3 Maximal Subgroups

This section is based on [33].

Suppose $G$ is a finite group, and $H \leq G$ is a maximal subgroup. Historically, the study of maximal subgroups (or, more precisely, pairs $(G, H)$) has been a principle topic in the theory of finite groups. For example, through the study of maximal subgroups, one may hope to obtain structural information about groups in general, through a recursive procedure. As another example, and one which is the principle motivating factor for the rest of these notes, is the role maximal subgroups play in the theory of permutation representations of finite groups. The group $G$ acts on $G/H$ not only transitively, but primitively, and the permutation representations associated to the pairs $(G, H)$ for $H$ running over all maximal subgroups of $G$ constitute the building blocks for all permutation representations of $G$ (analogous for nonlinear representations to the role played by the irreducible representations in the linear case). Finite automata theory provides one interesting modern application of permutation representations; see Chapters 6 and 7 of [14], entitled “Covering by permutation and reset machines” and “The theory of Krohn and Rhodes.” We remark that any permutation machine may be “covered,” in the terminology of [14], by a “cascade” (wreath-like) product of primitive permutation machines.

Determining maximal subgroups of an arbitrary finite group reduces to the case of solving this problem for simple or nearly simple groups by a theorem of Aschbacher and Scott, which we loosely paraphrase below.

**Theorem 3.1.** (Aschbacher–Scott, 1985 [4]) The determination (up to conjugacy) of all pairs $(G, M)$, $G$ a finite group and $M \leq G$ a maximal subgroup, reduces modulo “smaller or easier” problems to the following:

1. $G$ is almost simple (and $M$ is maximal in $G$)

2. $G = H.V$ a semidirect product of a quasisimple finite group $H$ and one of its irreducible modules $V$ over $\mathbb{F}_p$, and $M$ is a complement to $V$. In this case, the conjugacy classes in $G$ of such maximal subgroups $M$ correspond bijectively to elements of the cohomology group $H^1(H, V)$.

**Remark.**

1. Recall a finite group $G$ is almost simple if $G$ can be sandwiched as $G_0 \leq G \leq \text{Aut}(G_0)$ for a finite simple group $G_0$ and its automorphism group. By the Schreier Conjecture (now a theorem), $\text{Aut}(G_0)$ is a solvable group; usually it is “fairly small”.

2. Recall a finite group $G$ with center $Z(G)$ is quasisimple if $G/Z(G)$ is simple, and if $G$ is equal to its commutator subgroup (i.e., $G$ is perfect).

As mentioned previously, the finite groups of Lie type constitute a ‘large’ subcollection of the finite simple groups. The finite groups of Lie type split into two collections: those arising from the classical groups (associated to root systems of types $A, B, C, D$) and those of exceptional type (associated to root systems of types $E_6, E_7, E_8, F_4, G_2$). The following (very roughly phrased) theorem of Aschbacher’s thus determines the maximal subgroups for a very large selection of all finite groups.
Theorem 3.2 (Aschbacher, 1984 [3]). Let $G$ be a finite classical group associated to a vector space $V$, and $M \leq G$ a maximal subgroup. Then one of the following holds:

1. $M$ belongs to a natural list subgroups of $G$ (suspected maximal subgroups, constructed in relatively obvious ways), or to a small list of non-natural cases.

2. $M$ is the normalizer in $G$ of a quasisimple subgroup $H \leq GL(V)$ acting irreducibly on the vector space $V$.

Remark.

1. Item (2) of Theorem 3.2 is sometimes called “Dynkin’s principle”, since Dynkin pioneered this idea in the Lie theoretic context; a paper of Dynkin’s in the 1950s actually classified maximal connected closed subgroups of classical Lie groups through this idea. Dynkin eventually determined all maximal connected closed subgroups of semisimple complex Lie groups. An analogous program for finite groups was proposed by Scott in [34].

2. O’Nan and Scott determined candidate maximal subgroups for the alternating groups [34], the first general result of this type. Candidate maximal subgroups for sporadic and exceptional groups have also been given, cf. references in [33] pages 3–4. As remarked in [33], many “candidates” have been shown to be maximal (or nearly so).

3. Aschbacher’s theorem 3.2 is fundamental to the ‘geometric approach’ to finite linear groups in computational group theory (see [30] §3).

A significant problem stemming from part (2) of Theorem 3.2 is that, while $H$ and $M$ may both be finite subgroups of Lie type, one may arise as $G(q)$ and the other as $G(q')$ for some prime powers $q = p^m$, $q' = (p')^n$ but with $p' \neq p$. In this manner naturally arises the problem of determining modular representations $V$ for a finite group $G(q)$ of Lie type in the ‘cross-characteristic’ (or ‘non-describing’) case, that is, when $V$ is a $G(q)$-representation over a field of characteristic $p'$ that does not divide $q$. For this problem, the whole idea in the defining characteristic, i.e., relating representations for $G(q)$ to modules for $G$ and its Frobenius kernels $G_r$, as outlined in Section 2.4, is not applicable, and other methods must be employed.

Problem 3. Describe all the irreducible modules over a field $k$ of characteristic $p$, $p \nmid q$, of a finite group of Lie type $G(q)$.

By and large, current progress on the modular representation theory of finite groups of Lie type in the non-describing case is constrained to $G$ of type $A$, e.g., $G = GL_n(q)$ or $G = SL_n(q)$. Dipper and James [9] described all of the irreducible representations over a field $k$ of characteristic $p$, $p \nmid q$, of $GL(n, q)$. (Dipper and James also considered $SL_n(q)$, but there some issues remain.) Their approach used the $q$-Schur algebra; this concept and the related theories of Hecke algebras and quantum groups (i.e., quantum enveloping algebras) will be discussed in the next section.
be written \(\tilde{w}(-\rho) - \rho = \tilde{w}.(-2\rho)\), where \(w \to \tilde{w}\) is the automorphism fixing each ordinary Weyl group element, but taking a translation to its negative. The twist by the automorphism is often ignored in the literature, incorrectly, in some cases. (I am grateful to Jens Jantzen for alerting me to this issue, and explaining that such an inaccuracy occurs in his book [J, p. 294], copied in [CPS2] and elsewhere.) If we want to ignore the twist, we can, if we change the generating set of fundamental reflections: View the original generating reflections of the affine Weyl group, commonly viewed as reflections in hyperplanes containing the walls of a dominant alcove [J], [CPS2], as instead occurring in hyperplanes containing the walls of an anti-dominant alcove. We adopt that point of view here, and it is this action that we have used in writing the weight equations such as \(\lambda = \tilde{w}_0 \tilde{w}(-\rho) - \rho = \tilde{w}_0 \tilde{w}.(-2\rho)\) above. Our polynomials \(P_{\mu,\lambda} = P_{\mu,\lambda}\) would still be denoted \(P_{\mu,\lambda}\) in [CPS2], but be identified there as Kazhdan-Lusztig polynomials \(P_{\mu,\lambda}\). This is not inconsistent, because the generating reflections used in the two cases are different, and the recursive definition of the polynomials depends on the underlying list of fundamental reflections used.

Returning to \(\mathcal{G} = \text{SL}(6,F_q)\), we will let \(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\) denote the fundamental weights associated to the underlying root system.

**Proposition 3** (McDowell) Write Kazhdan-Lusztig polynomials as above, using representative weights in \(p\)-alcoves for \(p = 7\). (Thus \(P_{\mu,\lambda}\) for \(\mu = y, -2\rho\) is written \(P_{\mu,\lambda}\) for \(\mu = y, -2\rho\).) Then there is an affine Weyl group element \(w\) with \(w. -2\rho = \lambda = 4\lambda_1 + 5\lambda_2 + 4\lambda_3 + 5\lambda_4 + 4\lambda_5\). If \(\mu = 0 = w_0. -2\rho\), we have

\[
P_{\mu,\lambda}(t^2) = 1 + 8t^2 + 25t^4 + 51t^6 + 80t^8 + 87t^{10} + 70t^{12} + 38t^{14} + 14t^{16} + 3t^{18},
\]

where \(t^2\) is indeterminate (the usual “\(q\)”). The length \(\ell(\lambda) = \ell(w) - \ell(w_0)\) defined above is 19.

**Corollary 4** Assume \(p\) is large enough for the Lusztig conjecture to hold for \(\mathcal{G} = \text{SL}(6,F_q)\). Put \(V = L(w. -2\rho)\) with \(w\) as above. For all sufficiently large powers \(q\) of \(p\), we have

\[
\dim H^1(\mathcal{G}(q), V) \geq 3,
\]

Moreover, the center \(Z(q)\) of \(\mathcal{G}(q)\) acts trivially on \(V\), and the dimension of \(H^1(\mathcal{G}(q)/Z(q), V)\) is the same as that of \(H^1(\mathcal{G}(q), V)\). The module \(V\) is a faithful irreducible module for the group \(G = \mathcal{G}(q)/Z(q)\).

Thus, we have the main result of this paper:
Lemma 1.5. (E. Cline) For a restricted weight $\lambda \in X^+_1$, there is a surjective morphism $\hat{Z}_1(\lambda) \to \Delta(\lambda)$ in $G_1B$-mod.

Proof. Let $v^+ \in \Delta(\lambda)$ be a high weight vector. There is a natural $G_1B$-module map $\hat{Z}_1(\lambda) \to \Delta(\lambda)$ which sends $u \otimes 1 \in \hat{Z}_1(\lambda)$ to $u \cdot v^+$, $u \in g$. The image $E := u(g) \cdot v^+ = u(b) \cdot v$ of this map is a $u(g)$-submodule of $\Delta(\lambda)$. Of course, $v^+ \in E$, and we must show that $\Delta(\lambda) = kG \cdot v^+ \subseteq E$.

