COMPONENTS OF HILBERT SCHEMES

RECORDED ON WHITEBOARD BY IZZET COSKUN, AND EDITED BY LI LI

Following are the open problems raised/discussed during the AIM workshop *Components of Hilbert Schemes*, July 19 to 23, 2010, organized by Robin Hartshorne, Diane Maclagan, and Gregory G. Smith.

Notation and convention. Denote by $\mathrm{Hilb}^d(\mathbb{A}^n)$ the Hilbert scheme of d points in the affine space $\mathbb{A}^n$. Denote by $\mathrm{Hilb}_{d,g}(\mathbb{P}^r)$ the Hilbert scheme of curves of degree d and genus g in the projective space $\mathbb{P}^r$. By *component* we always mean an irreducible component.

Problem 1. *Describe the singularities of the smoothable component of* $\mathrm{Hilb}^d(\mathbb{A}^n)$*. To be more specific,*

- *How large can the dimension of the Zariski tangent space to this component get?*
- *Does the maximum occur in the intersection of components? Does it occur in the smoothable component?*

Problem 2. *Can you describe the Zariski tangent space to the smoothable component of* $\mathrm{Hilb}^d(\mathbb{A}^n)$*?*

Problem 3. *Fix* $d, g, r, e > 0$*. Let* $\mathrm{Hilb}^{sm}_{d,g}(\mathbb{P}^r)$ *be the open subscheme of the Hilbert scheme* $\mathrm{Hilb}_{d,g}(\mathbb{P}^r)$ *that parameterizes smooth curves. For each point* $[C] \in \mathrm{Hilb}^{sm}_{d,g}(\mathbb{P}^r)$*, we define the Gauss map* $C \to \mathrm{Gr}(1, r)$ *sending a point of* C *to the tangent line at that point. Define*

$$Z^e := \{[C] \in \mathrm{Hilb}^{sm}_{d,g}(\mathbb{P}^r) |\ \textit{the Gauss map of the curve } C \textit{ is inseparable of degree } p^e\}.$$

Then $\bigcup_{e\geq 0} Z^e = \mathrm{Hilb}^{sm}_{d,g}(\mathbb{P}^r)$*.*

What can we say about the set Z^e*? Can we construct exotic components (i.e. components that only exist in characteristic* p*) using this stratification? Study the action by the Galois group* $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$*.*

Remark. There are some other stratifications one may consider, e.g. the one obtained by the topological type of the ramification divisor.

Problem 4. *(1) Is there a component of* $\mathrm{Hilb}^d(\mathbb{A}^n)$ *that exists only in characteristic* p *for some* p*?*
(2) Same question for the Hilbert schemes of curves in $\mathbb{P}^3$*.*

Problem 5. *Is there a nonreduced component of* $\mathrm{Hilb}^d(\mathbb{A}^n)$*? If so, find it.*

Problem 6. *(1) Give an explicit example (or show it does not happen) of a geometrically irreducible component of* $\mathrm{Hilb}^d(\mathbb{A}^n)$*, which is not fixed under the action of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*.*
(2) Same question for the Hilbert schemes of curves in $\mathbb{P}^3$*.*

Problem 7. *Does there exist a nonrational component of* $\mathrm{Hilb}^d(\mathbb{A}^n)$*?*

Problem 8. *If a component of the Hilbert scheme contains a smooth Borel fixed point, does the component have to be rational?*

Remark. If the ideal is a segment ideal (i.e. a monomial ideal generated in some degree s, where $s \leq$ its regularity, by the maximal monomials with respect to some term ordering), then the component is known to be rational, c.f. [P. Lella, M. Roggero, *Rational components of Hilbert schemes*, arXiv:0903.1029]. Note that every segment ideal is a Borel fixed ideal, but the converse is not true.

