THE PAVING CONJECTURE IS EQUIVALENT TO THE
PAVING CONJECTURE FOR TRIANGULAR MATRICES

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Abstract. We resolve a 25 year old problem by showing that The Paving Conjecture is equivalent to The Paving Conjecture for Triangular Matrices.

1. Introduction

The Kadison-Singer Problem [14] has been one of the most intractable problems in mathematics for nearly 50 years.

Kadison-Singer Problem (KS). Does every pure state on the (abelian) von Neumann algebra \( \mathbb{D} \) of bounded diagonal operators on \( \ell_2 \) have a unique extension to a (pure) state on \( B(\ell_2) \), the von Neumann algebra of all bounded linear operators on the Hilbert space \( \ell_2 \)?

A state of a von Neumann algebra \( \mathcal{R} \) is a linear functional \( f \) on \( \mathcal{R} \) for which \( f(I) = 1 \) and \( f(T) \geq 0 \) whenever \( T \geq 0 \) (i.e. whenever \( T \) is a positive operator). The set of states of \( \mathcal{R} \) is a convex subset of the dual space of \( \mathcal{R} \) which is compact in the \( w^* \)-topology. By the Krein-Milman theorem, this convex set is the closed convex hull of its extreme points. The extremal elements in the space of states are called the pure states (of \( \mathcal{R} \)). The Kadison-Singer Problem had been dormant for many years when it was recently brought back to life in [9] and [10] where it was shown that KS is equivalent to fundamental unsolved problems in a dozen different areas of research in pure mathematics, applied mathematics and engineering.

A significant advance on the Kadison-Singer Problem was made by Anderson [2] in 1979 when he reformulated KS into what is now known as the Paving Conjecture (Lemma 5 of [14] shows a connection between KS and Paving). Before we state this conjecture, let us introduce some notation. For an operator \( T \) on \( \ell_2 \), its matrix representation \( (\langle Te_i, e_j \rangle)_{i,j \in I} \) is with respect to the natural orthonormal basis. If \( A \subset \{1,2,\cdots,n\} \), the diagonal projection \( Q_A \) is the matrix all of whose entries are zero except for the \( (i,i) \) entries for \( i \in A \) which are all one.

1991 Mathematics Subject Classification. Primary: 47A20, 47B99; Secondary: 46B07.
The first author was supported by NSF DMS 0405376.
Paving Conjecture (PC). For \( \epsilon > 0 \), there is a natural number \( r \) so that for every natural number \( n \) and every linear operator \( T \) on \( l_2^n \) whose matrix has zero diagonal, we can find a partition (i.e. a paving) \( \{ A_j \}_{j=1}^r \) of \( \{1, \cdots, n\} \), so that
\[
\| Q_{A_j} T Q_{A_j} \| \leq \epsilon \| T \| \quad \text{for all } j = 1, 2, \cdots, r.
\]
It is important that \( r \) not depend on \( n \) in PC. We will say that an arbitrary operator \( T \) satisfies PC if \( T - D(T) \) satisfies PC where \( D(T) \) is the diagonal of \( T \). It is known that the class of operators satisfying PC (the pavable operators) is a closed subspace of \( B(\ell_2) \). Also, to verify PC we only need to verify it for any one the following classes of operators \([1, 10, 8]\): 1. unitary operators, 2. positive operators, 3. orthogonal projections (or just orthogonal projections with \( 1/2 \)'s on the diagonal), 4. Gram operators of the form \( T^* T = (\langle f_i, f_j \rangle)_{i,j \in I} \) where \( \| f_i \| = 1 \) and \( T e_i = f_i \) is a bounded operator. The only large classes of operators which have been shown to be pavable are “diagonally dominant” matrices \([3, 4, 12]\), matrices with all entries real and positive \([5, 13]\) and matrices with small entries \([6]\).
Since the beginnings of the paving era, it has been a natural question whether PC is equivalent to PC for triangular operators This question was formally asked several times at meetings by Gary Weiss and Lior Tzafriri and appeared (for a short time) on the AIM website (http://www.aimath.org/The Kadison-Singer Problem) as an important question for PC. In this paper we will verify this conjecture. Given two conjectures \( C_1, C_2 \) we say that \( C_1 \) implies \( C_2 \) if a positive answer to \( C_1 \) implies a positive answer for \( C_2 \). They are equivalent if they imply each other.

