \Delta(w \cdot \lambda)$$

for some $b_w \in \mathbb{Z}$, relating the characters of the simple G-module $L(\lambda)$ to the characters of the Weyl modules $\Delta(w \cdot \lambda)$ with w an element of the affine Weyl group $W_p = p\mathbb{Z}\Phi \rtimes W$. Since the characters $\mathrm{ch}\,\Delta(\mu)$ are given by Weyl's Character Formula (i.e., by Equation 3 of Theorem 1.3.2 with $\Delta(\mu)$ substituted for $L(\mu)$), cf. [20] Section II.5.10, the solution of Problem 1 above amounts to the determination of the integers b_w.

Lusztig's conjecture, below, asserts that the coefficients b_w are in effect given by the values at 1 of certain polynomials $P_{y,w}$, called Kazhdan-Lusztig polynomials, associated with the Coxeter group W_p. Though Lusztig's formula is known to be correct for $p \gg h$, where $h = 1 + \langle \rho, \alpha_0^\vee \rangle$ is the Coxeter number of Φ (α_0 is the longest short root of Φ), a lower bound for p is not known. Before stating Lusztig's conjecture we need some terminology: We say that a dominant weight μ lies in the Jantzen region if $\langle \mu + \rho, \alpha_0^\vee \rangle \leq p(p - h + 2)$.

Conjecture 2.4.1 (Lusztig, 1979)**.** Let λ be a weight in the Jantzen region (which includes all restricted weights if $p \geq 2h - 2$, h the Coxeter number of Φ). Then if $p \geq h$, $\dim L(\lambda)_\nu$ is given as follows: Choose w in the affine Weyl group $W_p = p\mathbb{Z}\Phi \rtimes W$ such that $\lambda = w \cdot \lambda_0$, for some λ_0 (unique) with $-p \leq (\lambda_0 + \rho)(H_\alpha) \leq 0$ for all $\alpha \in \Phi^+$. (We say that λ_0 is in the antidominant lowest alcove). Let w_0 denote the longest element of W. Then

$$\dim L(\lambda)_\gamma = \sum (-1)^{l(w)-l(y)} P_{y,w}(1) \dim \Delta(w_0 y \cdot \lambda_0)_\gamma \tag{5}$$

where the sum is taken over all $y \in W$ such that $w_0 y \cdot \lambda_0$ is dominant and $w_0 y \cdot \lambda_0 \leq w_0 w \cdot \lambda_0 = \lambda$, $\Delta(w_0 y \cdot \lambda_0)$ is the Weyl module of highest weight $w_0 y \cdot \lambda_0$, and $P_{y,w}$ is a Kazhdan-Lusztig polynomial associated with the Coxeter group W_p.

In a helpful strengthening of the conjecture, Kato has proposed that formula (5) always holds for $\lambda \in X_1(T)$ provided $p \geq h$, thus not requiring $p \geq 2h - 2$. This strengthening seems to hold up empirically in the one (meager) case in which the result is known, i.e., for SL_5 over $\overline{\mathbb{F}}_5$, cf. [35]. The result remains an open problem for SL_6 and SL_7 over $\overline{\mathbb{F}}_7$.

3 Maximal Subgroups

This section is based on [33].

Suppose G is a finite group, and $H \leq G$ is a maximal subgroup. Historically, the study of maximal subgroups (or, more precisely, pairs (G, H)) has been a principle topic in the theory of finite groups. For example, through the study of maximal subgroups, one may hope to obtain structural information about groups in general, through a recursive procedure. As another example, and one which is the principle motivating factor for the rest of these notes, is the role maximal subgroups play in the theory of permutation representations of finite groups. The group G acts on G/H not only transitively, but primitively, and the permutation representations associated to the pairs (G, H) for H running over all maximal subgroups of G constitute the building blocks for all permutation representations of G (analogous for nonlinear representations to the role played by the irreducible representations in the linear case). Finite automata theory provides one interesting modern application of permutation representations; see Chapters 6 and 7 of [14], entitled "Covering by permutation and reset machines" and "The theory of Krohn and Rhodes." We remark that any permutation machine may be "covered," in the terminology of [14], by a "cascade" (wreath-like) product of primitive permutation machines.

Determining maximal subgroups of an arbitrary finite group reduces to the case of solving this problem for simple or nearly simple groups by a theorem of Ashbacher and Scott, which we loosely paraphrase below.

Theorem 3.1. (Aschbacher–Scott, 1985 [4]) The determination (up to conjugacy) of all pairs (G, M), G a finite group and $M \leq G$ a maximal subgroup, reduces modulo "smaller or easier" problems to the following:

1. G is almost simple (and M is maximal in G)

2. $G = H.V$ a semidirect product of a quasisimple finite group H and one of its irreducible modules V over $\mathbb{F}_p$, and M is a complement to V. In this case, the conjugacy classes in G of such maximal subgroups M correspond bijectively to elements of the cohomology group $H^1(H, V)$.