Let $\alpha \in -\Pi$ be the negative of a simple root and fix an $\alpha$-root vector $x_\alpha \in g$. For a positive integer $n$, let $x_\alpha^{(n)}$ be the corresponding divided power element in $h_v(G)$. Since $\lambda \in X^+_1$, for $n \geq p$, $\lambda - n\alpha$ is not a weight in $\Delta(\lambda)$. Thus, that $x_\alpha^{(n)} \cdot v^+ = 0$ for all $n \geq p$. (For example, as noted in the introduction, $\Delta(\lambda)$ is obtained by "reduction mod $p$" from the complex irreducible module of high weight $\lambda$, and $(\lambda, -\alpha^\vee) < p$.) Hence, $\exp(tx_\alpha) := \sum_{n \geq 0} t^n x_\alpha^{(n)}$ satisfies $\exp(tx_\alpha) \cdot v^+ = \sum_{n=0}^{n-1} t^n x_\alpha^{(n)} \cdot v^+ \in E$, for all $t \in k$. (For $n < p$, $x_\alpha^{(n)} \in u(g)$.) Thus, if $U_a \subseteq B$ is the root subgroup determined by $\alpha$, $U_a \cdot v^+ \subseteq E$. But $B$ is generated by $T$ and the $U_a$, $-\alpha \in \Pi$, so $kBB^+ \cdot v^+ \subseteq E$. Since $BB^+$ is dense in $G$, we get $kG \cdot v^+ \subseteq E$, as required. \qed

The following proposition is the $r = 1$ case of a result of Lin [28, Thm. 2.7]. It shows that the $\Delta_{red}$- and $\nabla_{red}$-construction behaves well with respect to tensor products. (The proof we give would also work for $r > 1$, and seems similar to Lin's proof which does not explicitly use Lemma 1.5, or its $r > 1$ analogue.) Let $g_C$ be the complex semisimple Lie algebra of the same type as $G$. There is a surjective "Frobenius morphism" $Fr : L \to U(g_C)$, where $U(g_C)$ is the universal enveloping algebra of $g_C$. Given any $g_C$-module $M$, $Fr^*M$ denotes the pull-back of $M$ to $L$.

Proposition 1.6. Suppose $\lambda = \lambda_0 + p\lambda_1$ where $\lambda_0 \in X^+_1$ and $\lambda_1 \in X^+$. Then $\Delta_{red}(\lambda) = \Delta_{red}(\lambda_0) \otimes \Delta(\lambda_1)^{(1)}$ and $\nabla_{red}(\lambda) = \nabla_{red}(\lambda_0) \otimes \nabla(\lambda_1)^{(1)}$.

Proof. It suffices to prove the first equality; the second then follows by a dual argument.

Let $v_0^+ \in \Delta_{red}(\lambda_0)$ and $v_1^+ \in \Delta(\lambda_1)^{(1)}$ be high weight vectors, so that $v_0^+ \otimes v_1^+$ is a high weight vector in $\Delta_{red}(\lambda_0) \otimes \Delta(\lambda_1)^{(1)}$. The latter module is the reduction mod $p$ of the lattice $\hat{L}_q^{min}(\lambda_0) \otimes \hat{L}_q^{min}(p\lambda_1)$ in $L_q(\lambda)$, so has the same dimension as $\Delta_{red}(\lambda)$ since $L_q(\lambda) \cong L_{\lambda_0}(\lambda) \otimes L_{\lambda_1}(p\lambda)$ by the quantum tensor product theorem. Also, we can arrange an inclusion $\hat{L}_q^{min}(\lambda) \subseteq \hat{L}_q^{min}(\lambda_0) \otimes \hat{L}_q^{min}(p\lambda_1)$ with reduction mod $p$ of the inclusion sending a high weight vector of $\Delta_{red}(\lambda)$ to $v_0^+ \otimes v_1^+$. Thus, it suffices to prove that $v_0^+ \otimes v_1^+$ generates $\Delta_{red}(\lambda_0) \otimes \Delta(\lambda_1)^{(1)}$.

Let $r = rad(\Delta_{red}(\lambda_0))$, so that $\Delta_{red}(\lambda_0)/r \cong L(\mu)$. Let $\mu = \mu_0 + p\mu_1 \in X^+$ with $\mu_0 \in X^+_1$ and $\mu_1 \in X^+$. Then $Hom_G(L(\lambda_0) \otimes \Delta(\lambda_1)^{(1)}, L(\mu)) \cong Hom_G(\Delta(\lambda_1), L(\mu_1)) \otimes Hom_G(L(\lambda_0), L(\mu_0))$, since $L(\mu) \cong L(\mu_0) \otimes L(\mu_1)^{(1)}$ by the tensor product theorem. Thus, $L(\lambda_0) \otimes \Delta(\lambda_1)^{(1)}$ has an irreducible head $L(\lambda_0) \otimes L(\lambda_1)^{(1)}$. The image of $v_0^+ \otimes v_1^+$ generates the quotient $\Delta_{red}(\lambda_0) \otimes \Delta(\lambda_1)^{(1)}/r \otimes \Delta(\lambda_1)^{(1)} \cong L(\lambda_0) \otimes \Delta(\lambda_1)^{(1)}$. In other words, if $E$ is the $G$-module generated by $v_0^+ \otimes v_1^+$ in $\Delta_{red}(\lambda_0) \otimes \Delta(\lambda_1)^{(1)}$, then $E + r \otimes \Delta(\lambda_1)^{(1)} = \Delta_{red}(\lambda_0) \otimes \Delta(\lambda_1)^{(1)}$.

Next, consider $\Delta_{red}(\lambda_0)$ as a $u(g), T$-module. By Lemma 1.5, $\Delta(\lambda_0)$ is a homomorphic image of $\hat{Z}_1(\lambda_0)$. Hence, $\Delta_{red}(\lambda_0)$ is also a homomorphic image of $\hat{Z}_1(\lambda_0)$. Since $\hat{Z}_1(\lambda_0)$
where \( \lambda_0 \in X_1^+ \) and \( \lambda \in X \). Then in \( G_1 B \)-mod or \( G_1 T \)-mod, \( \tilde{L}_1(\lambda) \cong p\lambda_1 \otimes \tilde{L}_1(\lambda_0) \), where \( p\lambda_1 \) denotes the one-dimensional module defined by the character \( p\lambda_1 \) on \( G_1 B \). In addition, \( \tilde{L}_1(\lambda_0) \cong \text{res}_{G_1 B}^G L(\lambda_0) \).

**Lemma 1.2.** For any rational \( G_1 B \)-module \( V \), we have \( R^n \text{ind}_{G_1 B}^G V(1) \cong (R^n \text{ind}_{B}^G V)(1) \) for any non-negative integer \( n \).

**Proof.** In degree 0,

\[
\text{ind}_{G_1 B}^G V(1) \cong (k[G] \otimes V(1))^{G_1 B} \\
\cong (k[G]^{G_1} \otimes V^{(1)})^B \\
\cong (k[G/G_1] \otimes V)^{B/B_1} \\
\cong (\text{ind}_{B}^G V)(1),
\]

functionally in \( V \). Let \( I \in G_1 B \)-mod be injective. Then \( I \) is also an injective object in \( B \)-mod, so that the required isomorphism holds in all degrees by dimension shifting.

Given \( \lambda \in X \), put \( \tilde{Z}_1(\lambda) := u(g) \otimes u(b^+) k_\lambda \). Then \( \tilde{Z}_1(\lambda) \in G_1 B \)-mod has dimension \( p^N \), where \( N = |\Phi^+| \), and irreducible head \( \tilde{L}_1(\lambda) \).

### 1.4. Character formulas.

Let \( \lambda \in X^+ \) and write \( \lambda = w \cdot \lambda^- \), where \( \lambda^- \in \overline{C^0} \) and \( w \) has minimal length among all elements \( w' \in W_p \) which satisfy \( w' \cdot \lambda^- = \lambda \). Because the isotropy subgroup of \( \lambda^- \) in \( W_p \) has the form \( W_J \) for some \( J \subset S_p \), \( w \) is uniquely determined as a distinguished left coset representative of \( W_J \) in \( W \). For \( y, w \in W_p \), let \( P_{y,w} \in \mathbb{Z}[t] \) be the associated Kazhdan-Lusztig polynomial.\(^1\) Define\(^2\)

\[
\chi_{KL}(\lambda) = \sum_{y \in W_p, y \cdot \lambda^- \in X^+} (-1)^{\ell(w) - \ell(y)} P_{y,w} (-1)^{\chi(y \cdot \lambda^-)}.
\]

The following result is proved by Kato for \( p \geq h \), but the argument works for all \( p \). (In Kato’s argument, replace the weight \( \lambda \) in the interior of an alcove by a weight in its closure.)