Problem 9. *Is the Hilbert scheme of local Cohen-Macaulay curves in* $\mathbb{P}^3$ *connected?*

Problem 10. *Let* $\rho = (p, q, r, 0, 0, \dots)$ *and define* $E_{p,q,r} := E_\rho$ *to be the moduli space of finite length graded modules of function* ρ*. For which* p, q, r *is* $E_{p,q,r}$ *irreducible?*

Remark. It is proved that if $4q < \max(6p + r, 6r + p)$ then $E_{p,q,r}$ is reducible (c.f. [M. Martin-Deschamps, D. Perrin, *Courbes gauches et modules de Rao*, J. Reine Angew. Math. 439 (1993), 103–145. page 119, Theorem 2.1]). Conjecturally, if $4q \geq \max(6p + r, 6r + p)$ then $E_{p,q,r}$ is irreducible.

Problem 11. *Describe the irreducible component of* E_ρ*, the moduli of finite length graded modules of function* ρ*.*

Problem 12. *What do properties of the Rao modules imply about* C*? For example, if* M_C *is Gorenstein or annihilated by a linear form, does* C *have any nice properties?*

Remark. We might have to require C to be minimal.

Problem 13. *Let* C_t *be a family of curves in* $\mathbb{P}^3$ *such that a general curve in this family is a smooth complete intersection, and the special curve* C_0 *is smooth. Does it imply that* C_0 *is also a complete intersection, assuming that the characteristic is 0?*

Remark. If the characteristic is $p > 0$, the answer is negative; if $n > 3$, the answer is negative; if C_0 is not smooth, the answer is negative. The reference is [P. Ellia, R. Hartshorne, *Smooth specializations of space curves: questions and examples*, Commutative algebra and algebraic geometry (Ferrara), 53–79, Lecture Notes in Pure and Appl. Math., 206, Dekker, New York, 1999].

Problem 14. *Give a geometric algebraic description of generic points of irreducible components of* $\mathrm{Hilb}^d(\mathbb{A}^n)$*.*

Problem 15. *Is the Gröbner fan a discrete invariant that distinguishes the components of* $\mathrm{Hilb}^d(\mathbb{A}^n)$*?*

Problem 16. *Does the set of monomial ideals contained in a component of* $\mathrm{Hilb}^d(\mathbb{A}^n)$ *determine that component? (Or more generally for* $\mathrm{Hilb}^P(\mathbb{P}^n)$ *for an arbitrary Hilbert polynomial* P*.)*

Remark. For the analogous question for the Hilbert scheme of curves, Richard Liebling might have given a counterexample in his thesis.

Problem 17. *Let $S \subset \mathbb{P}^3$ be an integral surface of degree d with a double curve D of degree e, and triple points, pinch point,. . . .*

Conjecture: there exists $n_0 \in \mathbb{Z}$ such that for any smooth curve $C \subset S$ of degree $n \geq n_0$, let H_C be the irreducible component of the Hilbert scheme containing C, then a general $C' \in H_C$ is also contained in a surface $S' \subset \mathbb{P}^3$ of degree d and a double curve D' of degree e.

Remark. The conjecture is posed in [P. Ellia, R. Hartshorne, *Smooth specializations of space curves: questions and examples*, Commutative algebra and algebraic geometry (Ferrara), 53–79, Lecture Notes in Pure and Appl. Math., 206, Dekker, New York, 1999]. It is proved to be true for $d = 3$ in [J. Brevik, F. Mordasini, *Curves on a ruled cubic surface*, Collect. Math. 54 (2003), no. 3, 269–281].

Problem 18. *Is it possible to define analogues of the Nakajima operators for the cohomology of a desingularization of the smoothable component of* $\mathrm{Hilb}^d(\mathbb{A}^n)$*?*

Problem 19. *What is the geography of locally Cohen-Macaulay surfaces in $\mathbb{P}^4$? To be more precise, what numerical invariants (degree, sectional genus, . . .) occur?*

Problem 20. *Let C be a curve, can* $\mathrm{Hilb}^d(C)$ *have a component of* $\dim < d$*?*

Remark. There exists a non-smoothable component of $\dim = d > 1$.