Remark.

1. Recall a finite group G is almost simple if G can be sandwiched as $G_0 \leq G \leq \mathrm{Aut}(G_0)$ for a finite simple group G_0 and its automorphism group. By the Schreier Conjecture (now a theorem), $\mathrm{Aut}(G_0)$ is a solvable group; usually it is "fairly small".

2. Recall a finite group G with center $Z(G)$ is quasisimple if $G/Z(G)$ is simple, and if G is equal to its commutator subgroup (i.e., G is perfect).

As mentioned previously, the finite groups of Lie type constitute a 'large' subcollection of the finite simple groups. The finite groups of Lie type split into two collections: those arising from the classical groups (associated to root systems of types A, B, C, D) and those of exceptional type (associated to root systems of types E_6, E_7, E_8, F_4, G_2). The following (very roughly phrased) theorem of Aschbacher's thus determines the maximal subgroups for a very large selection of all finite groups.

Theorem 3.2 (Aschbacher, 1984 [3])**.** Let G be a finite classical group associated to a vector space V, and $M \leq G$ a maximal subgroup. Then one of the following holds:

1. M belongs to a natural list subgroups of G (suspected maximal subgroups, constructed in relatively obvious ways), or to a small list of non-natural cases.

2. M is the normalizer in G of a quasisimple subgroup $H \leq GL(V)$ acting irreducibly on the vector space V.

Remark.

1. Item (2) of Theorem 3.2 is sometimes called "Dynkin's principle", since Dynkin pioneered this idea in the Lie theoretic context; a paper of Dynkin's in the 1950s actually classified maximal connected closed subgroups of classical Lie groups through this idea. Dynkin eventually determined all maximal connected closed subgroups of semisimple complex Lie groups. An analogous program for finite groups was proposed by Scott in [34].

2. O'Nan and Scott determined candidate maximal subgroups for the alternating groups [34], the first general result of this type. Candidate maximal subgroups for sporadic and exceptional groups have also been given, cf. references in [33] pages 3–4. As remarked in [33], many "candidates" have been shown to be maximal (or nearly so).

3. Aschbacher's theorem 3.2 is fundamental to the 'geometric approach' to finite linear groups in computational group theory (see [30] §3).

A significant problem stemming from part (2) of Theorem 3.2 is that, while H and M may both be finite subgroups of Lie type, one may arise as $G(q)$ and the other as $G(q')$ for some prime powers $q = p^m$, $q' = (p')^n$ but with $p' \neq p$. In this manner naturally arises the problem of determining modular representations V for a finite group $G(q)$ of Lie type in the 'cross-characteristic' (or 'non-describing') case, that is, when V is a $G(q)$-representation over a field of characteristic p' that does not divide q. For this problem, the whole idea in the defining characteristic, i.e., relating representations for $G(q)$ to modules for G and its Frobenius kernels G_r, as outlined in Section 2.4, is not applicable, and other methods must be employed.

Problem 3. Describe all the irreducible modules over a field k of characteristic p, $p \nmid q$, of a finite group of Lie type $G(q)$.

By and large, current progress on the modular representation theory of finite groups of Lie type in the non-describing case is constrained to G of type A, e.g., $G = GL_n(q)$ or $G = SL_n(q)$. Dipper and James [9] described all of the irreducible representations over a field k of characteristic p, $p \nmid q$, of $GL(n, q)$. (Dipper and James also considered $SL_n(q)$, but there some issues remain.) Their approach used the q-Schur algebra; this concept and the related theories of Hecke algebras and quantum groups (i.e., quantum enveloping algebras) will be discussed in the next section.

4 Hecke Algebras, Schur Algebras, and Quantum Enveloping Algebras

Throughout this section, take $G = GL_n(F)$ to be the general linear group over an algebraically closed field F, with Weyl group $W \cong \mathfrak{S}_n$; let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}_n(\mathbb{C})$. For $r \geq 1$, and the standard generating set $S = \{(1\,2), (2\,3), \ldots, (r-1\,r)\}$ for $\mathfrak{S}_r$, the pair $(W, S) = (\mathfrak{S}_r, S)$ is a Coxeter system, and for any such system one can define the Hecke algebra $H_q(W)$ over the Laurent polynomial ring $\mathcal{Z} := \mathbb{Z}[v, v^{-1}]$ in the indeterminate v to be the free $\mathcal{Z}$-module on basis symbols T_w, $w \in W$ with relations

$$\begin{cases} T_s T_w = T_{sw}, \quad l(sw) = 1 + l(w), \quad s \in S, w \in W; \\ (T_s + 1)(T_s - q) = 0, \quad s \in S, \end{cases}$$

where $q := v^2$. For a free $\mathcal{Z}$-module V of rank n, $H_q(W)$ acts naturally on $V^{\otimes r}$ by the right action determined by 'place permutations'. First defined in 1989 [9], the q-Schur algebra $S_q(n, r)$ over $\mathcal{Z}$ can then realized as the endomorphism algebra

$$S_q(n, r) = \mathrm{End}_{H_q(W)}(V^{\otimes r})$$

When $n = r$, we shall write $S_q(n)$ for the q-Schur algebra $S_q(n, n)$.

