Problems

August 24, 2006

0.1 " X-constants and free Poincare inequality" (Voiculescu)

Q: In a von Neumann algebra M with a faithful normal trace-state τ let $X = X^* \in M$ and let $1 \in B \subset M$ be an infinite-dimensional von Neumann subalgebra so that B and X are free in the algebraic sense and $M = W^*(X, B)$.

Assume that $\partial_{X:B}$ is closable in $L^2(M, \tau)$ (this is the case for instantce if X is a free semicircular perturbation $X = X_0 + \varepsilon S$, with S a semicircular free from X_0 and B).

Under what conditions are the L^2 solutions of

$$\overline{\partial_{X:B}} u = 0$$

in $L^2(B, \tau)$?

A related question about a stronger condition: when does the free Poincare inequality

$$C \|\partial_{X:B} \xi\|_2 \geq \|\xi - E_B \xi\|_2$$

hold for $\xi \in B\langle X \rangle$?

0.2 "Large Deviations", Guionnet, Hiai, Cabanal-Duvillard.

Q: Given a tracial state τ corresponding to a free stochastic process, does there exist a sequence of tracial states $\tau_n \to \tau$ with $\chi_p^*(\tau_n) \to \chi_p^*(\tau)$ where τ_n corresponds to the process $dA_i(t) = dS_i(t) + k_t(A_1(s), \ldots, A_m(s))_{s \leq t} dt$ with k_t stepwise constant in s, and χ_p^* denotes the quantity χ^* defined for processes in the paper of Guionnet and Cabanal-Duvillard.

Q: In the one variable case, if $A(t)$ follows a process $dA(t) = dS(t) + k_t(A(s))_{s \leq t}$ then replacing $A(t)$ with $A(t) + C_\epsilon$ (with C having Cauchy distribution and free from $A(t)$) then k_t is replaced by $k_t^\epsilon = \tau\left(k_t | A(t) + C_\epsilon\right)$. Thus, k_t^ϵ is smooth. Is there an analog of this smoothing in the several-variable case?

Q: We know that if $f : \mathbb{R} \to \mathbb{R}$ and A is an $n \times n$ Hermitian random matrix, then there exists a random matrix C_ϵ with Cauchy distribution such that $\mathbb{E} f(A + C_\epsilon) = P_\epsilon f(A)$ with $P_\epsilon f(x) = \int \frac{f(y)}{(y-x)^2 + i\epsilon^2} dy$ the usual Cauchy (Poisson) kernel. Can this be done for several variables?

Q: Given $x_1, \ldots, x_m \in (\mathcal{A}, \tau)$ a tracial unital vN algebra, do the conjugate variables belong to the L^2 closure of cyclic gradient space? i.e. do there exist $H_k \in \mathbb{C}\langle \alpha_1, \ldots, \alpha_m \rangle$ such that $\mathcal{J}(x_i) = \lim_k D_i H_k$ where $\partial_{x_i} : L^2(\mathcal{A}, \tau) \to L^2(\mathcal{A}, \tau) \otimes L^2(\mathcal{A}, \tau)$ by $x_j \mapsto \delta_{ij} 1 \otimes 1$ as a densely defined operator, $\mathcal{J}(x_i) = \partial_{x_i}^*(1 \otimes 1)$, and $D_i = m \circ \partial_{x_i}$ (m is the flip-multiplication $x \otimes y \mapsto yx$).

Q: Does the change of variables formula for χ also hold for χ^*?

Q: Is there a change of variables formula for processes? i.e. suppose that we start with random variables $x_1, \ldots, x_m \in (\mathcal{A}, \tau)$ which can be reached by a process $dA_i(t) = dS_i(t) + k_t(A_1(s), \ldots, A_m(s))_{s \leq t}$, $\mu_{A_1(1), \ldots, A_m(1)} = \mu_{x_1, \ldots, x_m}$. We define new random variables via functional calculus $y_1 = f_1(x_1, \ldots, x_m), \ldots, y_m = f_m(x_1, \ldots, x_m)$. Can we apply a function P to k_t to get $dB_i(t) = dS_i(t) + P(k_t(B_1(s), \ldots, B_m(s))_{s \leq t})$ such that $\mu_{B_1(1), \ldots, B_m(1)} = \mu_{y_1, \ldots, y_m}$.

