Ricci flow facts

0. Overview.

1. Short-time existence
2. maximal extension until curvature blows up
3. special solutions (spheres, cylinders, cigar soliton, Bryant's soliton)
4. evolution equation for the scalar curvature
5. maximum principle
6. Hamilton's maximum principle for the curvature operator
7. a priori estimates for blow-up time depending on bounds on initial $|Rm|$
8. short-time estimate on bounds on geometry
9. Shi's gradient estimate.

Unless otherwise mentioned:

- All manifolds and metrics are smooth.
- All results are due to Hamilton.

Part I. Special Solutions

1-0. The Ricci flow equation is:

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}.$$

I-1. Shrinking spheres. Let g_{can} denote the standard round metric on $\mathcal{S}^n$ of radius 1. The shrinking round sphere solutions are the ancient solutions

$$g(t) = r(t)^2 g_{\text{can}},$$

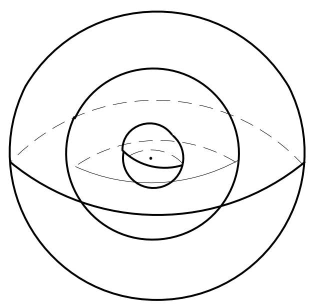
where

$$r(t) = \sqrt{r_0^2 - 2(n-1)t} = \sqrt{2(n-1)}\sqrt{T-t} \tag{1}$$

is obtained by solving the ODE

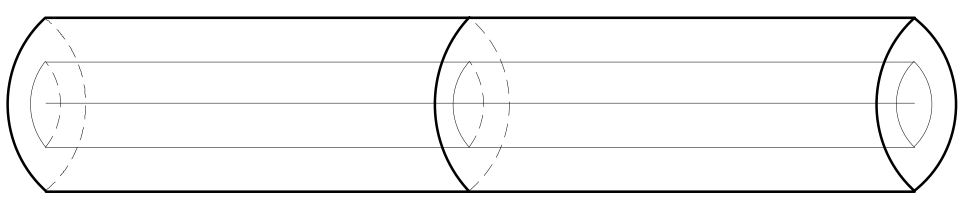
$$\frac{dr}{dt} = -\frac{n-1}{r}.$$

This solution is defined on the maximum time interval $(-\infty, T)$, where $T = \frac{r_0^2}{2(n-1)}$. More generally one has homothetic solutions on Einstein manifolds (manifolds of constant Ricci curvature).



I-2. Shrinking cylinders. Since $\mathbb{R}$ has no curvature we have the cylinder solutions where $\mathcal{M}^{n+1} = \mathcal{S}^n \times \mathbb{R}$ where

$$g_{\text{cyl}}(t) = r(t)^2 g_{\text{can}} + dy^2.$$



I-3. Cigar soliton. The **cigar metric** is defined on $\mathbb{R}^2$ by

$$g = \frac{dx^2 + dy^2}{1 + x^2 + y^2} = \frac{dr^2 + r^2\ d\theta^2}{1 + r^2} = ds^2 + \tanh^2 s\, d\theta^2$$

where $s = \operatorname{arcsinh} r = \log\left(r + \sqrt{1 + r^2}\right)$. Its scalar curvature is

$$R = \frac{4}{1 + r^2} = \frac{4}{\cosh^2 s} = \frac{16}{\left(e^s + e^{-s}\right)^2}.$$

This is also called the **Witten Black Hole**.

The cigar:

- is a stationary solution to the Ricci flow (steady soliton)
- has positive curvature
- asymptotic to a cylinder
- curvature decays exponentially
- is rotationally symmetric.

I-4. Ricci solitons. A Riemannian manifold (M, g) is a **gradient Ricci soliton** if it satisfies the equation

$$R_{ij} - \frac{r}{n} g_{ij} + \nabla_i \nabla_j f = 0$$

for some function f and where r is the average scalar curvature. When pulled back by the appropriate diffeomorphisms such a metric satisfies the normalized Ricci flow:

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} + \frac{2r}{n} g_{ij} = 2\nabla_i \nabla_j f = \mathcal{L}_{\nabla f} g_{ij}.$$

Such solutions are also called self-similar. When $r > 0$, $= 0$, or < 0, we say that the soliton is **shrinking**, **steady** or **expanding**.