Problem 21. *Is there a multigraded Hilbert scheme with a connected component isomorphic to a fat point?*

Problem 22. *Fix d, g, n. Find a good/sharp lower bound for the dimension of components of* $\mathrm{Hilb}_{d,g}(\mathbb{P}^n)$.

(For $\mathbb{P}^3$ a lower bound is $4d$; for $\mathbb{P}^4$ the lower bound is $5d + 1 - g$ which is obviously not a good bound for d fixed and g sufficiently large.)

Remark. In the range $d^3 \leq \lambda(n)g^2$ there is a better bound, c.f. [D. Chen, *On the dimension of the Hilbert scheme of curves*, Math. Res. Lett. 16 (2009), no. 6, 941–954. Theorem 1.3].

Problem 23. *(An old question of Joe Harris) Does there exist a nondegenerate rigid curve in $\mathbb{P}^n$ other than the rational normal curve? Here rigid means the only deformations of the curve are those induced by automorphisms of $\mathbb{P}^n$.*

Problem 24. *Is* $\mathrm{Hilb}^8(\mathbb{A}^4)$ *reduced? More generally, develop techniques to prove the reducedness of Hilbert schemes.*

Problem 25. *For $R = k[x_1, x_2, x_3]$. Is every multigraded Hilbert scheme connected?*

Remark. For $R = k[x_1, \dots, x_{26}]$, there are non-connected examples, c.f. [Francisco Santos, *Non-connected toric Hilbert-schemes*, MR2181765, arXiv:0204044]. Find examples with fewer variables.

Problem 26. *(i) Let C be of bidegree $(3, 7)$ on a nonsingular quadric surface in $\mathbb{P}^3$. Can C be connected to an extremal curve in* $\mathrm{Hilb}_{10,12}(\mathbb{P}^3)$*?*

(ii) Given 4 skew lines C_1 on a nonsingular quadric Q_1 in $\mathbb{P}^3$. Does there exists a family $Q_t \rightsquigarrow 2H$, and a family $\mathcal{C}_t \subset Q_t$ such that $C_0 \subset Q_0 = 2H$ is locally Cohen-Macaulay?

Remark. (ii)$\Rightarrow$(i).

Problem 27. *Are there necessary or sufficient conditions on Borel fixed monomial ideals such that they are the generic initial ideals of local Cohen-Macaulay curves?*

Problem 28. *What is the smallest d such that* $\mathrm{Hilb}^d(\mathbb{A}^3)$ *is reducible?*

Remark. $10 < d \leq 78$. The bound $d > 10$ needs the assumption that char$=0$ and follows from a recent paper of Sivic.

Problem 29. *Fix a Hilbert polynomial P, consider the moduli space $BrV_P\mathbb{P}^n$ of branchvarieties that are equidimensional and connected in codimension 1.*
Is $BrV_P(\mathbb{P}^n)$ connected when non-empty?

Problem 30. *Is there a rigid local Artinian algebra besides k^n?*

Problem 31. *Consider* $1, 4, 10, a$ *where* $6 \leq a \leq 10$. *The general Artinian algebra with this Hilbert function is nonsmoothable. What is the generic point of the component it lies on?*

Problem 32. *Fix d, g. Let $H \subset \mathrm{Hilb}_{d,g}(\mathbb{P}^3)$ be an irreducible component. What is the largest number of points in general position you can make these curves pass through?*

Remark. This is a hard problem. If the curves are arithmetically Cohen-Macaulay, it should be treatable.

Problem 33. *What is the best pair (d, g) (in the vague sense that d is as small as possible and g is close to* 0 *as possible) such that there is a component of* $\mathrm{Hilb}_{d,g}$ *whose general member has an embedded point.*

Remark. An example of a pair satisfying the above condition is $d = 4$, $g = -15$, c.f. [D. Chen, S. Nollet, *Detaching embedded points*, arXiv:0911.2221].