Theorem 3.1. (Aschbacher–Scott, 1985 [4]) The determination (up to conjugacy) of all pairs (G, M), G a finite group and $M \leq G$ a maximal subgroup, reduces modulo "smaller or easier" problems to the following:

1. G is almost simple (and M is maximal in G)

2. $G = H.V$ a semidirect product of a quasisimple finite group H and one of its irreducible modules V over $\mathbb{F}_p$, and M is a complement to V. In this case, the conjugacy classes in G of such maximal subgroups M correspond bijectively to elements of the cohomology group $H^1(H, V)$.

Theorem 3.2 (Aschbacher, 1984 [3])**.** Let G be a finite classical group associated to a vector space V, and $M \leq G$ a maximal subgroup. Then one of the following holds:

1. M belongs to a natural list subgroups of G (suspected maximal subgroups, constructed in relatively obvious ways), or to a small list of non-natural cases.

2. M is the normalizer in G of a quasisimple subgroup $H \leq GL(V)$ acting irreducibly on the vector space V.

Returning to $\overline{G}$ =SL(6,$\overline{F}_q$), we will let $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ denote the fundamental weights assoicated to the underlying root system.

Proposition 3 *(McDowell) Write Kazhdan-Lusztig polynomials as above, using representative weights in p−alcoves for $p = 7$. (Thus $P_{y,w}$ is written $P_{\mu,\lambda}$ for $\mu = y. - 2\rho, \lambda = w - 2\rho$.) Then there is an affine Weyl group element w with $w. - 2\rho = \lambda = 4\lambda_1 + 5\lambda_2 + 4\lambda_3 + 5\lambda_4 + 4\lambda_5$. If $\mu = 0$ $(= w_0. - 2\rho)$, we have*

$$P_{\mu,\lambda}(t^2) = 1 + 8t^2 + 25t^4 + 51t^6 + 80t^8 + 87t^{10} + 70t^{12} + 38t^{14} + 14t^{16} + 3t^{18},$$

where t^2 is indeterminate (the usual "q"). The length $\ell(\lambda) = \ell(w) - \ell(w_0)$ defined above is 19.

Corollary 4 *Assume p is large enough for the Lusztig conjecture to hold for $\overline{G}$ =SL(6,$\overline{F}_q$). Put $V = L(w. - 2\rho)$ with w as above. For all sufficiently large powers q of p, we have*

$$\dim H^1(\overline{G}(q), V) \geq 3,$$

Moreover, the center $Z(q)$ of $\overline{G}(q)$ acts trivially on V, and the dimension of $H^1(\overline{G}(q)/Z(q), V)$ is the same as that of $H^1(\overline{G}(q), V)$. The module V is a faithful irreducible module for the group $G = \overline{G}(q)/Z(q)$.

The following proposition is the $r = 1$ case of a result of Lin [28, Thm. 2.7]. It shows that the Δ^{red}- and ∇_{red}-construction behaves well with respect to tensor products. (The proof we give would also work for $r > 1$, and seems similar to Lin's proof which does not explicitly use Lemma 1.5, or its $r > 1$ analogue.) Let $\mathfrak{g}_{\mathbb{C}}$ be the complex semisimple Lie algebra of the same type as G. There is a surjective "Frobenius morphism" $\text{Fr} : U_\zeta \twoheadrightarrow U(\mathfrak{g}_{\mathbb{C}})$, where $U(\mathfrak{g}_{\mathfrak{C}})$ is the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$. Given any $\mathfrak{g}_{\mathbb{C}}$-module M, Fr^*M denotes the pull-back of M to U_ζ. For $\lambda \in X^+$, $\text{Fr}^* L_{\mathbb{C}}(\lambda) \cong L_\zeta(p\lambda)$, if $L_{\mathbb{C}}(\lambda)$ is the irreducible $\mathfrak{g}_{\mathbb{C}}$-module of high weight λ.