Specializing q to $1 \in F$ (that is, regarding F as a $\mathcal{Z}$ module via a morphism $\mathcal{Z} \to F, q \mapsto 1 \in F$, and taking the tensor product $H_q(W)_F := H_q(W) \otimes_{\mathcal{Z}} F$) there is an isomorphism of algebras $H := H_q(W)_F \cong FW \cong F\mathfrak{S}_n$, that is, the Hecke algebra is a 'deformation' of the group algebra of $\mathfrak{S}_n$. Moreover, in this case, the q-Schur algebra reduces to the Schur algebra $S(n, r) = \mathrm{End}_{F\mathfrak{S}_r}(V^{\otimes r})$ for the vector space $V \cong F^n$. Classical Schur-Weyl reciprocity relates the representation theory of $G = GL_n(\mathbb{C})$ to that of the Hecke algebra $H \cong \mathbb{C}\mathfrak{S}_r$ via the $(GL_n(\mathbb{C}), \mathfrak{S}_r)$-bimodule $(V^n)^{\otimes r}$ $(V = \mathbb{C}^n)$ which has a decomposition as a sum of certain tensor products $L(\lambda) \otimes S_\lambda$ of irreducible rational modules $L(\lambda)$ (that are 'polynomial' and homogeneous of degree r) for $GL_n(\mathbb{C})$ and irreducible modules S_λ (Specht modules, cf. [31]) for $\mathfrak{S}_r$. For $F = \mathbb{C}$ the representation theories of $S(n, r)$ and H are related by the famous Schur functors and the double centralizer property $S(n, r) = \mathrm{End}_H(V^{\otimes r}), H \cong {}_{S(n,r)}(V^{\otimes r})$. (For F of prime characteristic, the double centralizer property still holds, but the same decomposition of $V^{\otimes r}$ into terms $L(\lambda) \otimes S_\lambda$ does not necessarily hold, cf. [12], [10].)

As we shall discuss further below in this section, a generalization of this classical Schur-Weyl duality to the so-called 'quantum case' will make it possible to connect the representations of a 'quantum' analog U_q for the group $G = GL_n(F)$ to representations of the q-Schur algebra $S_q(n)$, and aspects of this picture will be central to gaining information about non-describing representations of the finite group $G(q)$ when $\mathrm{char}(F)$ is positive and q is some power of $\mathrm{char}(F)$. First, however, we provide an alternate formulation of the relevant (specialized) Hecke and q-Schur algebras that may better hint at the connections with non-describing representations and the theory of q-Schur algebras.

Under the assumption F has positive characteristic and specializing q to be a power of $\mathrm{char}(F)$, the Schur algebra $S_q(n)$ and the Hecke algebra $H_q(W)$ associated to the Weyl group W of G can be defined in the following way. Let $B(q)$ denote a Borel subgroup of $G(q)$ and let $P(q)$ denote a generic parabolic subgroup of $G(q)$. Then

$$H_q(W) \cong \mathrm{End}_{kG(q)}\left(F\uparrow_{B(q)}^{G(q)}\right) \qquad \text{and}$$

$$S_q(n,n) = \mathrm{End}_{kG(q)}\left(\bigoplus_{P(q)\geq B(q)} F\uparrow_{P(q)}^{G(q)}\right)$$

where the direct sum is taken over all parabolic subgroups of $G(q)$ containing $B(q)$. Moreover, we have

$$S_q(n,n) = \mathrm{End}_{H_q(W)}\left(\bigoplus F\uparrow_{H_q(W_J)}^{H_q(W)}\right)$$

with J ranging over the fundamental reflections in W.