**Lemma 1.3.** (Kato [25]) Let \( \lambda \in X^+ \) have expansion \( \lambda = \lambda_0 + p\lambda_1 \) where \( \lambda_0 \in X_1^+ \) and \( \lambda_1 \in X^+ \). Then

\[
\chi_{KL}(\lambda) = \chi_{KL}(\lambda_0)\chi(\lambda_1)\).
\]

Following [32], we say that \( \lambda \in X^+ \) satisfies the Lusztig character formula (LCF) provided that \( \text{ch} L(\lambda) = \chi_{KL}(\lambda) \). Also, we say that \( \lambda = w \cdot \lambda^- \) satisfies the homological LCF (hLCF) provided that

\[
t^{\ell(w) - \ell(y)} P_{y,w} = p_{y \cdot \lambda^- \cdot L(w \cdot \lambda^-)} = \sum_{n=0}^{\infty} \dim \text{Ext}_{G_1}^n(L(w \cdot \lambda^-), \nabla(y \cdot \lambda^-)) t^n,
\]

\(^1\)Recall that \( P_{y,w} \) is a polynomial in \( q := t^2 \). We prefer to regard \( P_{y,w} \) as a polynomial in \( t \), albeit one which is a polynomial also in \( t^2 \). Unless \( y \leq w \), \( P_{y,w} = 0 \). If \( y = w \), then \( P_{y,w} = 1 \). If \( y < w \), \( P_{y,w} \) has degree (in \( t \)) \( \leq \ell(w) - \ell(y) - 1 \). If \( y < w \), let \( \mu(y,w) \) be the coefficient of \( t^{\ell(w) - \ell(y)} \); otherwise, put \( \mu(y,w) = 0 \).

\(^2\)Let \( F \) be the unique facet containing \( \lambda \). Then, using [23, 6.11], \( F \) lies in the upper closure of a unique alcove \( C \). If \( C' \) is a second alcove satisfying \( F \subseteq \overline{C'} \), then \( C \cap C' \). If \( w \in W_p \) satisfies \( w \cdot C^- = C \), then \( w \) is the shortest element in \( W_p \) satisfying \( w \cdot \lambda^- = \lambda \). In the expression below, given \( \mu \in X^+ \), there may well exist several \( y \leq w \) such that \( y \cdot \lambda^- = \mu \).
By Lemma 1.1(b), for any \( \lambda \in X^+ \), \( \text{ch} \Delta(\lambda) = \sum_{\mu \in X^+} p_{\mu, \Delta(\lambda)}(-1) \text{ch} \Delta(\mu) \), where \( p_{\mu, \Delta(\lambda)} \) is the Poincare polynomial defined in (1.0.4) for \( M = \Delta(\lambda) \).

The following result should hold for \( p = h \). The third author and a University of Virginia undergraduate, Mark Rawls, have checked this result empirically for the case \( p = h = 7 \). The verification was obtained in the course of a general program to implement the proposition and the proof of Theorem 6.7 as a new algorithm for calculating the Kazhdan-Lusztig polynomials (for affine Weyl groups) appearing in the LCF.

**Proposition 4.2.** Assume that \( p > h \) and that \( \lambda, \mu \in X^+ \). Then \( p_{\mu, \Delta(\lambda)} = 0 \) unless \( \mu = w \cdot 0 + p\xi, w \in W, \xi \in X \). In this case, \( p_{\mu, \Delta(\lambda)} = \sum_{n=0}^{\infty} \sum_{x \in W} (-1)^{|x|} p_{n-n(\mu)}(x \cdot \lambda - \xi) t^n \)

where the sum is restricted to those integers \( n \) such that \( n \equiv l(w) \mod 2 \).

**Proof.** There is a Hochschild-Serre spectral sequence

\[
E_2^{s,t} = \text{Ext}_G^s(\Delta(\lambda), H^t(G_1, \nabla(\mu))(-1)) \Rightarrow \text{Ext}_G^{s+t}(\Delta(\lambda), \nabla(\mu)).
\]

If \( H^t(G_1, \nabla(\mu)) \neq 0 \), then Lemma 4.1 implies that \( \mu = w \cdot 0 + p\xi \) and \( t \equiv l(w) \mod 2 \). Also, \( H^t(G_1, \nabla(\mu))(-1) \) has a \( \nabla \)-filtration, so in (4.2.1), \( E_2^{s,t} = 0 \) unless \( s = 0 \) by Lemma 1.1(a). Thus, using Lemma 4.1

\[
\dim \text{Ext}_G^s(\Delta(\lambda), \nabla(\mu)) = \dim \text{Hom}_G(\Delta(\lambda), H^n(G_1, \nabla(\mu))(-1)) = [H^n(G_1, \nabla(\mu))(-1) : \nabla(\lambda)].
\]

Now apply (4.1.1).

Now we can answer the question posed by this section.

**Theorem 4.3.** Assume that \( p > h \). Let \( \lambda \in X^+ \). Then

\[
\Delta^\text{red}(p\lambda)[-l(p\lambda)] = \Delta(\lambda)[-l(\lambda)] \in \mathcal{E}^L \text{ and } \nabla(\lambda)[-l(\lambda)] = \nabla(\lambda)[-l(\lambda)] \in \mathcal{E}^R.
\]

(Here \( l(p\lambda) := l(t_{p\lambda}) = \sum_{\alpha \in \Phi^+} (\lambda, \alpha') \).

**Proof.** We prove that \( \Delta(\lambda)[-l(t_{p\lambda})] \in \mathcal{E}^L \), that \( \nabla(\lambda)[-l(t_{p\lambda})] \in \mathcal{E}^R \) is handled similarly. The composition factors \( L(\tau) \) of \( \Delta(\lambda) \) satisfy \( \tau \in W_p \cdot 0 \) and \( l(\tau) \equiv l(p\lambda) := l(t_{p\lambda}) \mod 2 \). We must show that if \( \mu \in X^+ \), then

\[
\text{Ext}_G^n(\Delta(\lambda), \nabla(\mu)) \neq 0 \implies n \equiv l(p\lambda) - l(\mu) \mod 2.
\]

We must have \( \mu = w \cdot 0 + p\xi, \) and \( n \equiv l(w) \mod 2 \). So, to conclude the proof, we must determine that \( l(p\lambda) \equiv l(p\xi) \mod 2 \). If \( \text{Ext}_G^n(\Delta(\lambda), \nabla(\mu)) \neq 0 \), then \( p\lambda \) and \( \mu = w \cdot 0 + p\xi \) belong to the same \( W_p \)-linkage class. Hence, \( p\lambda - p\xi \in \mathbb{Z} \Phi \). Since \( p > h \), \( X/\mathbb{Z} \Phi \) has no \( p \)-torsion, so \( p\lambda - p\xi + p\delta \), with \( \delta \in \mathbb{Z} \Phi \). Then \( t_{p\lambda} = t_p \xi t_p \delta \). Since \( l(t_{p\delta}) \) is even, \( l(p\lambda) \equiv l(p\xi) \mod 2 \), as required.

**Corollary 4.4.** Assume that \( p > h \). Suppose that \( \lambda \in X^+ \) and \( \Delta(\lambda) \cong L(\lambda) \). Then \( L(\lambda)[-l(p\lambda)] = L(p\lambda)[-l(p\lambda)] \in \mathcal{E}^E \cap \mathcal{E}^R \).

In type \( A_{n-1} \) (i.e., \( G = SL_n(k) \)), there is a determination of all \( \lambda \) for which \( \Delta(\lambda) = L(\lambda) \) given in [23, (8.21)].
5. Quantum Groups and Some Integral Representation Theory

We consider when \( \lambda \in X_{\text{reg}}^+ \) satisfies the hLCF, as defined in (1.3.1). We will say that \( X \in D \) satisfies the \( \widehat{E}^L \) (resp., \( \widehat{E}^R \)) condition provided that \( X \in \widehat{E}^L \) (resp., \( X \in \widehat{E}^R \)). Given \( \lambda \in X_{\text{reg}}^+ \), write \( \lambda \in \text{LCF} \) (resp., \( \lambda \in \text{hLCF} \), \( \lambda \in \widehat{E}^L \), \( \lambda \in \widehat{E}^R \)) provided that \( L(\lambda) \) satisfies the LCF (resp., the hLCF, the \( \widehat{E}^L \) condition, \( \widehat{E}^R \) condition). Of course, \( \lambda \in \widehat{E}^L \iff \lambda \in \widehat{E}^R \).

**Theorem 5.1.** For \( \lambda \in X_{\text{reg}}^+ \), \( \lambda \in \text{LCF} \) if and only if \( \lambda \in \text{LCF} \) and \( \lambda \in \widehat{E}^L \).

**Proof.** First, suppose that \( \lambda \in \text{hLCF} \). Write \( \lambda = w \cdot \lambda^-, \lambda^- \in \overline{C}_-^L \). If \( \mu \not\in W_p \cdot \lambda^- \), then \( \text{Ext}^n_G(L(\lambda), \nabla(\mu)) = 0 \) by the linkage principle. Hence, if the hLCF holds, then for \( \mu = y \cdot \lambda^- \), we have, by (1.3.1) that \( p_{y,\lambda^-}L(w,\lambda^-)(-1) = (-1)^{l(w)-l(y)} \mathcal{P}_{y,w}(-1) = (-1)^{l(w)-l(y)} P_{y,w}(-1) \). Therefore, Lemma 1.1(b) implies that \( L(\lambda) = \chi_{KL}(\lambda) \), so that \( \lambda \in \text{LCF} \). Also, since \( P_{y,w} \) is a polynomial in \( q = t^2 \), the validity of (1.3.1) implies that if \( \text{Ext}^n_G(L(\lambda), \nabla(\mu)) \neq 0 \), then \( l(\lambda) - l(\mu) \equiv n \mod 2 \). So, \( L(\lambda)[-l(\lambda)] \in \mathcal{E}_L \) and \( \lambda \in \widehat{E}^L \).