Proposition 1.6. *Suppose* $\lambda = \lambda_0 + p\lambda_1$ *where* $\lambda_0 \in X_1^+$ *and* $\lambda_1 \in X^+$. *Then* $\Delta^{\text{red}}(\lambda) = \Delta^{\text{red}}(\lambda_0) \otimes \Delta(\lambda_1)^{(1)}$ *and* $\nabla_{\text{red}}(\lambda) = \nabla_{\text{red}}(\lambda_0) \otimes \nabla(\lambda_1)^{(1)}$.

1.4. **Character formulas.** Let $\lambda \in X^+$ and write $\lambda = w \cdot \lambda^-$, where $\lambda^- \in \overline{C_{\mathbb{Z}}^-}$ and w has minimal length among all elements $w' \in W_p$ which satisfy $w' \cdot \lambda^- = \lambda$. Because the isotropy subgroup of λ^- in W_p has the form W_J for some $J \subset S_p$, w is uniquely determined as a distinguished left coset representative of W_J in W. For $y, w \in W_p$, let $P_{y,w} \in \mathbb{Z}[t]$ be the associated Kazhdan-Lusztig polynomial.[1] Define[2]

$$(1.2.1) \qquad \chi_{\mathrm{KL}}(\lambda) = \sum_{y \in W_p, y\cdot\lambda^- \in X^+} (-1)^{l(w)-l(y)} P_{y,w}(-1)\chi(y \cdot \lambda^-).$$

The following result is proved by Kato for $p \geq h$, but the argument works for all p. (In Kato's argument, replace the weight λ in the interior of an alcove by a weight in its closure.)

Lemma 1.3. *(Kato [25]) Let $\lambda \in X^+$ have expansion $\lambda = \lambda_0 + p\lambda_1$ where $\lambda_0 \in X_1^+$ and $\lambda_1 \in X^+$. Then*

$$\chi_{\mathrm{KL}}(\lambda) = \chi_{\mathrm{KL}}(\lambda_0)\chi(\lambda_1)^{(1)}.$$

Following [32], we say that $\lambda \in X^+$ satisfies the Lusztig character formula (LCF) provided that $\mathrm{ch}\, L(\lambda) = \chi_{\mathrm{KL}}(\lambda)$. Also, we say that $\lambda = w \cdot \lambda^-$ satisfies the homological LCF (hLCF) provided that

$$(1.3.1) \qquad t^{l(w)-l(y)}\overline{P}_{y,w} = p_{y\cdot\lambda^-, L(w\cdot\lambda^-)} = \sum_{n=0}^{\infty} \dim \mathrm{Ext}_G^n(L(w\cdot\lambda^-), \nabla(y\cdot\lambda^-))t^n,$$

[1] Recall that $P_{y,w}$ is a polynomial in $q := t^2$. We prefer to regard $P_{y,w}$ as a polynomial in t, albeit one which is a polynomial also in t^2. Unless $y \leq w$, $P_{y,w} = 0$. If $y = w$, then $P_{y,w} = 1$. If $y < w$, $P_{y,w}$ has degree (in t) $\leq \ell(w) - \ell(y) - 1$. If $y < w$, let $\mu(y, w)$ be the coefficient of $t^{\ell(w)-\ell(y)-1}$; otherwise, put $\mu(y, w) = 0$.

[2] Let F be the unique facet containing λ. Then, using [23, 6.11], F lies in the upper closure of a unique alcove C. If C' is a second alcove satisfying $F \subseteq \overline{C'}$, then $C \uparrow C'$. If $w \in W_p$ satisfies $w \cdot C^- = C$, then w is the shortest element in W_p satisfying $w \cdot \lambda^- = \lambda$. In the expression below, given $\mu \in X^+$, there may well exist several $y \leq w$ such that $y \cdot \lambda^- = \mu$.

The following result should hold for $p = h$. The third author and a University of Virginia undergraduate, Mark Rawls, have checked this result empirically for the case $p = h = 7$. The verification was obtained in the course of a general program to implement the proposition and the proof of Theorem 6.7 as a new algorithm for calculating the Kazhdan-Lusztig polynomials (for affine Weyl groups) appearing in the LCF.

