First passage percolation and related models

The American Institute of Mathematics

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Chapter A: Participant Contributions

A.1 Ahlberg, Daniel

In first-passage percolation a graph, typically the $\mathbb{Z}^d$ lattice for $d \geq 2$, is equipped with nonnegative weights on the edges. The resulting weighted graph gives rise to a random (pseudo-)metric on $\mathbb{Z}^d$, and it is the properties of this metic T we wish to understand. It is well known since the work of Richardson in 1973 and Cox and Durrett in 1981 that the ball B_t of radius t, once rescaled by $1/t$, converges almost surely to a compact and convex set. However, we know very little about this limiting shape as well as the fluctuations around it.

The lack of precise information about the limiting shape has consequences for more refined questions, such as the study of geodesics. Understanding the structure of finite and semi-infinite geodesics is a very attractive challenge as they may provide detailed information on the evolution of one or several competing growing entities. The best results to this date were obtained fairly recently by Damron and Hanson, and it would be interesting to hear their view on the current state of the art.

Another very interesting question in first-passage percolation is that of a columnar defect, which is closely related to the 'slow bond problem' for the TASEP, solved very recently by Basu, Sidoravicius and Sly. For FPP this problem may be stated as follows: Assign independent weights to the edges of the square lattice according to a distribution F, except for edges along the vertical axis which are assigned according to another distribution F', which is assumed to be stochastically dominated by F (that is $F(x) \leq F'(x)$ for all x). For any such F' which is not equal to F, does the defect line result in a macroscopic defect on the asymptotic shape? It would be interesting to learn from the people solving the slow bond problem their opinion regarding similar questions for FPP.

A question very much related to that of a columnar defect is one originating from a study of inhomogeneous first-passage percolation by Ahlberg, Damron and Sidoravicius. In this model, each half-plane (left and right) of the square lattice are assigned weights according to different distributions F_- and F_+. The authors showed that for many distributions the resulting limiting shape is given by the convex hull of the two half-shapes corresponding to F_- and F_+, but that in general the shape may equal the convex hull of these two half-shapes together with an additional line segment along the vertical axis. This means that the limiting shape would be equipped with a 'pyramid' in the vertical directions, indicating that the growth in the inhomogeneous model may benefit from low-weight edges in both environment. Examples are known which show that this indeed may occur, but it remains unknown how widespread the phenomena is. Obtaining a larger class of distributions for which this phenomena occurs would be very interesting.

A.2 Alexander, Ken

In first passage percolation (FPP) in two dimensions, it is conjectured that the passage time over a distance n has fluctuations of order n^χ, and the transverse fluctuations of the corresponding geodesic are of order n^ξ, with values $\chi = 1/3, \xi = 2/3$. This has been proved for some related models for special distributions that allow for exact solutions, including last passage percolation (Johansson, 2003) and directed polymer in a random environment (Seppalainen, 2012). The values $\chi = 1/3, \xi = 2/3$ come from the two exponent relations $\chi = 2\xi - 1, 2\chi = \xi$. Versions of the first exponent relation have been proved by Chatterjee

(2013) and Auffinger and Damron (2014) following work of Newman and Piza (1995). In these versions, one assumes that the exponents exist in a fairly strong sense and have nice properties (for example, that there is an exponential bound on the passage time fluctuations, on the scale n^{χ}), and shows that the relation must then hold.

The second relation $2\chi = \xi$ is more mysterious. It is related to the correlation structure of the increments, along the circle nS^1 of radius n, of the passage times from the origin to those points. Here to simplify, we suppose we are working with an asymptotically isotropic version of FPP on, say, a random lattice, so the limiting shape is a ball. Some heuristics are as follows. Let $Q_n(k)$ denote the passage time from the origin to a point at distance k from the axis counterclockwise along nS^1. The process $Q_n(\cdot)$ should have a correlation length n^{ξ}, and the relation $2\chi = \xi$ says roughly that its increments are nearly uncorrelated inside a correlation length (though they are necessarily correlated on longer scales, since $\mathrm{var}(Q_n(k))$ doesn't depend on k.) The relation $2\chi < \xi$ would mean negative correlations inside a correlation length, with $\xi < 2/3$ and $\chi < 1/3$, while $2\chi > \xi$ would mean positive correlations and the opposite bounds.

In more detail, heuristics suggest that the function $\mathrm{corr}((Q_n(0), Q_n(sn^{\xi}))$ should have a limit $\rho(s)$ as $n \to \infty$, satisfying $1 - \rho(s) \asymp s^{2\chi/\xi}$ as $s \to 0$, and $\rho(s) \asymp s^{-2\chi/(1-\xi)}$ as $s \to \infty$.