2. Preliminaries

Recall that a family of vectors \( \{ f_i \}_{i \in I} \) is a Riesz basic sequence in a Hilbert space \( \mathbb{H} \) if there are constants \( A, B > 0 \) so that for all scalars \( \{ a_i \}_{i \in I} \) we have:
\[
A^2 \sum_{i \in I} |a_i|^2 \leq \| \sum_{i \in I} a_i f_i \|^2 \leq B^2 \sum_{i \in I} |a_i|^2.
\]
We call \( A, B \) the lower and upper Riesz basis bounds for \( \{ f_i \}_{i \in I} \). If \( \epsilon > 0 \) and \( A = 1 - \epsilon, B = 1 + \epsilon \) we call \( \{ f_i \}_{i \in I} \) an \( \epsilon \)-Riesz basic sequence. If \( \| f_i \| = 1 \) for all \( i \in I \) this is a unit norm Riesz basic sequence. A natural question is whether we can improve the Riesz basis bounds for a unit norm Riesz basic sequence by partitioning the sequence into subsets.

\( R_\epsilon \)-Conjecture. For every \( \epsilon > 0 \), every unit norm Riesz basic sequence is a finite union of \( \epsilon \)-Riesz basic sequences.

The \( R_\epsilon \)-Conjecture was posed by Casazza and Vershynin \([11]\) where it was shown that KS implies this conjecture. It is now known that the \( R_\epsilon \)-Conjecture
is equivalent to KS [9]. We will show that PC for triangular operators implies a positive solution to the $R_{\epsilon}$-Conjecture. Actually, we need the finite dimensional quantitative version of this conjecture.

**Finite $R_{\epsilon}$-Conjecture.** Given $0 < \epsilon, A, B$, there is a natural number $r = r(\epsilon, A, B)$ so that for every $n \in \mathbb{N}$ and every unit norm Riesz basic sequence $\{f_i\}_{i=1}^n$ for $\ell_2^n$ with Riesz basis bounds $0 < A \leq B$, there is a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \ldots, n\}$ so that for all $j = 1, 2, \ldots, r$ the family $\{f_i\}_{i \in A_j}$ is an $\epsilon$-Riesz basic sequence.

There are standard methods for turning infinite dimensional results into quantitative finite dimensional results so we will just outline the proof of their equivalence. We will need a proposition from [7].

**Proposition 2.1.** Fix a natural number $r$ and assume for every natural number $n$ we have a partition $\{A_i^n\}_{i=1}^r$ of $\{1, 2, \ldots, n\}$. Then there are natural numbers $\{n_1 < n_2 < \cdots\}$ so that if $j \in A_i^{n_j}$ for some $i \in \{1, \ldots, r\}$, then $j \in A_i^{n_k}$ for all $k \geq j$. Hence, if $A_i = \{j | j \in A_i^{n_j}\}$ then

1. $\{A_i\}_{i=1}^r$ is a partition of $\mathbb{N}$.
2. If $A_i = \{j_1^i < j_2^i < \cdots\}$ then for every natural number $k$ we have $\{j_1^i, j_2^i, \ldots, j_k^i\} \subseteq A_i^{n_{ij}}$.

**Theorem 2.2.** The $R_{\epsilon}$-Conjecture is equivalent to the Finite $R_{\epsilon}$-Conjecture.

**Proof.** Assume the Finite $R_{\epsilon}$-Conjecture is true. Let $\{f_i\}_{i=1}^\infty$ be a unit norm Riesz basic sequence in $\mathbb{H}$ with bounds $0 < A, B$ and fix $\epsilon > 0$. Then there is a natural number $r \in \mathbb{N}$ so that for all $n \in \mathbb{N}$ there is a partition $\{A_i^n\}_{i=1}^r$ of $\{1, 2, \ldots, n\}$ and for every $j = 1, 2, \ldots, r$ the family $\{f_i\}_{i \in A_j^n}$ is an $\epsilon$-Riesz basic sequence. Choose a partition $\{A_j^r\}_{j=1}^r$ of $\mathbb{N}$ satisfying Proposition 2.1. By (2) of this proposition, for each $j = 1, 2, \ldots, r$, the first $n$-elements of $\{f_i\}_{i \in A_j}$ come from one of the $A_i^n$ and hence form an $\epsilon$-Riesz basic sequence. So $\{f_i\}_{i \in A_j}$ is an $\epsilon$-Riesz basic sequence.