Proposition 4.2. *Assume that* $p > h$ *and that* $\lambda, \mu \in X^+$. *Then* $p_{\mu,\Delta(\lambda)^{(1)}} = 0$ *unless* $\mu = w\cdot 0 + p\xi$, $w \in W$, $\xi \in X$. *In this case,* $p_{\mu,\Delta(\lambda)^{(1)}} = \sum_{n=0}^{\infty} \sum_{x\in W} (-1)^{l(x)} \mathfrak{p}_{\frac{n-l(w)}{2}} (x\cdot\lambda - \xi) t^n$ *where the sum is restricted to those integers* n *such that* $n \equiv l(w)$ *mod* 2.

Corollary 5.2. *Assume that* $p > h$. *For* $\lambda \in X^+$, *write* $p\lambda = x \cdot \tau^-$, $x \in W_p$ *and* $\tau^- \in C^-_{\mathbb{Z}}$. *Then* $\Delta(\lambda)^{(1)}$ *satisfies the* hLCF *condition, in the sense that*

$$t^{l(x)-l(y)}\overline{P}_{y,x} = \sum_{n=0}^{\infty} \dim \operatorname{Ext}^n_G(\Delta(\lambda)^{(1)}, \nabla(y \cdot \tau^-))t^n \tag{5.2.1}$$

for any $y \in W_p$ *such that* $y \cdot \tau^- \in X^+$*. In addition, we have*

$$\mu(y, x) = \dim \operatorname{Ext}^1_G(\Delta(\lambda)^{(1)}, \nabla(y \cdot \tau^-)) \leq 1, \tag{5.2.2}$$

where $\mu(y, x)$ *is the coefficient of* $t^{l(x)-l(y)}$ *in* $P_{y,x}$ *(cf. footnote 1).*

Theorem 5.4. *Assume that $p > h$ and the LCF holds for all regular restricted weights. Suppose λ, μ are regular dominant weights. Then*

$$\dim \operatorname{Ext}^n_G(\Delta^{\mathrm{red}}(\lambda), \nabla_{\mathrm{red}}(\mu)) = \sum_{m=0}^{n} \sum_{\nu} \dim \operatorname{Ext}^m_G(\Delta^{\mathrm{red}}(\lambda), \nabla(\nu)) \cdot \dim \operatorname{Ext}^{n-m}_G(\Delta(\nu), \nabla_{\mathrm{red}}(\mu)).$$

Furthermore, if $\lambda = x \cdot \lambda^-$, where $\lambda^- \in C^-_{\mathbb{Z}}$, then

$$t^{l(x)-l(y)} \overline{P}_{y,x} = \sum_{n=0}^{\infty} \dim \operatorname{Ext}^n_G(\Delta^{\mathrm{red}}(\lambda), \nabla(y \cdot \lambda^-)) t^n = \sum_{n=0}^{\infty} \dim \operatorname{Ext}^n_G(\Delta(y \cdot \lambda^-), \nabla_{\mathrm{red}}(\lambda)) t^n.$$

In particular,

$$\dim \mathit{Ext}^n(\Delta^{red}(\lambda), \nabla_{red}(\mu)) = \dim \mathit{Ext}^n_{\mathcal{C}_\zeta}(L_\zeta(\lambda), L_\zeta(\mu)),$$

$$E(\Phi) = \max \mu(y, w), \tag{7.2.1}$$

where the max ranges over all $y, w \in W_p$ with $l(y) = l(w_0) + l(w_0 y), l(w) = l(w_0) + l(w_0 w)$. (See (1.4.3.) It does not seem obvious from a Coxeter group viewpoint that $E(\Phi)$ is finite, though Lemma 7.6 shows this is the case. The precise value of $E(\Phi)$ is unknown, as is the value of a related constant $E_0(\Phi)$, which we define as

$$E_0(\Phi) = \max \mu(w_0, w), \tag{7.2.2}$$

where max ranges over all $w \in W_p$ with $l(w) = l(w_0) + l(w_0 w)$. That is, $E_0(\Phi)$ is defined like $E(\Phi)$, except with y fixed, as $y = w_0$. Thus, $y \cdot (-2\rho) = 0$, and $E_0(\Phi)$ is a bound, for $p > h$, on $\dim \mathrm{Ext}^1_{\mathcal{C}_\zeta}(L_\zeta(\lambda), L_\zeta(\mu))$ when $\lambda = 0$, the 1-cohomology case.

