OPEN PROBLEMS ON POWERS OF IDEALS

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Abstract. These are notes on the author's talk given at the workshop on Integral Closure, Multiplier Ideals and Cores, AIM, December 2006.

Problem 0.1. Let $R = \mathbb{C}[[x_1, \ldots, x_d]]$, and let $0 \neq f \in \mathfrak{m}_R$. Can f ever be a minimal generator of the ideal $I = \overline{(\partial f/\partial x_1, \ldots, \partial f/\partial x_d)}$?

It's well known that $f \in \overline{\mathfrak{m}(\partial f/\partial x_1, \ldots, \partial f/\partial x_d)}$.

Conjecture 0.2 (Eisenbud-Mazur)**.** If $\mathfrak{p} \in \operatorname{Spec}(R)$, then $\mathfrak{p}^{(2)} \subset \mathfrak{m}\mathfrak{p}$.

If Question 1 has a positive answer, then the conjecture holds: choose $f \in \mathfrak{p}^{(2)}$, then partials are in $\mathfrak{p}$, so $\overline{(\partial f/\partial x_1, \ldots, \partial f/\partial x_d)} \subset \mathfrak{p}$, which would imply that $f \in \mathfrak{m}\mathfrak{p}$. The Eisenbud-Mazur Conjecture is false in characteristic > 0. The best result to date is due to Takagi using multiplier ideals, Hochster-Huneke otherwise: If R is a regular local ring containing a field, and $\mathfrak{p} \in \operatorname{Spec}(R)$, then $\mathfrak{p}^{(c+1)} \subset \mathfrak{m}\mathfrak{p}$, where $c =$ height $\mathfrak{p}$.

Problem 0.3 (Hübl)**.** Let $(R, \mathfrak{m})$ be a regular local ring. If $\sqrt{I} = I$ and $f^n \in I^{n+1}$ for some n, is $f \in \mathfrak{m}I$? This implies Eisenbud-Mazur conjucture (Hübl).

If I is radical, then write $I = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_l$. Define $I^{(n)} = \mathfrak{p}_1^{(n)} \cap \ldots \cap \mathfrak{p}_l^{(n)}$. There is a theorem of Ein-Lazarsfeld-Smith (generalized by Hochster-Huneke): If $(R, \mathfrak{m})$ is local, regular, containing a field, and $\mathfrak{p} \in \operatorname{Spec}(R)$, then for all $n \geq 1, \mathfrak{p}^{(cn)} \subset \mathfrak{p}^n$, where $c =$ height $\mathfrak{p}$. It is also true if $\mathfrak{p}$ is replaced with $I = \sqrt{I}$ and $c = \max$ {height $\mathfrak{p}_1, \ldots,$ height $\mathfrak{p}_l$}. If $c = 2$, we get $I^{(4)} \subset I^2$.

Problem 0.4. If $R = k[x, y, z]$, and $I = I(X)$, where $X =$ some points, then is $I^{(3)} \subset I^2$?

I view these theorems as Briancon-Skoda, but for symbolic powers. The regular version of which is $\overline{I^{d+n-1}} \subset I^n$, where d is dimension of the ring, and $n \geq 1$ (Lipman-Sathaye). "What's true for integral closure by summing a constant is true for symbolic by multiplying by a constant".

Problem 0.5 (Understanding symbolic powers)**.** Let $\mathfrak{p}$ be a homogeneous prime in $k[x_1, \ldots, x_d]$, generated in degrees $\leq D$. Is $\mathfrak{p}^{(n)}$ generated in degrees $\leq Dn$?

Problem 0.6. Let $(R, \mathfrak{m})$ be a regular local ring of dimension d. What is the best integer $k \leq d$ such that for all $\mathfrak{m}$-primary ideals I, there is a Gorenstein ideal J with $I^k \subset J \subset I$?

Nonregular case:

Theorem 0.7. *If R is excellent, local, and reduced, then there is an integer k such that for all $I \subset R$, and all $n \geq 1$, $\overline{I^{n+k}} \subset I^n$.*

Problem 0.8. Is this true without local if $\dim R$ is finite?

Problem 0.9. If R is a complete local domain, is there a k so that for all $\mathfrak{p} \in \operatorname{Spec}(R)$, $\mathfrak{p}^{(kn)} \subset \mathfrak{p}^n$? I. Swanson proved that for any $\mathfrak{p}$ there is such a k.

Problem 0.10. Let I be an ideal generated by square-free monomials in a polynomial ring. (If $\sqrt{I} = I$, then $I^{(l)} = I^l$ for all $l \iff \operatorname{gr}_I(R)$ is reduced). When does a square-free monomial ideal satisfy $I^{(l)} = I^l$ for all l?

We say I has the packing property if after setting any subset of variables equal to 0 or 1, the new square free monomial ideal J has the property that there is a regular sequence of monomials having length equal to the height of J.

Conjecture 0.11 (Conforti-Cornuejols)**.** *I has the packing property if and only if $\operatorname{gr}_I(R)$ is reduced.* As far as I know the field is irrelevant. If the ideal is the ideal of a graph, the conjecture is true, and equivalent to the bipartite property of the graph.

An attempt to combine Frobenius and differentials: In characteristic 0, in $R = \mathbb{C}[[x_1, \ldots, x_d]]$, and I an ideal, if $f \in R$ and $(\partial f/\partial x_i) \subset I^n$ for all i, then $f \in I^{n-d+1}$, because $f \in \overline{(\partial f/\partial x_1, \ldots, \partial f/\partial x_d)} \subset \overline{I^n} \subset I^{n-d+1}$. In characteristic $p > 0$ this doesn't work out.

Problem 0.12. Let $R = \mathbb{Z}[x_1, \ldots, x_d]]$, and let $I \subset R$. Let $(-)_p$ denote reduction mod p. Does there exist a k so that for $p >> 0$, if $f \in R_p$ and $(\partial f/\partial x_i) \subset I_p^N$, then $f \in I_p^{N-k} + R_p^p$?