To prove the reverse direction, assume that \( \lambda \in \text{LCF} \) and \( \lambda \in \widehat{E}^L \). Then \( \lambda \in \text{hLCF} \) provided that

\[
\dim \text{Ext}^n_{U^L_K}(L_\zeta(\lambda), \nabla_\zeta(\mu)) = \dim \text{Ext}^n_G(L(\lambda), \nabla(\mu))
\]
holds for any \( \mu \in X_{\text{reg}}^+ \) and all non-negative integers \( n \). The left hand side of (5.1.1) is computed in the category of integrable, type 1 \( U^L_K \)-modules. See Remark 1.4.

Write \( L_\zeta(\lambda) = \widehat{L}_\zeta^{\text{min}}(\lambda) = U_\zeta \cdot v^+ \), and choose an admissible lattice \( \nabla_\zeta(\mu) \) for \( \nabla_\zeta(\mu) \). We can assume that \( \nabla_\zeta(\mu) / \pi \nabla_\zeta(\mu) \cong \nabla(\mu) \); see [17, p. 159] (which makes use of results of [4]).

Because the LCF holds for \( \lambda \), \( L(\lambda) \cong \widehat{L}_\zeta(\lambda) / \pi \widehat{L}_\zeta(\lambda) \). Thus, we have a short exact sequence

\[
0 \rightarrow \nabla_\zeta(\mu) \xrightarrow{p} \nabla_\zeta(\mu) \rightarrow \nabla(\mu) \rightarrow 0.
\]

By (1.4.5), \( \text{Ext}^\bullet_{U^L_\zeta}(\widehat{L}_\zeta(\lambda), \nabla(\mu)) \cong \text{Ext}^\bullet_G(L(\lambda), \nabla(\mu)) \).

Therefore, by the long exact sequence of \( \text{Ext} \) for \( \text{Hom}_{U^L_\zeta}(\widehat{L}_\zeta(\lambda), -) \), we obtain, for any non-negative integer \( n \), a long exact sequence

\[
\cdots \rightarrow \text{Ext}^n_{U^L_\zeta}(\widehat{L}_\zeta(\lambda), \nabla_\zeta(\mu)) \rightarrow \text{Ext}^n_{\widehat{U}^L_\zeta}(\widehat{L}_\zeta(\lambda), \nabla_\zeta(\mu)) \rightarrow \text{Ext}^n_{G}(L(\lambda), \nabla(\mu)) \rightarrow \cdots.
\]

But, by hypothesis, \( L(\lambda) \in \widehat{E}^L \), while \( L_\zeta(\lambda) \) belongs to the analogous category for \( U_K \). If \( n \not\equiv l(\lambda) - l(\mu) \mod 2 \), then \( \text{Ext}^n_{U^L_\zeta}(\widehat{L}_\zeta(\lambda), \nabla_\zeta(\mu)) \xrightarrow{p} \text{Ext}^n_{U^L_\zeta}(\widehat{L}_\zeta(\lambda), \nabla_\zeta(\mu)) \) is surjective. Since \( \text{Ext}^n_{U^L_\zeta}(\widehat{L}_\zeta(\lambda), \nabla_\zeta(\mu)) \) is a finite \( \mathcal{O} \)-module,\( \text{Ext}^n_{U^L_\zeta}(\widehat{L}_\zeta(\lambda), \nabla_\zeta(\mu)) \) is surjective. Since \( \text{Ext}^n_{U^L_\zeta}(\widehat{L}_\zeta(\lambda), \nabla_\zeta(\mu)) \) is a finite \( \mathcal{O} \)-module, \( \text{Ext}^n_{U^L_\zeta}(\widehat{L}_\zeta(\lambda), \nabla_\zeta(\mu)) \)xrightarrow{p} \text{Ext}^n_{U^L_\zeta}(\widehat{L}_\zeta(\lambda), \nabla_\zeta(\mu)) \rightarrow \text{Ext}^n_G(L(\lambda), \nabla(\mu)) \rightarrow 0 \). If \( n \) does not satisfy the congruence \( n \equiv l(\lambda) - l(\mu) \mod 2 \), the terms of (5.1.2) vanish. Thus, the finite \( \mathcal{O} \)-module \( \text{Ext}^n_{U^L_\zeta}(\widehat{L}_\zeta(\lambda), \nabla_\zeta(\mu)) \) is torsion-free (and possibly 0), so free. Therefore, (5.1.1) follows from (1.4.4). \( \square \)

**Corollary 5.2.** Assume that \( p > h \). For \( \lambda \in X^+ \), write \( p\lambda = x \cdot \tau^- \), \( x \in W_p \) and \( \tau^- \in C_{\tau}^- \).

Then \( \Delta(\lambda)^{(1)} \) satisfies the hLCF condition, in the sense that

\[
\ell^{l(x)-l(y)} \mathcal{P}_{y,x} = \sum_{n=0}^{\infty} \dim \text{Ext}^n_G(\Delta(\lambda)^{(1)}, \nabla(y \cdot \tau^-)) \ell^n
\]
for any \( y \in W_p \) such that \( y \cdot \tau^- \in X^+ \). In addition, we have

\[
(5.2.2) \quad \mu(y, x) = \dim \text{Ext}^1_G(\Delta(\lambda)^{(1)}, \nabla(y \cdot \tau^-)) \leq 1,
\]

where \( \mu(y, x) \) is the coefficient of \( t^{l(x) - l(y)} \) in \( P_{y,x} \) (cf. footnote 1).

**Proof.** Clearly, \( \tilde{L}_\zeta^{\min}(p\lambda) \) satisfies \( \tilde{L}_\zeta^{\min}(p\lambda)/\pi\tilde{L}_\zeta^{\min}(p\lambda) \cong \Delta(\lambda)^{(1)} \) (from the universal mapping property of \( \Delta(\lambda) \)). By Theorem 4.3, \( \Delta(\lambda)^{(1)} \in \mathcal{E}^{L} \). The result (5.2.1) now follows as in the proof of the theorem.

Finally, observe that (5.2.1) implies (5.2.2), by Proposition 4.2 and the fact that \( p_0(\sigma) = \delta_{\sigma,0} \) for \( \sigma \in X \).

For \( \lambda \in X_{\text{reg}}^+ \), let \( t = -1 \) in (5.2.1). Then Lemma 1.1(b) gives that \( \text{ch} \Delta(\lambda)^{(1)} = \chi(\lambda)^{(1)} = \chi_{\text{KL}}(p\lambda) \), which is just the special \( \lambda_0 = 0 \) case of Lemma 1.3. Neither Lemma 1.3 nor Corollary 5.2 depend in any way on the assumption that the LCF holds in an ideal.

The above discussion provides some evidence for a potentially far reaching question involving the category of rational \( G \)-modules. We begin with the following definition.

**Definition 5.3.** The left (resp., right) homological lattice property \( \text{hLP}^L \) (resp., \( \text{hLP}^R \)) holds for \( \lambda \in X_{\text{reg}}^+ \) provided that \( L_\zeta(\lambda) \) has an admissible lattice \( \tilde{L}_\zeta(\lambda) \) (resp., \( \tilde{L}_\zeta'(\lambda) \)) such that \( \text{Ext}_G^\bullet(\tilde{L}_\zeta(\lambda), \nabla_\zeta(\mu)) \) (resp., \( \text{Ext}_G^\bullet(\tilde{L}_\zeta'(\lambda), \nabla_\zeta(\lambda)) \)) is \( \mathcal{O} \)-torsion-free for any dominant weight \( \mu \leq \lambda \).

The two lattices \( \tilde{L}_\zeta(\lambda) \) and \( \tilde{L}_\zeta'(\lambda) \) appearing in the conditions \( \text{hLP}^L \) and \( \text{hLP}^R \) may not be the same. Equality does hold only when \( \tilde{L}_\zeta(\lambda)/\pi\tilde{L}_\zeta(\lambda) \) and \( \tilde{L}_\zeta'(\lambda)/\pi\tilde{L}_\zeta'(\lambda) \) are irreducible (and hence isomorphic to \( L(\lambda) \)), as discussed in more detail below. This irreducibility condition holds, along with \( \text{hLP}^L \) and \( \text{hLP}^R \), when the LCF holds on \( \Gamma \cap X_{\text{reg}}^+ \) for some finite ideal \( \Gamma \).

The \( n = 0, 1 \) case of the \( \text{hLP}^L \) implies there is an exact sequence

\[
0 \to \text{Hom}_{\tilde{G}}(\tilde{L}_\zeta(\lambda), \nabla_\zeta(\mu)) \xrightarrow{\pi} \text{Hom}_{\tilde{G}}(\tilde{L}_\zeta(\lambda), \nabla_\zeta(\lambda)) \to \text{Hom}_{G}(\tilde{L}_\zeta(\lambda)/\pi\tilde{L}_\zeta(\lambda), \nabla(\mu)) \to 0
\]

similar to (5.1.2). Thus, \( \tilde{L}_\zeta(\lambda)/\pi\tilde{L}_\zeta(\lambda) \) has simple head \( L(\lambda) \). By Nakayama’s lemma, \( \tilde{L}_\zeta(\lambda) \) is the lattice in \( L_\zeta(\lambda) \) generated by some vector \( v^+ \in \tilde{L}_\zeta(\lambda)/\pi\tilde{L}_\zeta(\lambda) \) of weight \( \lambda \), i. e., \( \tilde{L}_\zeta(\lambda) = \tilde{L}_\zeta^{\min}(\lambda) \).