Open Problem: Can we replace lim sup with lim inf in the microstates definition of the free entropy χ?

Q Hiai introduced the free pressure $\pi_R(h)$ for a self-adjoint element (regarded as a free hamiltonian) h of the universal free product C^*-algebra $\mathcal{A}^{(n)} = \bigstar_{i=1}^n C([-R, R])$, and defined a free entropy-like quantity $\eta_R(\tau)$ of a tracial state $\tau \in TS(\mathcal{A}^{(n)})$. The inequality $\eta_R(\tau) \geq \chi(\tau)$ holds. τ is called an equillibrium tracial state with respect to h if the variational equality $\eta_R(\tau) = \tau(h) + \pi_R(h)$ holds. Such a τ always exists for each h. For which h there is a unique equilibrium tracial state? A way to prove this is the free transportation inequality.

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Q: It was recently shown by Guionnet and Maurel-Segala that for the vN algebra $(\mathcal{A}, \tau)$ generated by m free semicirculars,

$$\sup_{\tau \in \mathcal{TS}(\mathcal{A})} \left\{ \chi(\tau) - \tau(\sum t_i q_i) \right\} = \sum_{p_1, \ldots, p_m} \prod_{k_1, \ldots, k_m} \frac{(t_i)^{p_i}}{k_i!} C(q, k_1, \ldots, k_m)$$

where $C(q, k_1, \ldots, k_m)$ enumerated planar maps with colored edges and vertices of types $q, k_1, \ldots, k_m$. Is there a similar interpretation for the non-microstates analog

$$\sup_{\tau \in \mathcal{TS}(\mathcal{A})} \left\{ \chi^*(\tau) - \tau(\sum t_i q_i) \right\} ?$$

0.3 "Free von Neumann Algebras", Dykema, Ricard.

Q: Given A, B free group factors with a common diffuse subalgebra $D \subset A, B$, what conditions on A, B, D guarantee that $A \star_D B$ is a free group factor?

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Q: for a regular weakly-rigid (in the sense of Popa) subalgebra of a von Neumann algebra, is the free entropy dimension ≤ 1?

Open Problem: for generators $\gamma_1, \ldots, \gamma_n \in \Gamma$ with the first L^2 -Betti number $\beta_1(\Gamma)$ large, is the microstates free entropy dimension of this family of generators large? (This is known for the non-microstates free entropy dimension [work of Mineyev-Shlyakhtenko]).

Q: Consider $\Delta = \sum_{i=1}^{m} \partial_{x_i}^* \partial_{x_i}$ and the corresponding completely positive map $\varphi_t = \exp(-t\Delta)$, where $(x_1, \ldots, x_m)$ have finite Free Fisher Information. Can φ_t converge uniformly to the identity map on the unit ball of $W^*(x_1, \ldots, x_m)$? If no, it follows that the von Neumann algebra generated by $(x_1, \ldots, x_m)$ is not weakly rigid if it is non-hyperfinite.

Q: Let $\Gamma_{q,n} = W^*(s_q(g) = l(g) + l(g)^* | g \in \mathcal{H}_{\mathbb{R}})$ with $n = \dim \mathcal{H}_{\mathbb{R}}$, $-1 < q < 1$ be the von Neumann algebra generated by fields operators acting on a q-deformed Fock space. Does $\Gamma_{q,n}$ depend on q? A way to approach this question could come from the following observation. In the free case, $q = 0$, the natural orthormal basis of the Fock space consists of vectors $e_{\underline{i}} = e_{i_1}^{\otimes \alpha_1} \otimes ... \otimes e_{i_k}^{\otimes \alpha_k}$ with $i_1 \neq ... \neq i_k$ and $\alpha_1 > 0$. This basis can be recoved from the algebra as $e_{\underline{i}} = T_{\alpha_1}(s_0(e_1))...T_{\alpha_k}(s_0(e_q))\Omega$, where T_k are Chebytchev polynomials. It would be interesting to find an analogue for these formulas in the general case and to unterstand the underlying combinatorics.

