## Perelman's argument

Phase I: Structure of finite time singularities.

Phase II: Flow with surgery is defined for all time.

Phase III: Long-time behavior of flow with surgery.



## Notation and Terminology

$R$ scalar curvature.

Rm curvature tensor.

$|\,$Rm$\,|$ norm of the curvature tensor.

$(M, g(\cdot))$ Ricci flow defined on some time interval $(a, b)$.

$(M, g(\cdot), x, t)$ Pointed Ricci flow, $x \in M$, $t \in (a, b)$.

$h(t) := a\, g(\frac{t}{a} + b)$ Parabolic rescaling of the flow by the scale factor $a > 0$.

$(M, g(t))$ Time $t$ slice.

$B(x,t,r)$ $r$-ball centered at $x \in M$ in the time $t$ slice $(M, g(t))$.

$P(x,t,r) = B(x,t,r) \times (t - r^2, t]$ Parabolic ball centered at $(x,t)$ of radius $r$.

$(B(x,t,r), r^{-2}g(t))$ Normalized ball.

Normalized curvature, normalized volume, normalized injectivity radius, etc of a ball refer to the corresponding quantities for the normalized ball. One has similar definitions for the normalized curvature of a parabolic ball.

All 3-manifolds will be orientable, for simplicity.



Let $(M, g(\cdot), x, t)$ and $(M', g'(\cdot), x', t')$ be pointed Ricci flows. Pick $\epsilon > 0$.

**Definition.** The two pointed Ricci flows are $\epsilon$-close in the $C^l$-topology if there are open sets $B(x, t, \frac{1}{\epsilon}) \subset U \subset M$ and $B(x', t', \frac{1}{\epsilon}) \subset U' \subset M'$, and a diffeomorphism $\phi : U \to U'$ such that for all $-\frac{1}{\epsilon^2} < \tau \leq 0$ we have

$$\left\| \frac{\partial^j(\phi^* g')}{\partial t^j}(t' + \tau) - \frac{\partial^j g}{\partial t^j}(t + \tau) \right\|_{C^l} < \epsilon$$

for all $0 \leq j \leq l$. In other words, modulo a diffeomorphism and a time shift, the $j^{th}$ time derivatives of $g$ and $g'$ are $C^l$ $\epsilon$-close in the on the parabolic ball $P(x, t, \frac{1}{\epsilon})$, for $0 \leq j \leq l$.

This definition allows one to make sense of convergence of a sequence of pointed flows $(M_k, g_k(\cdot), x_k, t_k)$ to a pointed flow $(M_\infty, g_\infty(\cdot), x$

Note that these notions ignore what happens to the flow after the marked time.

**Definition.** A Ricci flow $(M, g(\cdot)$ has **normalized initial conditions** if it is defined on an interval $[0, T)$, the curvature $|\,\mathrm{Rm}\,|$ is bounded by 1 at $t = 0$, and the volume of every unit ball at time zero is at least half the volume of a Euclidean unit ball. Any Ricci flow on a compact manifold can be normalized by parabolic rescaling.

Henceforth all Ricci flows will be 3-dimensional.

One of the main assertions in Phase I is that, roughly speaking, one of the following alternatives applies at each point $(x, t)$:

a. The geometry of a parabolic ball centered at $(x, t)$ of roughly unit scale is controlled.

b. Modulo rescaling by $R(x, t)$, the pointed flow $(M, g(\cdot), x, t)$ is close to a model flow (a "$\kappa$-solution").



**Main assertion.** (cf. I.12.1) Pick $\epsilon > 0$ and $T < \infty$. Then there are constants $R_0 = R_0(\epsilon, T)$, and $\kappa = \kappa(\epsilon, T)$, such that if $(M, g(\cdot))$ is a Ricci flow with normalized initial conditions, and $R(x, t) \geq R_0$, then the pointed flow $(M, g(\cdot), x, t)$, after being parabolically rescaled by $R(x, t)$, is $\epsilon$-close to a pointed $\kappa$-solution.