- The cigar soliton satisfies the steady gradient soliton equation with $f(s) = -\log(\cosh s)$.
- The sphere is a trivial shrinking soliton: $r > 0$ and we may take $f = 0$.
- The cylinder is a shrinking soliton: $r > 0$ and we may take $f =?$.

I-4. Rosenau solution. Let $\mathcal{M}^2 = \mathbb{R}\times\mathcal{S}^1$, where $\mathcal{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ is the circle of radius 1. The **Rosenau metric** is

$$g(x, \theta, t) = \frac{\sinh(-t)}{\cosh x + \cosh t}\left(dx^2 + d\theta^2\right) \tag{2}$$

for $t < 0$.

The Rosenau solution:

- Is **ancient**: defined for all $t \in (-\infty, 0)$.
- Is **Type II**: $\sup_{\mathcal{M}\times(-\infty,-1]} |t| R = \infty$.
- If we take a limit at either pole $x = \pm\infty$ as $t \to -\infty$, we get a copy of the cigar soliton.
- $\mathcal{M}^2 \times \mathbb{R}$ does not occur as a limit of a solution to the Ricci flow on a closed 3-manifold in finite time (by Perelman's no local collapsing theorem).

I-5. Bryant soliton.

- is a stationary solution to the Ricci flow

- has positive sectional curvature
- dimension reduces to a cylinder
- curvature decays linearly
- is rotationally symmetric
- is obtained by solving an ODE
- should occur as a limit of a degnerate neck pinch.

Part II. Basic Analytic Facts

II-1. Short-time existence.

Fact A. *On any closed Riemannian manifold* $(\mathcal{M}^n, g_0)$*, there exists a unique solution* $g(t)$ *to the Ricci flow defined on some positive time interval* $[0, \varepsilon)$ *such that* $g(0) = g_0$*.*

- There is a simplified proof due to Dennis DeTurck.

II-2. Maximal extension until curvature blows up (Long-time existence).

Fact B. *If* g_0 *is a metric on a closed manifold* M^n*, the unique solution* $g(t)$ *of the Ricci flow such that* $g(0) = g_0$ *exists on a maximal time interval* $0 \leq t < T \leq \infty$*. Moreover, if* $T < \infty$ *then*

$$\lim_{t \nearrow T} \left(\sup_{x \in \mathcal{M}^n} |\mathrm{Rm}(x,t)|_g \right) = \infty.$$

II-3. Doubling-time estimate

Let

$$M(t) \doteqdot \max_{x \in \mathcal{M}^n} |\mathrm{Rm}(x,t)|_{g(t)}$$

be the maximum of the norm of the curvature.

Fact C. *There exists $c > 0$ depending only on the dimension n such that if g_0 is a metric on a closed manifold M^n, then a unique solution $g(t)$ with $g(0) = g_0$ exists for $t \in [0, c/M(0)]$ and*

$$M(t) \leq 2M(0) \quad \textit{for all times} \quad 0 \leq t \leq \frac{c}{M(0)}.$$

See below for the proof.

Part III. Basic Analytic Techniques

III-1. Weak maximum principle (for scalars). Let $\mathcal{M}^n$ be a closed manifold, $g(t)$ and $X(t)$, $t \in [0, T)$, be a 1-parameter family of metrics and vector fields.

MP 1. *If a C^2 function $u : M^n \times [0, T) \to R$ satisfies*

$$\frac{\partial u}{\partial t} \geq \Delta_{g(t)} u + X(t) \cdot \nabla u$$

and $u(0) \geq \alpha$ for some $\alpha \in R$, then $u(x, t) \geq \alpha$ for all $x \in M^n$ and $t \in [0, T)$.

Application 1. Nonnegative scalar curvature is preserved since

$$\frac{\partial R}{\partial t} = \Delta R + 2|Rc|^2 \geq \Delta R.$$

MP 2. *Suppose*

$$\frac{\partial v}{\partial t} \leq \Delta_{g(t)} v + \langle X, \nabla v \rangle + F(v) \tag{3}$$

where $F : R \to R$ and $u(0) \leq C$. Let $\varphi(t)$ be the solution to the associated ODE

$$\frac{d\varphi}{dt} = F(\varphi)$$
$$\varphi(0) = C.$$

Then

$$u(x,t) \leq \varphi(t)$$

for all $x \in M^n$ and $t \in [0,T)$ such that $\varphi(t)$ exists.