The q-deformation leads to the commutation relations $l(e)^* l(f) = q l(f) l(e)^* + \langle f, e \rangle Id$. Instead consider themore general relations $l(e_i)^* l(e_j) = \sum_{s,t} t_{i,j}^{s,t} l(e_s) l(e_t)^* + \delta_{i,j} Id$. When does the C^*-algebra generated by these operators is an extension of a Cuntz algebra by compacts ? When does the fields operators associated to them produce a type II_1 factor ?

Consider the projection P_k from $\Gamma_{q,n}$ to its subspace consisting of x such that $x.\Omega$ has length at most k in the Fock space. Is $\|P_k\|_{cb}$ polynomially bounded in k ? This would prove the CBAP for the associated L_p spaces $(1 < p < \infty)$ and the exactness of the C^*-algebra generated by q-gaussians.

Q: To prove the existence of an embedding $\Gamma_{q,n} \to \mathcal{R}^\omega$, one uses Speicher's central limit theorem. In this procedure, is it possible to find explictely uniformly bounded matrix whose mixed moments approach those of q-gaussians ? More precisely, let $c_{i,j}$ be unitary generators of the CAR-algebra (or -1-gaussians), are the matrices $\frac{1}{\sqrt{n}}[c_{i,j}]_{i,j \leq n}$ uniformly bounded ?

Q: For the random matrix model $\exp(-nTr(p(A_1, A_1^*, \ldots, A_m, A_m^*))$ we know that the conjugate variables satisfy $\mathcal{J}_i = \mathcal{D}_i P$. Is the operator $\exp(-t \sum \partial_j^* \partial_j)$ compact in the limit $n \to \infty$ (where ∂_j is Voiculescu's partial difference quotient on the limit algebra with respect to the limit of A_j)? As a starting point, consider $P = \sum A_i^2 + \sum t_i q_i(A_1, \ldots, A_m)$ where Guionnet and Maurel-Segala have shown convergence of the model.

0.4 Focus Group on Free Entropy (day 3)

<u>Open Problem</u>: Is $\delta^* = \delta^\star$? Here

$$\delta^* = n - \limsup_{t \downarrow 0} \frac{\chi^*(x_1 + \sqrt{t}s_1, \ldots, x_n + \sqrt{t}s_m)}{\log t^{1/2}}$$

and

$$\delta^\star = n - \limsup_{t \to 0} \sum_{i=1}^{n} t\Phi^*(x_1 + \sqrt{t}s_1, \ldots x_m + \sqrt{t}s_m).$$

<u>Q</u>: What is the non-microstates analogue of free entropy in the presence, $\chi(x_1, \ldots, x_n : y_1, \ldots, y_n)$?

0.5 Focus Group on Operator Theory (day 3)

<u>Q</u>: What is the boundary behavior of the subordination functions which appear in free convolution of operator-valued random variables?

<u>Q</u>: What are examples/conditions for freely strongly unimodal variables, i.e. unimodal random variables that when freely convolved with a unimodal variables remain unimodal? (Unimodal means that the law of the random variable has a smooth density with a unique maximum; example: Gaussian law or the semicircle law).

<u>Q</u>: More specifically, if μ, ν are symmetric unimodal distribution, is $\mu \boxplus \nu$ unimodal?

0.6 "Invariant Subspaces for an Operator", Haagerup

<u>Q</u>: Let x, y be two free circular elements, and let S, T be two operators in a II_1 factor, which is free from x, y. In the Haagerup-Schultz estimate

$$(\star\star) \qquad \left\| (S + xy^{-1})^{-1} - (T + xy^{-1})^{-1} \right\|_p \le c(p) \left\| S - T \right\|_p < \infty$$

with $0 < p < \frac{2}{3}$, can one use x instead of xy^{-1}?