A later formulation of Dipper-James theory, e.g. by Takeuchi (for the unipotent representations, cf. [38]), and Cline, Parshall, and Scott (in general, cf. [7]), results in a categorical equivalence that guarantees that the irreducible modules for $G(q)$ in the cross-characteristic case can be recovered from knowledge of irreducible modules for q^a-Schur algebras $S_{q^a}(r_a)$:

Theorem 4.1 (cf. Theorem 9.17 of [7])**.** Let $\mathcal{O}$ be the ring of integers of a p-adic number field K, π a generator of the unique maximal ideal of $\mathcal{O}$, and $k = \mathcal{O}/\pi\mathcal{O}$ the residue field of characteristic p. Let $G(q) = GL(n,q)$, $p \nmid q$. Then there exists a quotient $\mathcal{O}G(q)/J(q)$, $J(q) \subseteq \mathrm{rad}\,\mathcal{O}G(q)$, such that $\mathcal{O}G(q)/J(q)$ is Morita equivalent to a direct sum of tensor products $\bigotimes_i S_{q^{a_i}}(r_i)$ of q^{a_i}-Schur algebras with $\sum_i a_i r_i = n$.

It follows from the theorem above that a parametrization of irreducible $G(q)$-modules in non-describing characteristic will follow from the same data for q-Schur algebras; likewise, character formulas for the q-Schur algebras will imply the same for $G(q)$-modules in cross-characteristic. It was known by the early 90s that the q-Schur algebra (in at least characteristic zero, though also later shown to be true in other characteristics) at q equal to an ℓ^{th} root of unity is a homomorphic image of a quantum enveloping algebra for $\mathfrak{g}_{\mathbb{C}}$ e.g., [11]. Set $\mathbf{U} = \mathbf{U}(\mathfrak{g}_{\mathbb{C}})$ to be the quantized enveloping algebra over $\mathbb{Q}(v)$ associated to the complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$ (that is, the quantum enveloping algebra $\mathbf{U}_v(\mathfrak{R})$ for a root datum realization $\mathfrak{R}$ associated to the Cartan matrix for $\mathfrak{g}_{\mathbb{C}}$; see below for a precise definition of the algebras $\mathbf{U}_v(\mathfrak{R})$ and a realization $\mathfrak{R}$ for $\mathfrak{g}_{\mathbb{C}}$). For $\mathbf{H} := \mathbb{Q}(v) \otimes_{\mathcal{Z}} H_q(W)$ and $\mathbf{S}_q(n,r) := \mathbb{Q}(v) \otimes_{\mathcal{Z}} S_q(n,r)$, and rank n free $\mathcal{Z}$-module V, there is a surjective algebra morphism $\delta_r : \mathbf{U} \to \mathrm{End}((V \otimes_{\mathcal{Z}} \mathbb{Q}(v))^{\otimes r})$ which factors through a natural surjective algebra morphism $\mathbf{U} \to \mathbf{S}_q(n,r)$; restricting δ_r to a particular integral form $\widetilde{U}_{\mathcal{Z}}$ of $\mathbf{U}$ yields a surjective morphism $\widetilde{U}_{\mathcal{Z}} \to \mathrm{End}(V^{\otimes r})$ with image $\mathrm{End}_{H_q(W)}(V^{\otimes r}) \cong S_q(n,r)$. From this 'integral' result one can base-change to get a version for any $\mathcal{Z}$-algebra in place of $\mathcal{Z}$; by this means, one obtains Schur-Weyl duality at q a root of unity via specialization, one step in a more difficult program to obtain quantum Schur-Weyl reciprocity in general [12], [8].

We now take a few moments to define the algebra $\mathbf{U}_v(\mathfrak{R})$, for $\mathfrak{R}$ a root-datum realization of an arbitrary $m \times m$ symmetrizable Cartan matrix C, with symmetrizing diagonal matrix D.[4] Our presentation is taken from [8]. Recall a matrix $C = (c_{i,j}) \in M_m(\mathbb{Z})$ is a Cartan

[4]For C the Cartan matrix of $\mathfrak{g}_{\mathbb{C}}$, D will be simply an identity matrix and the definitions below will simplify considerably, but because of the applications in these notes of the quantum enveloping algebras beyond type A, as well as applications to Kac-Moody algebras not discussed in these notes, we include the more general definition.

matrix if (i) $c_{i,i} = 2$ for $1 \le i \le m$; (ii) $i \ne j$ implies $c_{i,j} \le 0$; and (iii) $c_{i,j} = 0$ iff $c_{j,i} = 0$, for all i, j. A Cartan matrix is symmetrizable if there is a diagonal matrix $D = \operatorname{diag}(d_1, \dots, d_m) \in M_m(\mathbb{Z}^+)$ such that DC is symmetric.

By definition, a root datum realization $\mathfrak{R}$ of the $m \times m$ symmetrizable Cartan matrix C is the 4-tuple $\mathfrak{R} := (\Pi, X, \Pi^\vee, X^\vee)$ having the ingredients below.