Application 2. Suppose $\max_{x\in\mathcal{M}^n} |\mathrm{Rm}(x,0)|_{g(0)} \leq M(0)$. Since

$$\frac{\partial}{\partial t}|Rm|^2 \leq \Delta |Rm|^2 + C(n)|Rm|^3,$$

we have

$$\max_{x\in\mathcal{M}^n} |\mathrm{Rm}(x,t)|_{g(t)} \leq \frac{1}{M(0)^{-1} - \frac{1}{2}C(n)t}.$$

Hence for $t \leq M(0)^{-1}/C(n)$ we have (doubling-time estimate)

$$\max_{x\in\mathcal{M}^n} |\mathrm{Rm}(x,t)|_{g(t)} \leq 2M(0).$$

III-2. Tensor maximum principle.

MP 3. *Let $\alpha(t) \in C^\infty(T^*\mathcal{M}^n \otimes_S T^*\mathcal{M}^n)$ be a symmetric $(2,0)$-tensor satisfying*

$$\frac{\partial}{\partial t}\alpha \geq \Delta_{g(t)}\alpha + \beta,$$

*where $\beta(\alpha, g, t)$ is a symmetric $(2,0)$-tensor which satisfies the **null eigenvector assumption** that*

$$\beta(V,V)(x,t) = \left(\beta_{ij}V^iV^j\right)(x,t) \geq 0$$

whenever $V(x,t)$ is a null eigenvector of $\alpha(t)$, that is whenever

$$\left(\alpha_{ij}V^j\right)(x,t) = 0.$$

If $\alpha(0) \geq 0$ (that is, if $\alpha(0)$ is positive semidefinite), then $\alpha(t) \geq 0$ for all $t \geq 0$ such that the solution exists.

Application 3a. When $n = 3$ both nonnegative Ricci curvature and nonnegative sectional curvature are preserved.

$$\frac{\partial}{\partial t} R_{ij} = \Delta R_{ij} + Q_{ij}$$

where Q_{ij} is quadratic in the Ricci curvature.

Application 3b. $n = 3$. Let $\nu(x,t)$ denote the smallest eigenvalue of the curvature operator. If $\inf_{x \in \mathcal{M}^3} \nu(x,0) \geq -1$, then at any point $(x,t) \in \mathcal{M}^3 \times [0,T)$ where $\nu(x,t) < 0$, the scalar curvature is estimated by

$$\begin{aligned} R &\geq |\nu| (\log |\nu| + \log(1+t) - 3) \\ &\geq |\nu| (\log |\nu| - 3) . \end{aligned}$$

- $Rm \geq -\phi(R) R - C$ where $\phi(R) \to 0$ as $R \to \infty$. That is, $|\nu| \leq \phi(R) R + C$.
  (Since if $|\nu| \geq e^6$, then $R \geq \frac{1}{2} |\nu| \log |\nu|$.)

- To control $|Rm|$ in dimension 3, it suffices to bound R from above. I.e., if $R \leq C$, then $|Rm| \leq C$.
  (Since $R \leq C$ implies $|\nu| \leq C$ which implies $|Rm| \leq C$.)

III-4. Derivative estimates.

(i). Global derivative estimates. The following estimate is due to S. Bando and W.-X. Shi based on a general idea of Bernstein.

For each $\alpha > 0$ and every $m \in N$, there exists a constant C_m depending only on m, and n, and $\max\{\alpha, 1\}$ *such that if*

$$|\mathrm{Rm}(x,t)|_g \leq K \qquad \text{for all } x \in \mathcal{M}^n \text{ and } t \in [0, \frac{\alpha}{K}],$$

then

$$|\nabla^m \operatorname{Rm}(x,t)|_g \leq \frac{C_m K}{t^{m/2}} \qquad \textit{for all } x \in \mathcal{M}^n \textit{ and } t \in (0, \frac{\alpha}{K}].$$

This is used in the proof of the Long-time Existence Theorem (Fact B).

The following estimate is due to Shi.

(ii). Local first derivative estimate. *For any* $\alpha > 0$ *there exists a constant* $C(n,\alpha)$ *depending only on* n *and* α *such that if* M *is a* n*-manifold,* $p \in M$*, and* $g(t)$, $t \in [0,\tau]$, $0 < \tau \leq \alpha/K$, *is a solution to the Ricci flow on* $B = B_{g(0)}(p,r)$*, and if*

$$|Rm(x,t)| \leq K \textit{ for all } x \in B \textit{ and } t \in [0,\tau],$$

then

$$|\nabla Rm(p,\tau)| \leq C(n,\alpha) K \left(\frac{1}{r^2} + \frac{1}{\tau} + K\right)^{1/2}.$$

III-5. Harnack (Li-Yau-Hamilton) type estimates.