<u>Q</u>: (Brown measure of unbounded operators): As defined by (Haagerup and Schultz), $\Delta(T)$ makes sense for $T \in M^\Delta$ where $M^\Delta = \left\{ T \in \tilde{M} \middle| \int_0^\infty \log t \, d\mu_T(t) < \infty \right\}$. Then $\Delta(T) = \exp(\int_0^\infty \log t \, d\mu_T(t)) \in [0, \infty]$. Can one make sense of μ_T for such unbounded T?

<u>Q</u>: Does the main result of (Haagerup and Schultz) hold for $T \in L^p M$ (some or all p)? $T \in M^\Delta$? $T \in \tilde{M}$?

0.7 "Free Group Factors", Ozawa

Conj: if $\mathcal{H}$ an M-M bimodule $M = L\mathbb{F}_n$, and ${}_M\mathcal{H}_M \preceq L^2M \otimes L^2M$, (weak containment) then

$$\mathrm{Hom}({}_M\mathcal{H} \underset{M}{\otimes} \mathcal{H} \underset{M}{\otimes} \mathcal{H}_M, L^2M \otimes L^2M) \neq 0.$$

Note that the assumption of weak containment is equivalent that the map

$$x \otimes y \mapsto (\lambda(x)\rho(y) : \mathcal{H}_M \ni h \mapsto xhy) \in B({}_M\mathcal{H}_M)$$

is continuous for the min-tensor product on $M \otimes M$. Examples of bimodules with this property come from the basic construction

$${}_M\mathcal{H}_M = M \otimes_A M$$

over a hyperfinite subalgebra $A \subset M$.

0.8 Focus Group on Combinatorics of Random Matrix Models (day 4)

Given random matrices A_n and B_n with corresponding measures μ_{A_n} and μ_{B_n} on $M_n(\mathbb{C})$, we define their Itzykson-Zuber integral as

$$IZ(A_n, B_n) = \int \exp(-nTr(AU^*BU))d\mu_{A_n}(A)d\mu_{B_n}(B).$$

Thm (Guionnet and Zeitouni): if $\|A_n\| < c$, $\|B_n\| < c$ then $IZ(A_n, B_n) \sim \exp(-n\psi)$.

Q: There is another result that states that

$$\frac{\partial^n}{\partial t^n} \log IZ(tA_n, B_n)|_{t=0} \text{ converges.}$$

Does this expression match ψ above? Can we extend Guionnet and Zeitouni's result to complex parameters?

Q: Extend the model $\exp(-nTr(P(A_1, \ldots, A_m) + \frac{1}{2}\sum_{i=1}^m A_i^2))dA_1 \ldots dA_m$ of Guionnet and Maurel-Segala to non-selfadjoint P (i.e.polynomials with complex coefficients).

Q: Is there a combinatorial interpretation of free cumulants in terms of enumeration of maps and operations on maps?

Consider the spherical integrals

$$I_n(z, E_n) := \int \exp\{n\mathrm{tr}(UD_nU^*E_n)\}d_{m_n}(U),$$

where $D_n = \mathrm{diag}(z, 0, 0, \ldots, 0)$, $z \in \mathbb{C}$, and E_n is a sequence of $n \times n$ selfadjoint (diagonal) matrices, with spectrum uniformly bounded in n, and converging in distribution to μ_E

The sequence of functions of z

$$f_n(z) = \partial_z \frac{1}{n} \log I_n(z, E_n),$$

has been shown by Guionnet and Maida to converge to $R_{\mu_E}(z)$ for $|z|$ small enough.

Questions: What is the largest domain in the complex plane on which this convergence takes place? If μ_E is $\boxplus$-infinitely divisible, is the convergence happening on all the upper half-plane? Is there any possible generalization to measures with noncompact support? (one could probably approach this problem by trying to study the normality of the family/sequence f_n)

0.9 Focus Group on Invariant Subspaces (day 4)

If M is a II_1 factor, $T_1, \ldots, T_n \in M$, $[T_i, T_j] = 0$, then we have the "Brown Measure" defined as the unique measure on $\mathbb{C}^n$ such that

$$(\star) \qquad \log \Delta(1 - \sum \alpha_i T_i) = \int \log(1 - \sum \alpha_i \zeta_i) d\mu_{T_!, \ldots, T_n}(\zeta_1, \ldots, \zeta_n).$$

Q: Is $\mathrm{supp}\mu_{T_1, \ldots, T_n} \subset \sigma(T_1, \ldots, T_n)$, the Taylor spectrum of $T_1, \ldots, T_n$?

