The Atlas of Lie groups and representations: Character Table

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Outline

Fokko du Cloux - 12/20/1954 - 11/10/2006

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• The cross action and Cayley transforms of irreducible characters,

• Kazhdan-Lusztig polynomials for representations with regular integral infinitesimal character

Members of the Atlas of Lie Groups and Representations

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22. Gregg Zuckerman gregg@math.yale.edu

Outline

Representation Theory of Non Compact Lie Groups

G: connected Lie group with Lie algebra $\mathfrak{g}$

A **representation** of a Lie Group G on a complex Hilbert space $V \neq 0$ is a homomorphism π of G into the group of bounded Linear operators on V such that:

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The Atlas of Lie Groups and Representations group seeks to compute $\hat{G}$ when G is about the size of E_8. (benchmark!!)

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How does one represent such an infinite structure on a Computer?

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• A character table for G is a list of all the character values of all the irreducible representations.

This makes sense for finite and compact groups
(irreducible representations are finite dimensional)

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Theorem

(Weyl) Suppose

$$G = \{a + bi + cj + dk \quad | \quad a^2 + b^2 + c^2 + d^2 = 1\}$$

$$T = \{a + bi \quad | \quad a^2 + b^2 = 1\} = \{\exp(i\theta) : \theta \in \mathbb{R}\} \subset G.$$

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- $\Theta_{\pi_n}(\exp(i\theta)) = \sin(n\theta)/\sin(\theta)$

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  The character of an irreducible unitary representation of G is a generalized function on G
- To handle infinitely many π use the Jantzen-Zuckerman ”translation principle” which partitions the irreducible representations into finitely many families.

Kazhdan-Lusztig-Vogan Polynomials

Harish-Chandra, Langlands, Knapp, Zuckerman
Each irreducible character Θ_π has a unique expression

$$\Theta_\pi = \sum_{j=1}^{N(\lambda)} c_{(\pi,j)}^{(\lambda)} \gamma_j^{(\lambda)}$$

where the $c_{\pi,j}$ are all integers and γ_j's form a basis for the global solutions of a certain differential equations of characters solved by Harish-Chandra. Here λ refers to the infinitesimal character of π that is a homomorphism from the center of $\mathfrak{U}(\mathfrak{g})$ to $\mathbb{C}$.

How to compute the characters

Theorem

(Langlands and Knapp-Zuckerman) There is a natural bijection between the set of irreducible representations of G with infinitesimal character λ and Haris-Chandra's solutions to the differential equations of characters. Write π_i for each irreducible representations corresponding to the solution γ_i. Then the square matrix $c_{\pi_i,j}$ is a lower triangular integer matrix with 1's on the diagonal.

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G the complexification of $G_{\mathbb{R}}$
$K \subset G$ the complexification of $K_{\mathbb{R}}$

B Borel subalgebra of G

K acts on G/B: the flag variety

Theorem

Beilinson and Bernstein

Harish-Chandra solutions to the differential equations are naturally in one-to-one correspondence with pairs $(Z_0, \mathfrak{L})$ *consisting of a* K *orbit on* G/B *and a* K*-equivariant local system* $\mathfrak{L}$ *on* Z_0*.* $\mathfrak{L}$ *is a representation of the component group* $K^x/(K^x)^\circ$ *for* $x \in Z_0$

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i and j are local systems on K-orbits on G/B
The $p_{i,j}^m$'s are non negative and are zero when m is odd. So we do have polynomials.

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Versions of the Weil conjectures for intersection homology, proved by Bilinson, Bernstein, and Deligne give the tool to develop an algorithm for computing KLV polynomials by induction on the dimension of K orbits on G/B.

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Vogan gave a scheme for keeping track of these local systems. The end product in some cases is a set of inductive formulae for sum of K-L polynomials corresponding to two local systems. The major difficulty is to find ways to solve the resulting collection of equations.

The atlas software knows how to compute KVL polynomials using results from:

Vogan: *The Kazhdan-Lusztig conjectures for real reductive groups* Representation theory of reductive groups Proceedings of the conference held at the University of Utah, Park City, Utah, April 16-20, 1982. Edited by P. C. Trombi. Progress in Mathematics, 40, Birkhäuser, Boston MA. 1983 (223-264)

Fokko du Cloux: *Computing Kazhdan-Lusztig Polynomials for Arbitrary Coxeter Groups* Experimental Mathematics 11:3 2001 (371-381)

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The Character table of E_8

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By 2002 it was possible via the work of Dan Barbasch to think that one can find a finite description of the unitary duals of all classical Lie groups. The exceptional groups could certainly be dealt with in a finite way. Therefore the E8 case should be considered a benchmark for measuring the effectiveness of any representation-theoretic software.

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In november 2005 Fokko had a version of the software which calculates the character tables for all the real forms of the exceptional simple Lie groups except for $E_{8(8)}$, the split real form of E_8. However this incredible success came with some sadness. Around that time Fokko was diagnosed with ALS and was to die within a year. Until the end he continued to work with David Vogan and Marc van Leeuwen to finish.

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More technical considerations and advice from Birne Binegar and Noam Elkies lead to estimate the number of distinct polynomials to be around 800 millions and that the computation should be carried modulo m and then use Chinese Remainder Theorem.

William Stein gave us access to his 64G RAM and 75G SWAP machine SAGE. And after many trials the computation finished on January 8, 2007 just before 9:AM

- 60 gigabytes of files containing the KLV polynomials
- All irreducible characters are written down
- The biggest coefficient that appears is 11,808,808 in a polynomial P whose value at 1 is 60,779,787.

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• Explain the unitary dual result using Kirillov-Kostant orbit method, or Langlands' ideas about functoriality , or even something entirely new and different for the software is starting to point us in many unexpected directions.

Thank you to:

David Vogan : The Character of E_8 (to appear in AMS Notices)

Fokko du Cloux: who lives in his software

And all the members of the Atlas

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END