Remark: Due to Hamilton-Ivey curvature pinching, we have $|\,\mathrm{Rm}\,| < CR$ when $|\,\mathrm{Rm}\,|$ is large. This means that a point has large scalar curvature if and only if the curvature tensor has large norm.

## $\kappa$-solutions (section I.11)

A Ricci flow $(N, h(\cdot))$ is a $\kappa$-solution if:

• It is **ancient**: it is defined on an interval of the form $(-\infty, t]$ for some $t \in \mathbb{R}$.

• It has nonnegative curvature: Rm $\geq 0$

• The curvature $|\text{Rm}|$ (or equivalently the scalar curvature $R$) is bounded on each time slice.

• $(N, h(\cdot))$ has everywhere positive scalar curvature.

• $(N, h(\cdot))$ is $\kappa$-noncollapsed: if the normalized curvature of a parabolic ball $P(x, t, r)$ is $\leq 1$, then the normalized volume of the ball $B(x, t, r)$ is at least $\kappa$.

(An effectively equivalent definition of being $\kappa$-noncollapsed is: if the normalized curvature of a parabolic ball $P(x, t, r)$ is $\leq 1$, then the normalized injectivity radius of $B(x, t, r)$ is at least $\kappa$.)



The main assertion leads to the question: "What do $\kappa$-solutions look like?" Before discussing this, I first want to explain some of the ideas in the proof of the Main assertion.

## Neck structure in $\kappa$-solutions

**Definition.** Say that a point $(x,t)$ is the center of an $\epsilon$-neck if after parabolic rescaling by $R(x,t)$, the flow is $\epsilon$-close to round cylindrical flow on $S^2 \times \mathbb{R}$.

For all $\epsilon > 0$, there is a $D = D(\kappa, \epsilon) < \infty$ such that

- If $(M, g(\cdot))$ is a noncompact $\kappa$-solution defined at time $t \in \mathbb{R}$, then there is a point $x \in M$ such that all points lying outside the ball $B(x, t, DR(x,t)^{-\frac{1}{2}})$ are centers of $\epsilon$-necks at time $t$. Furthermore, unless $(M, g(\cdot))$ is round cylindrical flow on $S^2 \times \mathbb{R}$, then $x$ can be chosen so that the metric ball $B(x, t, DR(x,t)^{-\frac{1}{2}})$ is a 3-ball or a twisted line bundle over $\mathbb{R}P^2$.



- If $(M, g(\cdot))$ is a compact $\kappa$-solution defined at time $t$, then there is a pair of points $x_1, x_2 \in M$ such that points in $M$ lying outside the union of the two balls $B(x_1, t, DR(x_1,t)^{-\frac{1}{2}}) \cup B(x_2, t, DR(x_2,t)^{-\frac{1}{2}})$ are centers of $\epsilon$-necks at time $t$. Note that $M$ is diffeomorphic to a spherical space form by Hamilton's theorem on 3-manifolds with positive Ricci curvature.

## Further properties of $\kappa$-solutions

• The collection of pointed $\kappa$-solutions $(M, g(\cdot), x, t)$ with $R(x,t) = 1$ is compact. In particular, for any $r$, any space-time derivative of curvature will be uniformly bounded on the parabolic ball $P(x,t,r)$ on any such $\kappa$-solution.

• There is constant $\eta = \eta(\kappa)$ such that $0 \leq \frac{d}{dt}R \leq \eta R^2$, and $|\nabla R| \leq \eta R^{\frac{3}{2}}$.

• (Volume controls curvature) The normalized curvature of any ball $B(x,t,r)$ is bounded above by a decreasing function of its normalized volume.

• (Curvature controls volume) The normalized volume of any ball $B(x,t,r)$ is bounded below by a positive decreasing function of the normalized scalar curvature at the center point $(x,t)$.



The actual proof of the main assertion is a much more subtle blow-up argument.

## Assembling the local information

Using the fact that the large curvature part of the manifold is close to a $\kappa$-solution locally at the scale determined by the curvature, one concludes that the large curvature part consists of

• "Thin" components: $S^2 \times S^1$'s, $S^3$'s, $RP^3$'s, and $RP^3 \# RP^3$'s.