Q: Which functions on $\mathbb{C}^n$ have an integral representation as in $(\star)$?

Q: M a II_1 factor and $T \in M$. Define

$$K(T, r) = \left\{ \xi \in \mathcal{H} | \exists \xi_n \in \mathcal{H} \text{ s.t. } \|\xi_n - \xi\|_2 \to 0 \text{ and } \limsup \|T^n \xi_n\|^{1/n} \to 0 \right\},$$

$$\text{and } E(T, r) = \left\{ \xi \in \mathcal{H} | \limsup \|T^n \xi_n\|^{1/n} \to 0 \right\}.$$

Does $K(T, r) = E(T, r)$? The DT quasinilpotent operator may be a counterexample.

Q: Let c be a circular element ($\sigma(c) = \bar{\mathbb{D}}$), and let $f \in C^\infty(\mathbb{C})$. Can we make sense of $f(c)$ as an (unbounded) operator affiliated with $\{c\}''$?

Q: Let (Γ, τ) be a II_1 factor, $T \in \Gamma$, $\mu_T = \delta_0$. Does T have a non-trivial invariant subspace affiliated with Γ?

Q: Let $B_c$be a band limited operator obtained from c a circular element, and let D be the band limited operator obtained from the identity. Then D is uniformly distributed on $[0, 1]$ and $\star$-free from $\{B_c, B_c^*\}$. Is $D \in W^*(B_c)$? Or is $W^*(B_c) = L\mathbb{F}_t$ with $t = 1 + 2c(1 - \frac{c}{2})$?

0.10 "Infinite Divisibility", Nica.

Q: Given $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$ in a vNa such that $\{x_1, \ldots, x_k\}$ is tensor-independent of $\{y_1, \ldots, y_k\}$ and such that $\mu_{x_1,\ldots,x_k}, \nu_{y_1,\ldots,y_k}$ are freely infinitely divisible, we can apply the Fourier transform to get the power-series of the classical convolution of $\mu_{x_1,\ldots,x_k}$ and $\nu_{y_1,\ldots,y_k}$. How do such power-series relate to the noncommutative power series obtained from free convolution? (In other words how does the set of classically obtainable power-series relate to the set of freely obtainable power-series?)

Q: Can we make sense of the R-transform for x_1, x_2 unbounded (power-series are insufficient to encode all the information)? Easier question is for infinitely divisible unbounded operators.

Q: If c is unbounded R-diagonal, what is the R-transform of c, c^*?

0.11 Focus Group on Dirichlet Forms, from Classical to Quantum (day 5)

Q: For the q-deformed semicircular, the analogue of $\partial^*\partial$ exists (it is the number operator). Describe explicitly the associated ∂ (which exists by the work of Sauvageot).

Q: More generally, given a negative definite function on a group Γ (i.e. a Dirichlet form), we know it gives a representation by affine actions on $L^2\Gamma$. When is it a multiple of the left regular representation? What conditions on the negative definite function guarantee this?

Q: What conditions on a Dirichlet form $\delta^*\delta$ guarantee that the bimodule associated to δ embeds into $\bigoplus L^2N \otimes L^2N$?

Q: What is the analogue of the Bakry-Emery criterion in the noncommutative case? i.e. what is Γ_2 for noncommutative Dirichlet forms?

Q: Let $\partial : M \to L^2(M)\bar{\otimes}L^2(M^o)$ be a closable derivation, and let $\Delta = \partial^*\partial$, $S_t = \exp(-t\Delta)$. If the semigroup S_t converges uniformly to the identity in $\|\cdot\|_2$ on the unit ball, is the derivation inner when considered with values in the algebra of unbounded operators affiliated to $M\bar{\otimes}M^o$?