- a free $\mathbb{Z}$-module $X^\vee$ of finite rank $m+s$ having an ordered basis $\{\alpha_1^\vee, \dots, \alpha_m^\vee, b_1, \dots, b_s\}$, where s is a fixed positive integer
- the set of simple coroots $\Pi := \{\alpha_1^\vee, \dots, \alpha_m^\vee\} \subset X^\vee$
- the linear dual $X := \operatorname{Hom}_{\mathbb{Z}}(X^\vee, \mathbb{Z})$, also a free $\mathbb{Z}$-module of rank $m + s$, called the weight lattice of C (or its realization)
- the simple roots $\Pi := \{\alpha_1, \dots, \alpha_m\}$ determined via duality pairing $X \times X^\vee \to \mathbb{Z}$, $(\alpha, h) \mapsto \langle \alpha, h\rangle = \alpha(h)$ and $\langle \alpha_i, \alpha_j^\vee\rangle = c_{j,i}$ for all i, j
- assume also that for all i, j, $a_{j,i} := \langle \alpha_i, b_j\rangle \in \{0, 1\}$ are chosen so that the $(m + s) \times m$ matrix $\left(\begin{smallmatrix} C \\ A \end{smallmatrix}\right)$ has rank m, where $A := (a_{i,j})$; consequently, Π consists of linearly independent vectors.

The root datum realization $\mathfrak{R}$ is minimal if $s = m - \operatorname{rank}(C)$; it follows from the item above that $s \ge m - \operatorname{rank}(C)$. The root lattice of the realization $(\Pi, X, \Pi^\vee, X^\vee)$ of C is $R(\Pi) := \bigoplus_{i=1}^n \mathbb{Z}_{\alpha_i} \subset X$, $\mathfrak{h} = \mathfrak{h}_{\mathbb{R}} := X^\vee \otimes_{\mathbb{Z}} \mathbb{R}$.

Example 4.2. For $n > 1$, let $\kappa_1, \dots, \kappa_n$, denote basis elements for a free $\mathbb{Z}$-module $X^\vee$ with dual space $X := \operatorname{Hom}_{\mathbb{Z}}(X^\vee, \mathbb{Z})$, and corresponding dual basis $\epsilon_1, \dots, \epsilon_n$. Setting $\alpha_i^\vee := \kappa_i - \kappa_{i+1}$ for $1 \le i \le (n-1)$ and setting $b_1 := \kappa_n$ gives a new $\mathbb{Z}$-basis $\{\alpha_1^\vee, \dots, \alpha_{n-1}^\vee, b_1\}$ for $X^\vee$. Taking $\alpha_i := \epsilon_i - \epsilon_{i-1}$ for $1 \le i \le (n-1)$ yields vectors satisfying $\langle \alpha_i, \alpha_j^\vee\rangle = c_{j,i}$ for $C = (c_{i,j})$ the Cartan matrix of $\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}_n(\mathbb{C})$, with $c_{i,i} = 2, c_{i,i+1} = c_{i+1,i} = -1$, for all $1 \le i \le n-1$, and $c_{i,j} = 0$ otherwise. Taking $\Pi = \{\alpha_1, \dots, \alpha_{n-1}\}$ and $\Pi = \{\alpha_1^\vee, \dots, \alpha_{n-1}^\vee\}$, the 4-tuple $\mathbb{R} = (\Pi, X, \Pi^\vee, X^\vee)$ gives a root datum realization for C. Replacing $X^\vee$ by the $\mathbb{Z}$-span $X'^\vee := \sum_{i=1}^{n-1} \mathbb{Z}\alpha_i^\vee$ and X by $X' := \sum_{i=1}^{n-1} \mathbb{Z}\alpha_i$ yields a minimal root datum realization of C, this time corresponding at the Lie algebra level to $\mathfrak{sl}_n(\mathbb{C})$.

More generally, every root datum realization $\mathfrak{R}$ of a (symmetrizable) Cartan matrix C gives rise to a complex Lie algebra, i.e., the associated Kac-Moody Lie algebra; see [8] or [6].

With the notion of a root datum realization $\mathfrak{R}$ of a symmetrizable Cartan matrix C in hand, we need just a bit more notation in order to define the associated quantized enveloping algebra $\mathbf{U}_v(\mathfrak{R})$. For any positive[5] integers n, m set

$$[n] := \frac{v^n - v^{-n}}{v - v^{-1}}$$

$$[n]! := [n][n-1]\cdots[2][1] = \prod_{i=1}^{n} \frac{v^i - v^{-i}}{v - v^{-1}}$$

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n][n-1]\cdots[n-m+1]}{[m][m-1]\cdots[2][1]},$$

[5] Actually, n any integer can be assumed for the first and third definitions below.

where $[0]! := 1$ and $\left[{n \atop 0}\right] := 1$. Let $[n]_\zeta$ (resp., $[n]_\zeta!, \left[{n \atop m}\right]_\zeta$) denote the outcome of replacing v in $[n]$ (resp., $[n]!, \left[{n \atop m}\right]$) with ζ.