• "Thin" components with boundary: $S^2 \times I$'s, $RP^3 \setminus B^3$'s, and $B^3$'s.

• Components whose diameter is comparable to the radius of curvature. After rescaling to unit diameter, these components have controlled geometry, and by Hamilton's theorem are diffeomorphic to space forms.



## Extinction results

• $R > 0$.

• $R \geq 0$.

• Perelman, Colding-Minicozzi. If $M_0$ has no aspherical summands in its prime decomposition, then $M_t = \emptyset$ for large $t$.

## The long-time picture

We look at the time $t$ slice at scales $\leq \sqrt{t}$.

**The "Thick" part $M^{+}$.** A point $x \in M$ belongs to the thick part at time $t$ if there is a scale $r \simeq \sqrt{t}$ such that the parabolic ball $P(x, t, r)$ has controlled normalized geometry.

**The "thin" part $M^{-}$.** $x \in M$ belongs to $M^{-}$ if there is a scale $r \leq \sqrt{t}$ such that $B(x, t, r)$ has normalized curvature $\geq -1$ and small normalized volume.



## The thick-thin decomposition, with quantifiers

For all $w > 0$ there are $\lambda = \lambda(w) > 0$, $K = K(w) < \infty$ such that for all $x \in M_t$, one of the following holds:

- There exists an $r \leq \sqrt{t}$ such that the normalized ball $(B(x, t, r), r^{-2} g(t))$ has Rm $\geq -1$ and volume at most $w$.

- The parabolic ball $P(x, t, \lambda\sqrt{t})$ has normalized $|\,\mathrm{Rm}\,|$ at most $K$, and the normalized volume of each of its time slices is at least half the volume of the Euclidean unit ball.

In the second case, one concludes that the volume of $(M_t, g(t))$ is $\geq c\, t^{\frac{3}{2}}$.

## The Hamiltonian endgame

$$\frac{dR}{dt} = \Delta R = 2|\operatorname{Ric}^0|^2 + \frac{2}{3}R^2 \qquad (1)$$

where $\operatorname{Ric}^0$ is the traceless Ricci tensor.

$$R_{\text{min}} \geq \frac{2}{3}R_{\text{min}}^2 \qquad (2)$$

$$R_{\text{min}} \geq -\frac{3}{2}\left(\frac{1}{t+\frac{1}{4}}\right) \qquad (3)$$

using the fact that $R_{\text{min}} \geq 6$ at $t = 0$.

$$\frac{d}{dt}V = -\int R dV \leq R_{\text{min}}V. \qquad (4)$$

(3) and (4) imply that

$$\frac{V(t)}{\left(t+\frac{1}{4}\right)^{\frac{3}{2}}} \qquad (5)$$

is nonincreasing; we let $\bar{V}$ be its limit as $t \to \infty$.



Case A. If $\bar{V} = 0$, then for large $t$, $M^+(w,t)$ must be empty, since any point in $M^+(w,t)$ contributes a ball with volume comparable to $t^{\frac{3}{2}}$. Therefore $M = M^-(w,t)$. By the theory of collapsing with a lower curvature bound (Shioya-Yamaguchi), this implies that $M$ is a graph manifold. (I'm ignoring a special case here.)

Case B. $\bar{V} > 0$. Let $\hat{R} := R_{\text{min}}V^{\frac{2}{3}}$.

$$\frac{d\hat{R}}{dt} \geq \frac{2}{3}\hat{R}V^{-1}\int_{M_t}(R_{\text{min}} - R)dV \geq 0 \qquad (6)$$

since $\hat{R} < 0$. Let $\bar{R}$ be the limit of $\hat{R}$ as $t \to \infty$. It is not hard to check that $\bar{R}\bar{V}^{-\frac{2}{3}} = -\frac{3}{2}$, which implies that $R_{\text{min}}$ is asymptotic to $-\frac{3}{2t}$.