Similarly, if the \( \text{hLP}^R \) holds for \( \lambda \), then the required lattice \( \tilde{L}_\zeta(\lambda) \) is unique up to isomorphism, and can be taken to be \( L_\zeta^{\max}(\lambda) \).

**Theorem 5.4.** Assume that \( p > h \).

(a) For \( \lambda \in X_{\text{reg}}^+ \), \( \text{hLP}^L \) (resp., \( \text{hLP}^R \)) holds if and only if \( \Delta^{\text{red}}(\lambda)[-l(\lambda)] \in \mathcal{E}^{L} \) (resp., \( \nabla^{\text{red}}(\lambda)[-l(\lambda)] \in \mathcal{E}^{R} \)).

(b) Suppose that \( \text{hLP}^L \) (resp., \( \text{hLP}^R \)) holds for \( \lambda, \mu \in X_{\text{reg}}^+ \). Then

\[
\dim \text{Ext}_G^n(\Delta^{\text{red}}(\lambda), \nabla^{\text{red}}(\mu)) = \sum_{m=0}^{n} \sum_{\nu} \dim \text{Ext}_G^m(\Delta^{\text{red}}(\lambda), \nabla(\nu)) \cdot \dim \text{Ext}_G^{n-m}(\Delta(\nu), \nabla^{\text{red}}(\mu)).
\]
Theorem 5.4. Assume that \( p > h \) and the LCF holds for all regular restricted weights. Suppose \( \lambda, \mu \) are regular dominant weights. Then

\[
\dim \text{Ext}^n_G(\Delta^{\text{red}}(\lambda), \nabla_{\text{red}}(\mu)) = \sum_{m=0}^{n} \sum_{\nu} \dim \text{Ext}^m_G(\Delta^{\text{red}}(\lambda), \nabla(\nu)) \cdot \dim \text{Ext}^{n-m}_G(\Delta(\nu), \nabla_{\text{red}}(\mu)).
\]

Furthermore, if \( \lambda = x \cdot \lambda^- \), where \( \lambda^- \in C^-_Z \), then

\[
t^{l(\lambda)} - l(\mu) \overline{P}_{y,x} = \sum_{n=0}^{\infty} \dim \text{Ext}^n_G(\Delta^{\text{red}}(\lambda), \nabla(y \cdot \lambda^-)) t^n
= \sum_{n=0}^{\infty} \dim \text{Ext}^n_G(\Delta(y \cdot \lambda^-), \nabla_{\text{red}}(\lambda)) t^n.
\]

In particular,

\[
\dim \text{Ext}^n(\Delta^{\text{red}}(\lambda), \nabla_{\text{red}}(\mu)) = \dim \text{Ext}^n_G(L_\zeta(\lambda), L_\zeta(\mu)),
\]

as given in (1.4.2).

Proof. We first prove (a). If \( \Delta^{\text{red}}(\lambda)[-l(\lambda)] \in \mathcal{E}^L \), then, just as in the proof of Theorem 5.1, there is for every integer \( n \) a short exact sequence

\[
0 \to \text{Ext}^n_{U_\zeta}(\bar{L}_\zeta(\lambda), \bar{\nabla}_\zeta(\mu)) \xrightarrow{\pi} \text{Ext}^n_{U_\zeta}(\bar{L}_\zeta(\lambda), \bar{\nabla}_\zeta(\mu)) \to \text{Ext}^n_G(\Delta^{\text{red}}(\lambda), \nabla(\mu)) \to 0
\]

which implies that \( \text{hLP}^L \) holds for \( \lambda \). Conversely, if \( \text{hLP}^L \) holds for \( \lambda \), then we obtain the same short exact sequence, so that \( L(\lambda)[-l(\lambda)] \in \mathcal{E}^L \), using (1.4.3) and Remark 1.4. A similar argument applies for the other half of (a).

Finally, (b) follows immediately from (a) and [?, Thm. 3.5]. See also Remark 1.4. \( \square \)
There is a constant \( F \) for which the (7.2.1)\( E \) ≤ (7.3.1) 1
and see [3]. We will choose also assume, without loss, that According to (b), the conclusion of (a) holds, unless \( \dim \) semisimple, simply connected algebraic group over an algebraically closed field \( A \) groups of Lie type. Let \( \mu \) = \( G \) \( L \) \( \mathbb{C} \)
where max ranges over all \( y, w \in W_p \) with \( l(y) = l(w_0) + l(w_0) \), \( l(w) = l(w_0) + l(w_0 w) \). By the discussion preceding the theorem, which makes use of Lemma 7.6 proved below, the \( \dim \) LCF holds for all regular weights in \( X^+_1 \) then \( \dim H^1(G, L(\mu)) \) ≤ \( E_0(\Phi) \), \( \forall \mu \in X^+ \).

**Theorem 7.3.** There is a constant \( C = C(\Phi) \), depending only on \( \Phi \), such that if \( G \) is a semisimple, simply connected algebraic group over an algebraically closed field \( k \) with root system \( \Phi \), then \( \dim H^1(G, L(\mu)) \leq C, \forall \mu \in X^+ \). Also, if we consider only characteristics \( p > h \) for which the LCF holds for all regular weights in \( X^+_1 \) then \( \dim H^1(G, L(\mu)) \leq E_0(\Phi), \forall \mu \in X^+ \).

**Proof.** We can assume \( \Phi \) is irreducible. Also, Lemma 7.1(a),(b) may be applied with \( \lambda = 0 \). According to (b), the conclusion of (a) holds, unless \( \dim H^1(G, L(\mu)) = 1 \). We may, thus assume, without loss, that \( j = 0 \) and that \( \mu_0 \neq 0 \). In particular, \( H^0(G_1, L(\mu_0)) = 0 \), and \( H^1(G, L(\mu)) \) injects into \( H^1(G_1, L(\mu)) \); in fact, \( H^1(G, L(\mu)) \cong H^1(G_1, L(\mu))^{G/G_1} \). Write \( \mu = \mu_0 + p \mu_1, \mu_1 \in X^+ \). Then \( H^1(G_1, L(\mu))^{G/G_1} \cong \text{Hom}_G(L(\mu_1)^*, H^1(G_1, L(\mu_0))(-1)) \), where \( L(\mu_1)^* \) is the module dual to \( L(\mu_1) \). Hence, \( \dim H^1(G, L(\mu)) \) is bounded by the number of \( G \)-composition factors in \( H^1(G_1, L(\mu_0)) \).

Since there are only a finite number of possible restricted weights \( \mu_0 \), we have proved there exists a constant \( C(\Phi, p) \), depending on both \( \Phi \) and the prime \( p \), such that if \( \mu \in X^+ \), then \( \dim H^1(G, L(\mu)) \leq C(\Phi, p) \), when \( G \) is the semisimple, simply connected group over \( k = \mathbb{F}_p \) having root system \( \Phi \).

Next, we prove the (second part of) the theorem with \( p > h \). For each root system \( \Phi \), there exists a constant \( D(\Phi) \) such that if \( p > D(\Phi) \), the LCF holds for all regular \( \lambda \in X^+_1 \); see [3]. We will choose also \( D(\Phi) \geq h \) Assume that \( p > D(\Phi) \). Let \( \mu \in X^+ \) be so that \( \mu_0 \neq 0 \) and \( H^1(G, L(\mu)) \neq 0 \). By Theorem 7.2,
\[
1 \leq \dim H^1(G, L(\mu)) \leq \dim \text{Ext}^1_{\mathbb{C}^*}(L(\lambda), L(\mu)).
\]
By the discussion preceding the theorem, which makes use of Lemma 7.6 proved below, the right-hand side is bounded by \( E_0(\Phi) \). This proves the theorem for \( p > h \).

Finally, putting things together, we see that \( C(\Phi) := \max \{ E_0(\Phi), C(p, \Phi), p \leq D(\Phi) \} \) satisfies the requirements of the theorem.

We will now apply these results to generic cohomology for the infinite families of finite groups of Lie type. Let \( q = p^d \). These groups fall into several classes: (1) the split groups \( A_n(q), B_n(q), \ldots, E_8(q) \); (2) the (Steinberg) twisted groups \( 2A_n(q), \ldots, 2E_8(q) \); (3) the Suzuki

\[\text{(7.3.1)}\]

\[1 \leq \dim H^1(G, L(\mu)) \leq \dim \text{Ext}^1_{\mathbb{C}^*}(L(\lambda), L(\mu)).\]
groups ${^2}B_2(2^{n+1})$; and (4) the Ree groups ${^2}F_4(2^{2n+1})$ and ${^2}G_2(3^{2n+1})$. The reader is referred to [16, Ch. 3]. It will be convenient, however, to denote these groups as ${^i}G(q)$. For example, if $G = SL_n(k)$, then $2G(q)$ denotes $2A_{n-1}(q)$.9

Fix $G$ and an infinite family $\{iG(q)\}$. Here $i$ is fixed and $q$ is allowed to vary over appropriate powers of $p$, as indicated above. For any finite dimensional rational $G$-module $V$ and positive integer $n$, the generic cohomology of $V$ in degree $n$ is defined to be the common limit

$$H^n_{\text{gen}}(G, V) := \lim_{q \to \infty} H^n(iG(q), V) = \lim_{m \to \infty} H^n(G, V^{(m)}).$$

In [15], [18] it is shown that this limit is achieved for the split $i = 1$ cases for $q$ or $m$ sufficiently large. (The characteristic $p$ is fixed in [15], but allowed to vary in [18].) Paper [6] treats these results in the remaining cases $i > 1$.10 Applying Theorem 7.3 to (7.3.2), we obtain:

**Theorem 7.4.** The number $\dim H^1_{\text{gen}}(G, L)$ is, for all irreducible rational $G$-modules $L$, bounded by a constant depending only on $\Phi$, and not on $p$ and $L$.