Is it possible to establish the second exponent relation $2\chi = \xi$ by assuming enough "nice properties," analogously to what was done for the first relation? Is there a geometric picture that shows heuristically why the second relation is true, which can be turned into a theorem with enough such assumptions? A lesser goal would be to show $2\chi \leq \xi$ or $2\chi \geq \xi$, with the first corresponding to upper bounds on χ, ξ by the conjectured values, and the second corresponding to lower bounds.

A.3 Arguin, Louis-Pierre

The problem of first passage percolation can sometimes be formulated in terms of ground states of disordered ferromagnets. From this perspective, I am particularly interested in ground states of spin glasses which are disordered magnets with both ferromagnetic and antiferromagnetic interactions. In these systems, there is no monotonicity in the sense that the passage time along an edge is not necessarily positive. It is an open problem to show the uniqueness of ground states of spin glasses in two dimensions, which is the spin glass equivalent to showing that no infinite geodesics exist in first passage percolation. One promising approach to show uniqueness is to consider the fluctuation bounds for the free energy difference between two states. Such an approach was developed to prove the uniqueness of ground states for the random field Ising model by Aizenman and Wehr. Another more straightforward approach was used by Wehr and Wasielak to show uniqueness in a suitable sense for disordered ferromagnets. There are obstacles to carry this program for spin glasses, but there is hope to circumvent the lack of monotonicity using fluctuations of free energy differences.

A.4 Bhatnagar, Nayantara

I am interested in learning the techniques involved in the derivation of limit theorems for growth models, first passage times and related models. I am particularly interested in models which are not known to be integrable and what kind of methods can be used in these cases. I'm also interested in the connections of these topics with combinatorics. My interest in the workshop comes from working on longest monotone subsequences in a certain non-uniform model of random permutations.

A.5 Damron, Michael

I am interested in two main problems.

1. Show that with probability one, there is an infinite geodesic with an asymptotic direction. That is, there is an infinite geodesic with vertices $x_0, x_1, ...$ such that $arg\ x_n$ converges as$n \to \infty$.

Progress toward this question: This was solved under a global curvature condition by Newman and coauthors in the mid '90s. For $2d$, Jack Hanson and I reduced this condition to a local condition on the boundary of the limit shape: if there exists θ such that the limit shape boundary is exposed and differentiable in direction θ, then with probability one, there is an infinite geodesic with asymptotic direction θ. Unfortunately there is only one known type of distribution with this property and it is only in the direction $\pi/4$: any distribution on edge weights t_e satisfying $P(t_e = 1) = \vec{p_c}$ and $P(t_e < 1) = 0$. Here $\vec{p_c}$ is the critical probability for $2d$ oriented percolation. We also showed that there are deterministic sectors of aperture at most $\pi/2$ such that with probability one, there are infinite geodesics that are asymptotically directed in these sectors. This is a weaker notion of direction for a geodesic, since it does not rule out geodesics wandering throughout the sector.

2. Lower bounds on the variance of the passage time. Prove that for $d \geq 3$, $VarT(0, ne_1)$ diverges as $n \to \infty$.

Progress toward this question: It is known by Newman-Piza and Pemantle-Peres that in $2d$ and for most passage time distributions, this variance is bounded below by $C \log n$. These results were extended to general measures by Zhang and then to general directions for measures whose limit shapes have flat edges (but outside these edges) by Tuca Auffinger and me. However there is no lower bound for higher dimensions. Actually the Newman-Piza method does not even work in more general $2d$ settings. For example, it is not known that if T is the minimal passage time of any path that winds once horizontally around the $2d$ torus of size n, then $VarT$ diverges.

A.6 Deshayes, Aurelia

During my phd thesis, I studied random linear growth models for the spread of particles over time in the cubic lattice. This class of models includes for instance the first passage percolation and various extensions of the contact process. Like in the first-passage-percolation case (under appropriate hypothesis), these processes satisfy an asymptotic shape theorem. But a lot of questions remain open; for example, we have no information about the norms that appear in these theorems. To this end, I am aware that an important step is to understand the very foundations of the proof of the few results that exist in the first-passage-percolation case and to point out the differences with the contact process case. I am also interested in the study of the fluctuations which remains a difficult problem in FPP too. Finally, Regine Marchand and I are interested about a contact process with long range interaction, in continuation of the work of Chatterjee and Dey about long range first passage percolation.