For $\mathfrak{R} := (\Pi, X, \Pi^\vee, X^\vee)$ the root-datum realization for the $m \times m$ Cartan matrix $C = (c_{ij})$ (with symmetrizing matrix $D = \mathrm{diag}(d_1, \dots, d_m) \in M_m(\mathbb{Z}^+)$), the quantum enveloping algebra $\mathbf{U}_v(\mathfrak{R})$ is the $\mathbb{Q}(v)$-algebra generated by E_i, F_i $(1 \le i \le m)$, K_h $(h \in X^\vee)$ subject to the relations:

(QEA1) $K_0 := 1$ and $K_{h+h'} = K_h K_{h'}$, for $h, h' \in X^\vee$;

(QEA2) $K_h E_i = v^{\langle \alpha_i, h \rangle} E_i K_h$, for $h \in X^\vee$ and $1 \le i \le m$;

(QEA3) $K_h F_i = v^{-\langle \alpha_i, h \rangle} E_i K_h$, for $h \in X^\vee$ and $1 \le i \le m$;

(QEA4) $E_i F_j - F_j E_i = \delta_{i,j} \frac{\tilde{K}_i - \tilde{K}_i^{-1}}{v_i - v_i^{-1}}$, for $1 \le i, j \le m$;

(QEA5) $\sum_{s+t=1-c_{i,j}} (-1)^s \left[{1-c_{i,j} \atop s}\right]_{v_i} E_i^s E_j E_i^t = 0$, for $1 \le i \ne j \le m$;

(QEA6) $\sum_{s+t=1-c_{i,j}} (-1)^s \left[{1-c_{i,j} \atop s}\right]_{v_i} F_i^s F_j F_i^t = 0$, for $1 \le i \ne j \le m$;

where $\tilde{K}_i := K_{d_i \alpha_i^\vee}$ and $v_i := v^{d_i}$, for $\alpha_i \in \Pi$. The quantum Serre relations are those relations given by (QEA5) and (QEA6). As already suggested by the terminology, comparing the relations above with Serre's relations (cf. [16]) for defining a complex Lie algebra with a root datum $\mathfrak{R}$ show the algebra $\mathbf{U}_v(\mathfrak{R})$ to be a 'quantized' version of such a Lie algebra. (The same analogy also holds more generally when $\mathfrak{R}$ is a root datum realization associated to a Cartan matrix for a Kac-Moody Lie algebra.) We next consider an analogue of Kostant's $\mathbb{Z}$-form in the quantum case.

Define the divided powers $E_i^{(n)}, F_i^{(n)}$, by

$$E_i^{(n)} := \frac{E_i^n}{[n]_{v_i}!} \quad (6)$$

$$F_i^{(n)} := \frac{F_i^n}{[n]_{v_i}!} \quad (7)$$

The Lusztig integral form $\mathbf{U}_\mathcal{Z}(\mathfrak{R})$ is the $\mathcal{Z}-$subalgebra of $\mathbf{U}_v(\mathfrak{R})$ generated by all divided powers $E_i^{(n)}, F_i^{(n)}$ $(1 \le i \le m, n \ge 1)$, and elements K_h for $h \in X^\vee$. For $K_i := K_{\alpha_i^\vee}, 1 \le i \le m$ and $K_{m+j} := K_{b_j}, 1 \le j \le s$, one has in fact that $\mathbf{U}_\mathcal{Z}(\mathfrak{R})$ is generated by all $E_i^{(n)}, F_i^{(n)}, 1 \le i \le m, n \ge 1$ and $K_i, 1 \le i \le m+s, n \ge 1$.

Finally, for any commutative ring R and invertible element $q \in R$, there is a unique ring homomorphism $e_q : \mathcal{Z} \to k$ with $e_q(v) = q$. The specialization $U_{q,R}(\mathfrak{R}) := \mathbf{U}_\mathcal{Z}(\mathfrak{R}) \otimes_\mathcal{Z} R$ is obtained from the integral form $\mathbf{U}_\mathcal{Z}(\mathfrak{R})$. In the rest of our discussion, we will assume for the sake of simplicity that $\mathfrak{R}$ is the root datum for a semisimple algebraic group (utilizing other accompanying notation such as $X(T)^+, L(\lambda)$ etc., as before in these notes) and we write simply $U_{q,R}$ for $U_{q,R}(\mathfrak{R})$. Having been somewhat careful in giving definitions up to this point, we will now proceed to balance the resulting length and detail involved in that endeavor by simply sketching the representation theory for the relevant algebras $U_{q,K}$ and its applications of importance for these notes.