**Remarks 7.5.** (a) In [20], Guralnick conjectured that there exists a constant $C$ such that if $G$ is a finite group acting faithfully on an absolutely irreducible module $L$, then $\dim H^1(G, L) \leq C$. In [21], it is suggested that, in fact, $\dim H^1(G, L) \leq 2$ in all cases. Although this specific guess is now known to be false [36], the original conjecture on a universal bound remains open. In the conference report [22], Guralnick mentioned that the current highest dimension known is 3, and expressed the view, verbally (in his conference talk), that (even if it is found that there is no constant bound), it should be the case that the dimension grows very, very slowly (in an unspecified way). Theorem 7.4 provides very positive evidence for this philosophy and is even consistent with the original conjecture.

(b) This paper considers the cohomology of the finite groups of Lie type with coefficients in a non-trivial irreducible module $L$ in the defining characteristic $p$ of the ambient algebraic group $G$. The generic cohomology is only partly developed in the cross-characteristic case where the module $L$ is taken over a field of characteristic different from $p$. There is a reasonably satisfactory theory in type $A$ [13]. Considerably more is known about cross-characteristic cohomology with trivial coefficients [35] for $GL_n(q)$.

The following lemma completes the proof of Theorem 7.3 and is needed for the proof of Theorem 7.7. We conjecture that the assumption of regularity on $\lambda$ can be removed. Of course, in the quantum case, one need not require that $p$ be a prime, but we assume this to avoid the discussion of even more special cases.

**Lemma 7.6.** There is a constant $\widehat{C}(\Phi)$ depending only on the root system $\Phi$ with the following property. For any prime $p > 2$, and $> 3$ if $\Phi$ has a component of type $G_2$, let $U_\zeta$ be defined for the root system $\Phi$ and $p$th root of unity $\zeta$. Then $\dim \text{Ext}^1_{\zeta}(L_\lambda(\lambda), L_\zeta(\mu)) \leq \widehat{C}(\Phi)$, $\forall \lambda \in X^+_{\text{reg}}, \mu \in X^+$.

**Proof.** Since $\lambda \in X^+_{\text{reg}}, \mu \in X^+$ by the linkage principle. By the translation principle, we can assume $\lambda, \mu \in W_p \cdot (-2\rho)$. As noted above Theorem 7.3,

$$\dim \text{Ext}^1_{\zeta}(L_\lambda(y \cdot (-2\rho)), L_\zeta(w \cdot (-2\rho))) = \mu(y, w)$$

---

9 We are using the notation of [16]. Some authors would denote this group by $^2A_{n-1}(q^2)$.

10 In [9] it is shown (in general) that, for any $m$, $H^n(G, V^{(m)}) \hookrightarrow H^n_{\text{gen}}(G, V)$, so that $H^n_{\text{gen}}(G, V)$ is the (finite) directed union of the cohomology of the twisted modules $V^{(m)}$. 
Remark 6.9. Observe that the proof of Theorem 6.8 shows that

\[(6.9.1) \quad \Ext^1_G(D(\lambda), \nabla_{\text{red}}(\omega)) = 0\]

for all \(\omega \in X^+\) satisfying \(\omega < \lambda\) and \(l(\omega) \neq l(\lambda) \mod 2\). In a later paper, we will show that (assuming the LCF holds for restricted regular weights) the category of all rational \(G\)-modules with composition factors having regular dominant weights of a fixed parity forms a highest weight category whose standard and costandard modules are \(\Delta_{\text{red}}(\mu)\) and \(\nabla_{\text{red}}(\mu)\), respectively. Since all composition factors of \(D(\lambda)\) are regular and have parity opposite to \(\lambda\) and are smaller than \(\lambda\), the above vanishing (6.9.1) result is precisely the standard criterion that \(D(\lambda)\) (or \(\tilde{D}(\lambda)\)) have a \(\Delta_{\text{red}}\)-(or \(\tilde{\nabla}_{\text{red}}\))-filtration in the highest weight category. This remark helps provide some conceptual insight into the above proof.

One can conjecturally extend the main conclusion of Theorem 6.8(a) to the entire radical series of \(\Delta_\zeta(\lambda)\). For \(\lambda \in X^+_\reg\) and \(n \geq r\), put \(E^r_\zeta(\lambda) := \Delta_\zeta(\lambda)/\rad^r\Delta_\zeta(\lambda)\), and let \(\tilde{E}^n(\lambda)\) be the image of \(\tilde{L}^{\min}(\lambda)\) in \(E^n_\zeta(\lambda)\).

Conjecture 6.10. Assume that \(p > h\) and let \(\lambda \in X^+_\reg\). Let \(\tilde{D}^{n-1}(\lambda)\) be the kernel of the natural surjection \(\tilde{E}^n(\lambda) \to \tilde{E}^{n-1}(\lambda)\) \((n \geq 2)\). Then \(\tilde{D}^{n-1}(\lambda)\) has a filtration with sections \(\tilde{L}^{\min}(\mu), \mu \in X^+_\reg\).

7. Applications to degree one cohomology

Throughout this section, \(G\) denotes a semisimple, simply connected group, defined and split over \(\mathbb{F}_p\). We apply the results of the previous section to obtain new results on the bounds of 1-cohomology for finite groups \(G(q)\) of Lie type. We also obtain several results on \(\Ext^1_G(L, L')\) and \(H^1(G, L)\) for the algebraic group \(G\) and irreducible modules \(L, L'\), relating these groups to quantum analogues. Given \(\lambda \in X^+_1\), write \(\lambda = \sum_{i=0}^\infty p^i \lambda_i\), where \(\lambda_i \in X^+_1\). We make no assumption on \(p\), except those explicitly noted below. Put

\[\lambda^{(i)} = \sum_{j=i}^\infty p^{j-i} \lambda_j.\]

By [1], \(\Ext^1_{G_1}(\tilde{L}, \tilde{L}) \neq 0\) for some irreducible \(G_1\)-module \(\tilde{L}\) if and only if \(p = 2\) and \(\Phi\) has a component of type \(C_n\) \((n \geq 1)\). These statement holds if \(\tilde{L}\) is replaced by an irreducible \(G\)-module \(L(\lambda)\). When \(\Phi\) is irreducible of type \(C_n\) and \(p = 2\), we have \(\Ext^1_{G_1}(L(0), L(0))^{(-1)} \cong \nabla(\varpi_1) \cong L(\varpi_1)\), the irreducible 2\(n\)-dimensional (standard) module of high weight \(\varpi_1\); see, e. g., [23, 12.2]. On the other hand, in all cases, if \(L\) is irreducible, then \(\Ext^1_G(L, L) = 0\).

The following result (at least part (a)) is essentially contained in [1]. Part (a) is a step in the proof of [1, Theorem 5.6], but is a step not requiring the hypothesis \(p \geq 3h - 3\) of that theorem.

Lemma 7.1. Let \(\lambda, \mu \in X^+_1\), and let \(j\) be minimal so that \(\lambda_j \neq \mu_j\). (If \(\lambda = \mu\), put \(j = \infty\) and \(\lambda^{(j)} = 0\).)

(a) We have

\[(7.1.1) \quad \Ext^1_G(L(\lambda), L(\mu)) \cong \Ext^1_G(L(\lambda^{(j)}), L(\mu^{(j)})),\]

unless \(p = 2\) and \(\Phi\) has a component of type \(C_n\) \((n \geq 1)\).
We prove (a). We can assume $\Phi$ is irreducible. Since $\text{Ext}^1_{\mathbb{C}}(L(\lambda), L(\mu)) \cong H^1(G, L(\mu)) \cong k$, while $\text{Ext}^1_G(L(\lambda(1)), L(\mu(1))) \cong H^1(G, L(\omega_1)) = 0$.

Proof. We prove (a). We can assume $\Phi$ is irreducible. Since $\text{Ext}^1_G(L(\lambda), L(\lambda)) = 0$, we can assume that $\lambda \neq \mu$ and $j < \infty$. It suffices to show that if $\lambda_0 = \mu_0$, then $\text{Ext}^1_{\mathbb{C}}(L(\lambda), L(\mu)) \cong \text{Ext}^1_G(L(\lambda(1)), L(\mu(1)))$. In the Hochschild-Serre spectral sequence

$$E^{s,t}_2 = \text{Ext}^s_G(L(\lambda(1)), \text{Ext}^t_{G_1}(L(\lambda_0), L(\lambda_0))^{(-1)} \otimes L(\mu(1))) \Rightarrow \text{Ext}^{s+t}_G(L(\lambda), L(\mu)),$$

the $(s, t) = (0, 1)$-term on the left is zero, unless $p = 2$ and $\Phi$ has type $C_n$, as remarked above. Otherwise, $\text{Ext}^1_{G_1}(L(\lambda_0), L(\lambda_0)) = 0$ and the equation (7.1.1) follows. This proves (a)

Now assume the hypothesis of (b). First, assume that (7.1.1) fails. By remarks before the statement of the lemma, we must have

$$E^{0,1}_2 = \text{Hom}_G(L(0), \text{Ext}^1_G(L(0, L(0))^{(-1)} \otimes L(\mu(1))) \cong \text{Hom}_G(L(\mu(1)), \text{Ext}^1_{G_1}(L(0), L(0))^{(-1)}) \neq 0.$$

It follows that $\mu(1) = \omega_1$ (so $\mu = p \omega_1$), since $\text{Ext}^1_{G_1}(L(0), L(0))^{(-1)} \cong L(\omega_1)$. Also, $E^{1,0}_2 \cong E^{2,0}_2 = 0$ because $L(\omega_1) \cong \nabla(\omega_1)$ and $H^1(G, \nabla(\omega_1)) = 0$. Thus, $\text{Ext}^1_{G_1}(L(\lambda), L(\mu)) \cong \text{Ext}^1_{G_1}(L(0), L(\mu)) \cong E^{0,1}_2 \cong 0$, as required. The converse statement follows similarly. \qed

In the following result, the requirement $p > h$ is needed for Theorem 6.7. Also, the assumption $p > h$ is required to assume that the LCF holds for $U_\zeta$; cf. Remark 1.4.