A.7 Kubota, Naoki

I am interested in the following two topics:

- Gaussian concentrations for crossing random walks in i.i.d. nonnegative potentials on $\mathbb{Z}^d$ $(d \geq 2)$
- Lower bounds for the variance of first-passage percolation on $\mathbb{Z}^d$ $(d \geq 2)$

In particular, I will focus on the first topic. The main object is the travel cost

$$a(0,x) := -\log E^0\left[\exp\left\{-\sum_{k=0}^{H(x)-1}\omega(S_k)\right\}, H(x) < \infty\right],$$

where $\omega(z)$, $z \in \mathbb{Z}^d$, are i.i.d. nonnegative potentials, $(S_k)_{k=0}^{\infty}$ is the simple random walk on $\mathbb{Z}^d$ starting at 0, P^0 is its law, and $H(x)$ is the first passage time through x. Gaussian concentrations for $a(0,x)$ have already been studied for potentials with bounded and strictly positive support, see the following references more precisely:

(1) D. Ioffe and Y. Velenik, Stretched polymers in random environment, Probability in Complex Physical Systems, 2012
(2) S. Sodin, Positive temperature versions of two theorems on first-passage percolation, Lecture Notes in Mathematics, 2014

Then, I would like to consider Gaussian concentrations for potentials with unbounded and (not strictly) positive support under proper moment conditions.

First-passage percolation is directly related to the first topic. In first-passage percolation, the investigation of concentrations progresses recently, and techniques taken in the following references seem useful for the first topic:

(3) M. Damron, J. Hanson and P. Sosoe, Sublinear variance in first-passage percolation for general distributions, Probability Theory and Related Fields, 2013
(4) M. Damron, J. Hanson and P. Sosoe, Subdiffusive Concentration in First-Passage Percolation, Electron. J. Probab, 2014
(5) M. Damron, N. Kubota, Gaussian concentration for the lower tail in first-passage percolation under low moments, arXiv

If $d \geq 3$ and potentials have at least second moment, then we can extend the concentration for the lower tail to the case where potentials have unbounded and (not strictly) positive support. If $d = 2$, then we need strict positivity, but the concentration for the lower tail can be improved for unbounded potentials. See the following more precisely:

(6) N. Kubota, Large deviations for the travel cost of the simple random walk in random potentials, arXiv

With these observations, the next questions arises:

(Q1) In $d = 2$, can we remove the strict positivity for potentials?
(Q2) What about Gaussian concentration for the upper tail?

For (Q1), the concentration for the lower tail is partially true under an exponential moment. This means that the large deviation inequality for the lower tail holds under an exponential moment condition, see reference (6). Therefore, the strict positivity may be removed under some moment conditions. The large deviation inequality also holds for the upper tail, so that (Q2) may be true for suitable settings, see (6) again. Reference (5) mentioned that in first-passage percolation, a subdiffusive concentration holds for the upper tail, and we may obtain a similar estimate for the travel cost $a(0,x)$ in the case where potentials have unbounded and (not strictly) positive support.

In particular, (Q1) is related to the Lyapunove exponent and the nonrandom fluctuations. It is well known that for $x \in \mathbb{Z}^d$, the limit

$$\alpha(x) := \lim_{n\to\infty} \frac{1}{n} a(0, nx)$$

exists almost surely, and it is called the Lyapunov exponent. By the concentration for the lower tail, the similar argument to reference (7) below gives an upper bound for the nonrandom fluctuation $\mathbb{E}[a(0, n\xi)] - \alpha(n\xi)$, where ξ is any coordinate vector on $\mathbb{R}^d$:

(7) A.-S. Sznitman, Distance fluctuations and Lyapounov exponents, The Annals of Probability, 1996

Unfortunately, it does not work very well to obtain a bound for the nonrandom fluctuation for any direction. This is because Sznitman assumes that potentials are invariant under rotation. Our model treats $\mathbb{Z}^d$ as the underling space, so we usually does not assume it.

Finally, I will comment on the second topic stated at the beginning. For the variance of first-passage percolation, the gap of its lower and upper bounds seems large in the present stage. In fact, a lower bound is logarithmic and an upper bound is sublinear. See the following references:

(8) C. M. Newman and M. S. T. Piza, Divergence of shape fluctuations in two dimensions, The Annals of Applied Probability, 1995

(9) A. Auffinger and M. Damron, Differentiability at the edge of the percolation cone and related results in first-passage percolation, Probability Theory and Related Fields, 2013

(10) N. Kubota, Upper bounds on the non-random fluctuations in first passage percolation with low moment conditions, to appear in Yokohama Mathematical Journal

I have no idea about this topic, and it seems that the lower bound should be improved under suitable settings.