Suppose K is a field of characteristic 0 and q is a primitive ℓ^{th} root of unity $\ell \geq 3$ (and $\gcd(\ell, 3) = 1$ if the root system has a component of type G_2). Then without loss of generality, in studying the representation theory of $U_{q,K}$, we can restrict attention to 'integrable type 1' $U_{q,K}$-modules (see e.g., the brief exposition in [20], based largely on the fundamental papers [29],[2]). For each $\lambda \in X(T)^+$ there is a simple and finite dimensional integrable $U_{q,k}$-module $L_q(\lambda)$ of type 1 generated by a vector $v \in L_q(\lambda)_\lambda$, $v \neq 0$ such that $E_i^{(n)} v = 0$ for all $n > 0$ and for all $1 \leq i \leq m$, and each simple finite-dimensional type 1 module for $U_{q,k}$ is isomorphic to some $L_q(\lambda)$. In general the finite-dimensional type 1 $U_{q,k}$-modules are not completely reducible, but they are direct sums of their weight spaces running over weights $X(T)$. This (characteristic zero) representation theory of $U_{q,K}$ (q a primitive ℓ^{th} root of unity) models (crudely, but still significantly) the modular representation theory of algebraic groups, and hence gives applications to the defining characteristic representations of finite groups of Lie type (more on this connection at the end of this section). At the same time, for the type A case when $\mathfrak{R}$ corresponds to $\mathfrak{g}_{\mathbb{C}}$, the integral form $\widetilde{U}_{\mathcal{Z}}$ introduced earlier is closely related to the Lusztig integral form $\mathbf{U}_{\mathcal{Z}}(\mathfrak{g}_C)$ of $\mathbf{U}(\mathfrak{g}_{\mathbb{C}})$ (cf. [8, Ch. 14] for more details), and the connections between $U_{q,K}(\mathfrak{g}_{\mathbb{C}}) = \mathbf{U}_{\mathcal{Z}}(\mathfrak{g}_{\mathbb{C}}) \otimes_{\mathcal{Z}} K$ and q-Schur algebras $S_q(n,r)_K$ will prove instrumental to analyzing the non-describing representations of finite groups of Lie type, the story to which we now return.

It was in the early 1990s, using many deep results, that Kazhdan-Lusztig [24] [25] [26] [27] [28] and Kashiwara-Tanisaki [22] [23] determined that the (integrable, type 1) irreducible modules for $U_{q,K}$ for any characteristic zero field K with q an ℓ^{th} root of unity, parametrized by dominant weights $X(T)^+$, have character formulas very similar to Lusztig's formula for algebraic group representations as in Section 2.4. In fact, there are quantum Weyl modules $V_q(\lambda)$ (as well as quantum induced modules, e.g., [2]) and the character for $L_q(\lambda)$, an irreducible $U_{q,K}$-module associated to $\lambda \in X(T)^+$ (in a sufficiently restricted region) is given by the very same formula as Conjecture 2.4.1 Equation 5, with the affine Weyl group W_ℓ acting in place of W_p. However, compared with the algebraic group case, information about $L_q(\lambda)$ is much more complete, that is, with a few limitations, the character formula for an irreducible $L_q(\lambda)$ is not a conjecture, but a theorem (see, e.g., [20] H.12 for a brief sketch and further comments on the references given above).

In particular, putting this information on irreducible $U_{q,K}$-modules together with the identification of $S_q(n)$-modules as $U_{q,K}(\mathfrak{g}_{\mathbb{C}})$-modules obtained via specialization from the integral quantum Schur-Weyl duality setting produces a parametrization of irreducible characteristic zero $S_q(n)$-modules at q an ℓ^{th} root of unity, along with character formulas for them. What is needed now to complete our story for cross-characteristic representations in type A is a relationship between the ordinary and modular representation theories for the q-Schur algebras, and this is provided by James' conjecture (stated before the advent of quantum enveloping algebras); see [33] for a discussion and further references.

Conjecture 4.3 (James)**.** Return to the hypotheses that $\mathcal{O}$ is the ring of integers of a p-adic number field K, π a generator of the unique maximal ideal of $\mathcal{O}$, and $k = \mathcal{O}/\pi\mathcal{O}$ the residue field of characteristic p. Let $G(q) = GL_n(q)$, $p \nmid q$. For $\ell p > n$, ℓ fixed, q an ℓ^{th} root of unity in K, the irreducible representations of the q-Schur algebra in characteristic zero at an ℓ^{th}

root of unity reduce to those in k. More precisely, for $S_q(n)_{\mathcal{O}}$ the $q-$Schur algebra over $\mathcal{O}$ (with q specialized a primitive ℓ^{th}-root of unity), all irreducible $S_q(n)_{\mathcal{O}}$-lattices in irreducible $S_q(n,n)_K$-modules reduce modulo $\pi\mathcal{O}$ to given irreducible $S_q(n)_k$-modules.[6]

For $p \gg 0$, with the size of p depending upon a given n, Gruber-Hiss (drawing from observations of Geck) noted the validity of James' Conjecture. James [21] showed that the conjecture holds for $n \leq 10$. See [33] for more details.