**Theorem 7.2.** Assume that $p > h$ and that the LCF holds for all regular weights in $X^+_1$. Let $\lambda, \mu \in X^+_1$ be distinct weights with $\lambda > \mu$ and let $j$ minimal so that $\lambda_j \neq \mu_j$. Suppose that $\lambda(j) \in X^+_\text{reg}$. Then \[ \dim \text{Ext}^1_G(L(\lambda), L(\mu)) = \dim \text{Ext}^1_{\mathbb{C}}(L(\lambda(j)), L(\mu(j))) \leq \dim \text{Ext}^1_{\mathbb{C}}(L(\zeta(\lambda(j)), L(\zeta(\mu(j))))). \]

**Proof.** Using Lemma 7.1, we can assume that the restricted weights $\lambda_0, \mu_0$ are distinct and regular. The hypothesis and Proposition 1.6 imply that $\Delta^\text{red}(\lambda) \cong L(\lambda_0) \otimes \Delta(\lambda(1))^{(1)}$. Therefore, the composition factors of $\Delta^\text{red}(\lambda)$ have the form $L(\lambda_0) \otimes L(\tau)^{(1)}$, where $\tau \in X^+$ satisfies $\tau \leq \lambda(1)$. Hence, $\text{Hom}_G(\text{rad } \Delta^\text{red}(\lambda), L(\mu)) = 0$, so, by the long exact sequence of cohomology, we have an injection $\text{Ext}^1_G(L(\lambda), L(\mu)) \hookrightarrow \text{Ext}^1_G(\Delta^\text{red}(\lambda), L(\mu))$. Also, the inclusion $L(\mu) \hookrightarrow \nabla(\mu)$ induces an inclusion $\text{Ext}^1_G(\Delta^\text{red}(\lambda), L(\mu)) \hookrightarrow \text{Ext}^1_G(\Delta^\text{red}(\lambda), \nabla(\mu))$ since there is no nonzero morphism $\Delta^\text{red}(\lambda) \hookrightarrow \nabla(\mu)$ because $\lambda > \mu$. Therefore, composing these inclusions gives an inclusion $\text{Ext}^1_G(L(\lambda), L(\mu)) \hookrightarrow \text{Ext}^1_G(\Delta^\text{red}(\lambda), \nabla(\mu))$. By Theorem 6.7, \[ \dim \text{Ext}^1_G(\Delta^\text{red}(\lambda), \nabla(\mu)) = \dim \text{Ext}^1_{\mathbb{C}}(L(\zeta(\lambda)), L(\zeta(\mu))). \]

We will show later in Lemma 7.6 that $\dim \text{Ext}^1_{\mathbb{C}}(L(\zeta(\lambda)), L(\zeta(\mu)))$ is bounded by a constant $\tilde{C}(\Phi)$ depending only on the $\Phi$, for all $\lambda \in X^+_\text{reg}$ and $\mu \in X^+$. Assuming $p > h$ ($p \geq h$ is

\footnote{It is quite possible that this restriction of regularity can be removed, at least in many cases. For results and conjectures comparing $\text{Ext}^1$ between irreducible modules with singular high weights and $\text{Ext}^1$ for irreducible modules with related regular high weights, see [37, §4].}
is independent of $p$, if $p > h$. The conditions discussed on $y$, $w$ (immediately following (7.2.1)) are equivalent to the assertions that $y \cdot (-2\rho), w \cdot (-2\rho) \in X^+$, so that (7.6.1) applies to all groups $\text{Ext}^1_{\mathcal{C}}(L_\zeta(\lambda), L_\zeta(\mu))$. Thus, it suffices simply to bound $\dim \text{Ext}^1_{\mathcal{C}}(L_\zeta(\lambda), L_\zeta(\mu))$ for each prime $p$. Such a bound is easy to find, without even requiring regularity: Without loss, we may assume $\lambda \leq \mu$. Let $St_\zeta \in \mathcal{C}$ denote the irreducible (quantum Steinberg) module with high weight $(p - 1)\rho$. Then $St_\zeta$ is well-known to be a (self-dual) projective module, as is $St_\zeta \otimes St_\zeta \otimes L_\zeta(\lambda)$. Therefore, $St_\zeta \otimes St_\zeta \otimes L_\zeta(\lambda)$ contains the projective indecomposable cover of $L_\zeta(\lambda)$ as a direct summand. Consequently, $\dim \text{Ext}^1_{\mathcal{C}}(L_\zeta(\lambda), L_\zeta(\mu))$ is at most the dimension of the $\mu$-weight space in $St_\zeta \otimes St_\zeta \otimes L_\zeta(\lambda)$. This multiplicity is at most the number of ways of writing $\lambda + 2(p - 1)\rho - \mu$ as an integral sum of positive roots. Since $\mu \geq \lambda$, we may add $\mu - \lambda$ to each of these sums and obtain at least one way of writing $2(p - 1)\rho$ as a sum of positive roots. Thus, if $p$ is the Kostant partition function, $\hat{C}(\Phi, p) = p(2(p - 1)\rho)$ bounds $\dim \text{Ext}^1_{\mathcal{C}}(L_\zeta(\lambda), L_\zeta(\mu))$. This completes the proof of the lemma. \hfill $\Box$

**Theorem 7.7.** There is a constant $\hat{C} = \hat{C}(\Phi)$ such that depending only on the root system $\Phi$ with the following property: Let $G$ is a semisimple, simply connected algebraic group over an algebraically closed field $k$ of characteristic $p > 0$. If $\lambda \in X^+$ with $\lambda_j$ regular for each index $j \geq 0$, then $\dim \text{Ext}^1_{\mathcal{G}}(L(\lambda), L(\mu)) \leq \hat{C} \forall \mu \in X^+$. If $p > h$ is such that $\text{LCF}$ holds for all regular weights in $X_1^+$, we may take $\hat{C} = E(\Phi)$ (defined in terms of Kazhdan-Lusztig polynomials in (7.2.1)).

**Proof.** Necessarily, $p \geq h$, since the hypothesis states that regular weights exist. If $p = 2$, then $h = 2$ and all irreducible components of $\Phi$ are of type $A_1$. In this case, all weight spaces in all Weyl modules $\Delta(\lambda)$ have dimensions $\leq 1$. In general, if $\lambda \leq \mu$, $\dim \text{Ext}^1_G(L(\lambda), L(\mu))$ is at most the multiplicity of the module $L(\lambda)$ as a composition factor of $\nabla(\mu)$, or, equivalently, of $\Delta(\mu)$. So, the theorem certainly holds in this case.

If $p \neq 2$, part (a) of Theorem 7.2 applies to $\text{Ext}^1_G(L(\lambda), L(\mu))$. Thus, we may assume $\lambda_0 \neq \mu_0$. Also, applying the Hochschild-Serre sequence, we find that

$$\text{Ext}^1_G(L(\lambda), L(\mu)) \sim \text{Ext}^1_{G_1}(L(\lambda), L(\mu))^{G/G_1} \tag{7.7.1}$$

Let $St$ denote the irreducible and self-dual (Steinberg) module with high weight $(p - 1)\rho$. Then $St$ is projective as a $G_1$-module, as is any module $St \otimes V$, for $V \in G_1$-mod. The $G$-module $S := St \otimes St \otimes L(\lambda)$ is $G_1$-projective. Let $N$ be the kernel of the natural map $\eta : S \rightarrow k \otimes L(\lambda) \cong L(\lambda)$ (defined by the self-duality of $St$), and form the short exact sequence $0 \rightarrow N \rightarrow S \overset{\eta}{\rightarrow} L(\lambda) \rightarrow 0$ of $G$-modules. If $M$ is any $G_1$-module, let $r(M)$ denote the $G_1$-radical of $M$, the smallest $G_1$ submodule of $M$ whose corresponding factor module is completely reducible. If $M$ is a $G$-module, then so is $r(M)$. We claim that the natural sequence of restriction maps

$$\text{Hom}_{G_1}(S, L(\mu)) \xrightarrow{\alpha} \text{Hom}_{G_1}(N, L(\mu)) \xrightarrow{\beta} \text{Hom}_{G_1}(r(S) \cap N, L(\mu)) \rightarrow 0 \tag{7.7.2}$$

is an exact sequence.