A.8 Mourrat, Jean-Christophe

— A list of questions —

- List all conceivable heuristics leading to the prediction of the fluctuation and wandering exponents of two-dimensional FPP.

- Find a “robust” proof that the variance of the first passage time on a binary tree is O(1). The proof should be robust e.g. to the addition of a sparse set of extra branches, and should not use the actual value of the time constant.

- Can one prove Kesten’s bound (or anything non-trivial) for two-dimensional FPP, assuming that the passage times have a finite range of dependence? (i.e. without assuming an i.i.d. product structure.)

A.9 Newman, Chuck

I have a long-standing interest in two particular open problems/issues about first passage percolation. The first is specific to two-dimensional models; the second is more general although chances of progress are most likely in two dimensions. Both of these are quite widely known.

1. Over the past 15 years or so, starting with the work of Baik, Deift and Johansson on longest increasing subsequences, there have been wonderfully precise results for certain two-dimensional models about scaling exponent values (= 2/3 and $\chi = 1/3$) and about related random variable scaling limits in terms of Tracy-Widom type distributions. It's widely recognized that there should be rather robust universality for the values of the exponents and the nature of the limiting distributions but these have been hard to develop. What seems to be needed is an approach or technique that avoids relying on exact solubility or combinatorial identities.

2. Geodesics in first-passage percolation are finite, semi-infinite, or doubly infinite (bi-infinite) paths such that all their finite segments are minimizing. It is easy to construct (or rather, prove the existence of) semi-infinite geodesics. Not only is it harder to construct bi-infinite ones, but the standard conjecture is that, at least in low dimensions (including two), they almost surely do not exist. The open problem is to prove this, at least for $d = 2$. There is an old straightforward heuristic argument that in any dimension where the wandering exponent strictly exceeds 1/2, there should almost surely be no bi-geodesics. Thus two dimensions where equals or should equal 2/3 would be a good place to start.

A.10 Rassoul-Agha, Firas

I am interested in polymer models, both positive and zero temperature. This includes both oriented and standard last (and first) percolation models. Of interest to me are Busemann functions, variational formulas for the limiting free energy and time constants, and fluctuation exponents. Tools that are of interest include concentration inequalities, large deviation techniques, and homogenization.

A.11 Wang, Xuan

Consider the model of first-passage percolation on $\mathbb{Z}^2$ with iid edge weights with distribution function F. Let T_n be the passage time between the origin and the point $(n, 0)$. Under the assumption $F(0) < p_c = 1/2$ and certain moment conditions, we have $\lim_{n\to\infty} T_n/n = \mu \in (0, \infty)$ in L^1 and almost surely. The magnitude of $\mathrm{Var}(T_n)$ is of great interest. Under minimal assumptions, the best known upper bound is of order $n/\log n$ (Damron, Hanson and Sosoe '13), while the best known lower bound is of order $\log n$ (Newman and Piza '95). I am interested in improving the low bound.

This problem is also related to the scaling exponents. The fluctuation exponent, denoted by χ, is defined to be such that $\mathrm{Var}(T_n) \sim n^{2\chi}$. The wandering exponent ξ is such that n^{ξ} is the order of fluctuation, for the geodesic between the origin and the point $(n, 0)$, about the straight line that goes through the two points. It is widely believed that $\chi = 1/3$ and $\xi = 2/3$ and the relation $\chi = 2\xi - 1$ was proved under certain assumptions. Under a stronger assumption that $\xi = 2/3$, the method used by Newman and Piza can produce a better low bound, which eventually proves $\chi \geq 1/6$. There is still a gap between $\chi \geq 1/6$ and the conjectured $\chi = 1/3$. An attempt of filling this gap may inspire new ideas and insights about the model.

I hope to sort out the ideas and main difficulties in improving the lower bound. I also would like to learn more techniques, results and problems about first-passage percolations during the workshop.

A.12 Wierman, John

My dissertation and early career research was on first-passage percolation. Several papers and a 1978 research monograph with Robert Smythe established some of the fundamental results on the topic. My research has focused on classical percolation since the early 1980s, with occasional papers on first-passage percolation and applications of subadditive processes to statistics. There has been a new wave of research in first-passage percolation in recent years, which I am eager to learn about. I will be starting a sabbatical year in summer 2015, and the workshop will give me an excellent opportunity to see the new methods, results, problems, and people in the field.