Theorem 4.4 (Gruber-Hiss [15])**.** Return to the hypotheses that $\mathcal{O}$ is the ring of integers of a p-adic number field K, π a generator of the unique maximal ideal of $\mathcal{O}$, and $k = \mathcal{O}/\pi\mathcal{O}$ the residue field of characteristic p. Let $G(q) = GL_n(q)$, $p \nmid q$. For $p \gg n$, ℓ fixed, q an ℓ-th root of unity in K, the irreducible representations of the q-Schur algebra in characteristic zero at an ℓ-th root of unity reduce to those in k.

For non-describing characteristic representations of finite groups of Lie type beyond type A, it is not clear that q-Schur algebras (and hence quantum enveloping algebras) will be the right objects to use. Raphael Rouquier has proposed utilizing in place of the q-Schur algebras an analogous class of algebras (the Cherednik algebras), and work of Michelle Broué linking Hecke-type algebras for complex reflection groups and blocks for unipotent characters arising from ('cuspidal' characters) in 'Deligne-Lusztig induction' provides one suggestive replacement to substitute for the type A connection between $GL_n(\mathbb{C})$ and H (or between $\mathbf{U}(\mathfrak{g}_{\mathbb{C}})$ and $\mathbf{H}$ and their associated specializations at q an ℓ^{th} root of unity) but it is not yet clear whether these will do the trick[7]. Currently, for non-describing representation theory in the non-type A case, there is not even a parametrization of irreducible $G(q)$-modules, much less character formulas.

Although quantum enveloping algebras may not provide the 'right stuff' for the cross-characteristic case (non-type A), we shall close these notes with a comment on their importance for the defining characteristic case (in any type). Building on the work of Kazhdan-Lusztig and Kashiwara-Tanisaki (see references given above) that established the validity of Lusztig's character formula for the irreducible $U_{q,K}$-modules $L_q(\lambda)$ (q a primitive ℓ^{th} root of unity in the characteristic 0 field K), Andersen, Jantzen, and Soergel [1] proved the Lusztig Conjecture 2.4.1 for irreducible modular representations of algebraic groups, in that for each fixed root system[8] there is some number such that if p is a prime greater than the number, the Lusztig conjecture for any group G over k with $\operatorname{char} k = p$. A key fact is that simple G-modules $L(\lambda)$ for $\lambda \in X(T)_1$, can be obtained via reduction modulo p from simple modules $L_q(\lambda)$ for $U_{q,K}$. The Kazhdan-Lusztig polynomials $P_{y,w}$, whose values appear as the needed coefficients in Lusztig's Conjecture 2.4.1, can themselves be characterized as coefficients arising from base changing from a standard basis $T_w, w \in W_\ell$ to a 'Kazhdan-Lusztig basis' in an appropriately defined Hecke algebra (the Iwahori-Hecke algebra) associated to the affine Weyl group W_ℓ; see [19] Chapter 7 for details.

[6] Actually, there are some hypotheses on ℓ; see e.g., [33] Conjecture 2.2 for details. Also, James' Conjecture was stated for defining as well as cross characteristic representations.

[7] We hope before the start of the AIM workshop, or at least by its end, to add further comments on this front to these notes.

[8] Assume that the rank of the root system is > 3.

Disclaimers

These notes are only scratching the surface of the topic, and then only lightly. Many key themes have, at present, been omitted entirely or almost entirely, including, but not limited to, the cohomology of finite groups of Lie type (in both describing and non-defining characteristics) and its connections with the Lusztig conjecture via Kazhdan-Lusztig theory, the deep geometric underpinnings essential to current progress on Lusztig's conjectures (e.g., perverse sheaves and intersection cohomology), the theory of quivers and Ringel-Hall algebras as related to quantum groups, more on the the role of symmetric group representations and their analogues in the theory of Hecke algebras, the theory of highest weight categories, quasi-hereditary algebras and stratified algebras, Alperin's Conjecture and Broué's Conjecture, calculations in small primes, support varieties, and more. We hope to see these topics added to these notes in the future.

The authors welcome suggestions and corrections; please e-mail Terrell Hodge at terrell.hodge@wmich.edu. Any omission of important topics or references are the result of constraints on time and the effort to limit the survey to some significant background material to serve as a common base for the June 2007 AIM Workshop "Representations and Cohomology of Finite Groups of Lie Type: Computations and Consequences", and lack of knowledge on the authors' parts, not intentional slight. The authors thank Brian Parshall and Len Scott for reviewing drafts of these notes, but claim remaining inaccuracies as their own.

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