First, any $G_1$-module map $S \rightarrow L(\mu)$ has $r(S)$ in its kernel, so that $\beta \circ \alpha = 0$. Next, $r(N) \subseteq r(S)$, so that $r(S) \cap N$ is a direct summand of $N/r(N)$ as a $G_1$-module. Thus, $\beta$ is surjective. Also, $N/(r(S) \cap N) \cong N + r(S)/r(S)$ is naturally isomorphic to a direct summand of $S/r(S)$. If $f : N \rightarrow L(\mu)$ is a $G_1$-module map vanishing on $r(S) \cap N$, then there is a map...
In Theorem 7.7, we conjecture the assumption that the constant $\Delta$ of Theorem 7.3, making use of Lemma 7.6. We leave these details to the reader.

The standard “projective resolution” definition of $\text{Ext}^1_{G_1}$ gives a $G$-module isomorphism $\text{Hom}_{G_1}(r(S) \cap N, L(\mu)) \cong \text{Ext}^1_{G_1}(L(\lambda), L(\mu))$. Taking $G$-fixed points, and using (7.7.1), $\text{Hom}_{G}(r(S) \cap N, L(\mu)) \cong \text{Ext}^1_{G}(L(\lambda), L(\mu))$, so $\dim \text{Ext}^1_{G}(L(\lambda), L(\mu))$ is at most the dimension of the $\mu$-weight space of $S$, just as in the argument for Lemma 7.6. Continuing as in that argument, taking $\lambda \leq \mu$, we find that $p(2(p-1)\rho)$ bounds $\dim \text{Ext}^1_{G}(L(\lambda), L(\mu)) = \dim \text{Ext}^1_{G}(L(\mu), L(\lambda))$. Thus, the theorem holds for individual primes. The remaining details, for $p > h$ large enough so that the LCF holds for restricted weights, parallel the proof of Theorem 7.3, making use of Lemma 7.6. We leave these details to the reader. \qed

**Remarks 7.8.** (a) In Theorem 7.7, we conjecture the assumption that the $\lambda_j$ be regular is not needed, at least for $j > 0$.

(b) We do not know if there are similar results to Theorem 7.7 or to Theorem 7.3 for higher degree $\text{Ext}^\bullet_G$-groups or cohomology (in a fixed degree). The quantum analogue seems very likely to hold, at least for regular weights, because the $\text{Ext}^\bullet_A$ algebra of a Koszul algebra $A$ is generated by its terms in degree one.

(c) In [1, Theorem 5.6], Andersen gives a nice reduction, if $p \geq 3h - 3$, of the calculation of $\text{Ext}^1_G(L(\lambda), L(\mu))$ to a calculation of tensor products and certain other $\text{Ext}^1_G$-groups. In our notation, his formula is

$$
\text{Ext}^1_G(L(\lambda), L(\mu)) = \sum_\nu \dim \text{Ext}^1_G(L(\lambda_j + \nu), L(\mu_j)) \dim \text{Hom}_G(L(\lambda^{j+1}), L(\nu) \otimes L(\mu^{j+1})),
$$

where, as in Theorem 7.2, $j$ is the smallest index $i$ with $\lambda_i \neq \mu_i$. The weight $\nu$ ranges over dominant weights required to satisfy the inequality $\lambda_j + \nu \leq 2(p-1)\rho + w_0(\mu)$. These weights $\nu$ are quite small (though [1, Remark 5.7(i)] incorrectly, but inconsequentially, overstates the case for this—$\nu$ need not be a sum of distinct fundamental weights). For such a $\nu$, $\dim \text{Ext}^1_G(L(\lambda_j + \nu), L(\mu_j))$ may be calculated in terms of Kazhdan-Lusztig polynomials, provided $\lambda_j$ is regular (again, [1, Remark 5.7(iv)] is too optimistic, and the assumption of regularity is still needed, though may be, one day, removed). At the time of [1] quantum groups did not exist, but it seems likely that the same methods should yield a similar formula, with even weaker assumptions, to (7.8.1) in the quantum case, if $j = 0$. The right-hand Hom terms in (7.8.1) would then be taken over the enveloping algebra $U(\mathfrak{g}_\mathbb{C})$. Comparison of the Hom terms in (7.8.1) would provide an alternate proof of Theorem 7.2 without the use of Theorem 6.7, though it would require the additional assumption $p \geq 3h - 3$. Even without such a quantum version, one may use (7.8.1) and the weight space philosophy of [1, Remark 5.7(v)] to obtain an alternate proof of some version of Theorems 7.3 and 7.7, albeit without the same bounds, and with the additional assumption $p \geq 3h - 3$.

The following consequence of Theorem 7.7 is stated without further proof.

**Theorem 7.9.** There is a constant $\tilde{C} = \tilde{C}(\Phi)$, depending only on $\Phi$, such that, for any $G$ over an algebraically closed field $k$ of characteristic $p > h$ having root system $\Phi$, we have

$$
\dim \text{Ext}^1_{G,\text{gen}}(L(\lambda), L(\mu)) := H^1_{\text{gen}}(G, \text{Hom}_k(L(\lambda), L(\mu)) \leq \tilde{C} \text{ for } \lambda, \mu \in X^+ \text{ such that } \lambda_j
$$
regular for each \( j \). If \( p > h \) is such that the LCF holds for all regular weights in \( X_1^+ \), we may take \( \tilde{C} = E(\Phi) \).

We conclude this paper with the following result and a remark.

**Theorem 7.10.** Assume that \( p > h \). Let \( \lambda = \tau + p\nu \in X^+ \), with \( 0 \neq \tau \in X_1^+ \) and \( \nu \in X^+ \). Suppose that \( \dim H^1(G, L(\lambda)) > 1 \). Then \( \tau > \nu^* \), where \( \nu^* = -w_0(\nu) \in X^+ \) (the image of \( \nu \) under the opposition involution).

**Proof.** By Theorem 7.2, the hypotheses imply that

\[
\dim H^1(U_\zeta, L_\zeta(\lambda)) = \dim \text{Ext}^1_{C_\zeta}(L_\zeta(p\nu^*), L_\zeta(\tau)) > 1.
\]

By Corollary 5.2 and (1.4.1), \( \dim \text{Ext}^1_{C_\zeta}(L_\zeta(p\nu^*), \nabla_\zeta(\tau)) = \dim \text{Ext}^1_{C_\zeta}(\Delta(\nu^*)^{(1)}, \nabla(\tau)) \leq 1 \). Therefore, if we set \( Q_\zeta(\tau) := \nabla_\zeta(\tau)/L_\zeta(\tau) \), the long exact sequence of cohomology forces the conclusion that \( \text{Hom}_{C_\zeta}(L_\zeta(p\nu^*), Q_\zeta(\tau)) \neq 0 \). Hence, \( p\nu^* \prec \tau \). \( \Box \)

**Remarks 7.11.** (a) Assume that \( p > h \) and the LCF holds for all regular weights in \( X_1^+ \). Using Lemma 7.1 and Theorem 7.10, we see that if \( \dim H^1(G, L(\lambda)) > 1 \), then \( \lambda = p^r\sigma \), with \( \sigma = \sigma_0 + p\sigma_1 \), with \( 0 \neq \sigma_0 \in X_1^+ \) and \( \sigma_1 \in X^+ \) where \( p\sigma_1^* \prec \sigma_0 \). In addition, \( p\sigma_1^* \) must be \( W_p \)-linked to \( \sigma_0 \).

(b) Suppose \( G = SL_6 \) for a field \( k \) of characteristic \( p > h = 6 \) sufficiently large that the LCF holds for all regular weights in \( X_1^+ \). There is a natural isomorphism \( W_p \to W_7 \), \( w \mapsto w' \), of Coxeter groups, taking the simple reflections for \( W_p \) to their exact analogues in \( W_7 \). Choose \( w \in W_p \) so that \( w' \cdot (-2\rho) = 4\varpi_1 + 5\varpi_2 + 4\varpi_3 + 5\varpi_4 + 4\varpi_5 \). Let \( \lambda = w \cdot (-2\rho) \). By a calculation of C. McDowell reported in [36, Prop. 3, Cor. 4], \( \mu_{w_0, w} = 3 \). Thus, \( \dim H^1(SL_6(p^3), L(\lambda)) \geq 3 \) for \( d \) sufficiently large.

(c) The bounds given by Theorem 7.7 (or Theorem 7.9) are roughly exponential in terms of the rank of the root system (and so not very good!). For example, the proof of Lemma 7.6 shows that

\[
\tilde{C}(\Phi) \leq p(2(p - 1)h),
\]

for any prime \( p > h \). Actually, the same statement holds with a similar proof replacing \( p \) by \( h + 1 \). Then one can easily bound \( p(2(\rho - 1)h) \) by \( N!P(N(h - 1)) \), where \( N = rh/2 \) is the number of positive roots (and \( r \) is the rank of \( G \)), and \( P \) is the usual partition function. One can also argue with the recursive definition of Kazhdan-Lusztig polynomials to get a bound

\[
\tilde{C}(\Phi) \leq 2^{2|W|},
\]

where \( W = W(\Phi) \) is the Weyl group. Presumably none of these estimates is even close. N. Xi [41] has provided elegant theoretic arguments which show the relevant Kazhdan-Lusztig polynomial coefficients for ordinary Weyl groups are often 0 or 1 for good reason. His result implies that \( \text{Ext}^1 \)-groups for the BGG category \( \mathcal{O} \) (for a complex semisimple Lie algebra) are often of dimension \( \leq 1 \). This phenomenon is very similar to what Guralnick (and subsequently many others) have empirically observed for finite groups. It is similar in spirit to Theorem 7.10.