(A) $n = 2$.

(Stronger) Matrix:

$$\nabla_i \nabla_j \log R + \frac{1}{2} R g_{ij} + \frac{1}{2t} g_{ij} \geq 0.$$

(Medium) Trace:

$$\frac{\partial}{\partial t} \log R - |\nabla \log R|^2 + \frac{1}{t} = \Delta \log R + R + \frac{1}{t} \geq 0.$$

(Weaker) Monotonicity:

$$\frac{\partial}{\partial t}(tR) \geq 0.$$

If we have an ancient solution, then the above inequalities hold with the time term removed.

(B) $n \geq 2$.

(Stronger) Matrix: for every 2-form U and 1-form W

$$M_{ij} W_i W_j + 2 P_{ijk} U_{ij} W_k + R_{ijk\ell} U_{ij} U_{k\ell} \geq 0$$

where

$$P_{ijk} = D_i R_{jk} - D_j R_{ik}$$

and

$$M_{ij} = \triangle R_{ij} - \frac{1}{2} \nabla_i \nabla_j R + 2 R_{ikj\ell} R_{k\ell} - R_{ik} R_{jk} + \frac{1}{2t} R_{ij} \, .$$

(Medium) Trace: for every vector V

$$\frac{\partial R}{\partial t} + \frac{R}{t} + 2 \nabla R \cdot V + 2 R_{ij} V^i V^j \geq 0.$$

(Weaker) Monotonicity:

$$\frac{\partial}{\partial t} (tR) \geq 0.$$

- P. Li and S.-T. Yau: gradient estimate method to prove Harnack inequalities for the heat equation.
- S.-C. Chu: Harnack = space-time curvature (as conjectured by Hamilton).
- Integrated version (comparing curvatures at different points and times): taking the minimizing $V^i = -\frac{1}{2} \left(Rc^{-1} \right)^{ij} \nabla_j R$, we have

  $$\frac{\partial R}{\partial t} + \frac{R}{t} - \frac{1}{2} \left(Rc^{-1} \right)^{ij} \nabla_i R \nabla_j R \geq 0$$

  so

  $$\frac{\partial}{\partial t} \log (tR) - \frac{1}{2R} \left(Rc^{-1} \right)^{ij} \nabla_i R \nabla_j R \geq 0.$$

Integrating over a path $\gamma : [t_1, t_2] \to M$ joining x_1 and x_2

$$\begin{aligned}
\log\left(\frac{t_2 R(x_2,t_2)}{t_1 R(x_1,t_1)}\right) &= \int_{t_1}^{t_2} \frac{d}{dt}\log\left(tR(\gamma(t),t)\right)dt \\
&= \int_{t_1}^{t_2}\left(\frac{\partial}{\partial t}\log(tR) + \nabla\log R\cdot\dot\gamma\right)dt \\
&\geq \int_{t_1}^{t_2}\left(\frac{1}{2R}\left(Rc^{-1}\right)^{ij}\nabla_i R\nabla_j R + \frac{1}{R}\nabla R\cdot\dot\gamma\right)dt \\
&\geq -\int_{t_1}^{t_2}\frac{1}{2R}R_{ij}\dot\gamma^i\dot\gamma^j dt.
\end{aligned}$$

Since $R_{ij} \leq R g_{ij}$ $(Rc \geq 0)$, we have

$$\frac{t_2 R(x_2,t_2)}{t_1 R(x_1,t_1)} \geq e^{-\frac{1}{2}A},$$

where

$$A = A(x_1,t_1,x_2,t_2) = \inf_\gamma \int_{t_1}^{t_2}\left|\frac{d\gamma}{dt}\right|^2 dt,$$

and the infimum is taken over all C^1-paths $\gamma : [t_1,t_2] \to \mathcal{M}^2$ joining x_1 and x_2. Since $\left|\dot\gamma(t)\right|_{g(t)} \leq \left|\dot\gamma(t)\right|_{g(t_1)}$, we conclude

$$R(x_2,t_2) \geq \frac{t_1}{t_2}\; e^{-d(x_1,x_2,t_1)^2\,/\,2(t_2-t_1)} R(x_1,t_1)\,.$$