Theorem 7.3. *There is a constant $C = C(\Phi)$, depending only on Φ, such that if G is a semisimple, simply connected algebraic group over an algebraically closed field k with root system Φ, then $\dim H^1(G, L(\mu)) \leq C$, $\forall \mu \in X^+$. Also, if we consider only characteristics $p > h$ for which the LCF holds for all regular weights in X_1^+) then $\dim H^1(G, L(\mu)) \leq E_0(\Phi)$, $\forall \mu \in X^+$.*

Theorem 7.4. *The number* $\dim H^1_{\mathrm{gen}}(G,L)$ *is, for all irreducible rational* G*-modules* L*, bounded by a constant depending only on* Φ*, and not on* p *and* L*.*

these groups to quantum analogues. Given $\lambda \in X^+$, write $\lambda = \sum_{i=0}^{\infty} p^i \lambda_i$, where $\lambda_i \in X_1^+$. We make no assumption on p, except those explicitly noted below. Put

$$\lambda^{(i)} = \sum_{j=i}^{\infty} p^{j-i} \lambda_j.$$

Theorem 7.2. *Assume that* $p > h$ *and that the LCF holds for all regular weights in* X_1^+*. Let* $\lambda, \mu \in X^+$ *be distinct weights with* $\lambda > \mu$ *and let* j *minimal so that* $\lambda_j \neq \mu_j$*. Suppose that* $\lambda^{(j)} \in X^+_{\mathrm{reg}}$*. Then*[7] $\dim \mathrm{Ext}^1_G(L(\lambda), L(\mu)) = \dim \mathrm{Ext}^1_G(L(\lambda^{(j)}), L(\mu^{(j)})) \leq \dim \mathrm{Ext}^1_{\mathcal{C}_\zeta}(L_\zeta(\lambda^{(j)}), L_\zeta(\mu^{(j)}))$.

Theorem 7.7. *There is a constant $\widehat{C} = \widehat{C}(\Phi)$ such that depending only on the root system Φ with the following property: Let G is a semisimple, simply connected algebraic group over an algebraically closed field k of characteristic $p > 0$. If $\lambda \in X^+$ with λ_j regular for each index $j \geq 0$, then $\dim \operatorname{Ext}^1_G(L(\lambda), L(\mu)) \leq \widehat{C}\ \forall \mu \in X^+$. If $p > h$ is such that LCF holds for all regular weights in X_1^+, we may take $\widehat{C} = E(\Phi)$ (defined in terms of Kazhdan-Lusztig polynomials in (7.2.1)).*

Theorem 7.9. *There is a constant $\widehat{C} = \widehat{C}(\Phi)$, depending only on Φ, such that, for any G over an algebraically closed field k of characteristic $p > h$ having root system Φ, we have* $\dim \operatorname{Ext}^1_{G,\mathrm{gen}}(L(\lambda), L(\mu)) := H^1_{\mathrm{gen}}(G, \operatorname{Hom}_k(L(\lambda), L(\mu)) \leq \widehat{C}$ *for $\lambda, \mu \in X^+$ such that λ_j is* *regular for each j. If $p > h$ is such that the LCF holds for all regular weights in X_1^+, we may take $\widehat{C} = E(\Phi)$.*

We conclude this paper with the following result and a remark.

Theorem 7.10. *Assume that $p > h$. Let $\lambda = \tau + p\nu \in X^+$, with $0 \neq \tau \in X_1^+$ and $\nu \in X^+$. Suppose that $\dim H^1(G, L(\lambda)) > 1$. Then $\tau > p\nu^\star$, where $\nu^\star = -w_0(\nu) \in X^+$ (the image of ν under the opposition involution).*