Now assume the the Finite $R_{\epsilon}$-Conjecture fails. Then there is some $0 < \epsilon, A, B$, natural numbers $n_1 < n_2 < \cdots$ and unit norm Riesz basic sequences $\{f_i^n\}_{i=1}^{n_r}$ for $\ell_2^n$ so that whenever $\{A_j\}_{j=1}^r$ is a partition of $\{1, 2, \ldots, n\}$ one of the sets $\{f_i^n\}_{i \in A_j}$ is not an $\epsilon$-Riesz basic sequence. Considering

$$\{f_i\}_{i=1}^\infty = \{f_i^n\}_{i=1, r=1}^{n_r, \infty} \in \left(\bigoplus_{r=1}^{\infty} \ell_2^{n_r}\right)^{1/2},$$

we see that this family of vectors forms a unit norm Riesz basic sequence with bounds $0 < A, B$ but for any natural number $r$ and any partition $\{A_j^r\}_{j=1}^r$ of $\mathbb{N}$ one of the sets $\{f_i\}_{i \in A_j}$ is not an $\epsilon$-Riesz basic sequence. □
3. The Main Theorem

Our main theorem is:

**Theorem 3.1.** The Paving Conjecture is equivalent to the Paving Conjecture for Triangular matrices.

**Proof.** Since a paving of $T$ is also a paving of $T^*$, we only need to show that The Paving Conjecture for Lower Triangular Operators implies the Finite $\epsilon$-Conjecture. Fix $0 < \epsilon, A, B$, fix $n \in \mathbb{N}$ and let $\{f_i\}_{i=1}^n$ be a unit norm Riesz basis for $\ell_2^n$ with bounds $A, B$. We choose a natural number $r \in \mathbb{N}$ satisfying:

$$1 - \frac{B^4}{A^4} \geq 1 - \frac{\epsilon}{2}.$$ 

We will do the proof in 5 steps.

**Step 1:** There is a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \ldots, n\}$ so that for every $j = 1, 2, \ldots, r$ and every $i \in A_j$ and every $1 \leq k \neq j \leq r$ we have:

$$\sum_{i \neq \ell \in A_j} |\langle f_i, f_\ell \rangle|^2 \leq \sum_{\ell \in A_k} |\langle f_i, f_\ell \rangle|^2.$$ 

The argument for this is due to Halpern, Kaftal and Weiss ([13], Proposition 3.1) so we will outline it for our case. Out of all ways of partitioning $\{1, 2, \ldots, n\}$ into $r$-sets, choose one, say $\{A_j\}_{j=1}^r$, which minimizes

$$\sum_{j=1}^r \sum_{i \in A_j} \sum_{i \neq \ell \in A_j} |\langle f_i, f_\ell \rangle|^2.$$ 

We now observe that for each $1 \leq j \leq r$, each $i \in A_j$ and all $1 \leq k \neq j \leq r$ we have:

$$\sum_{i \neq \ell \in A_j} |\langle f_i, f_\ell \rangle|^2 \leq \sum_{\ell \in A_k} |\langle f_i, f_\ell \rangle|^2.$$ 

To see this, assume this inequality fails. That is, for some $j_0, i_0, k_0$ as above we have

$$\sum_{i_0 \neq \ell \in A_{j_0}} |\langle f_{i_0}, f_\ell \rangle|^2 > \sum_{\ell \in A_{k_0}} |\langle f_{i_0}, f_\ell \rangle|^2.$$ 

We define a new partition $\{B_j\}_{j=1}^r$ of $\{1, 2, \ldots, n\}$ by: $B_j = A_j$ if $j \neq j_0, k_0$; $B_{j_0} = A_{j_0} - \{i_0\}$; $B_{k_0} = A_{k_0} \cup \{i_0\}$. It easily follows that

$$\sum_{j=1}^r \sum_{i \in B_j} \sum_{i \neq \ell \in B_j} |\langle f_i, f_\ell \rangle|^2 < \sum_{j=1}^r \sum_{i \in A_j} \sum_{i \neq \ell \in A_j} |\langle f_i, f_\ell \rangle|^2,$$

which contradicts the minimality of Equation 3.1.
Step 2: For every $j = 1, 2, \cdots, r$ and every $i \in A_j$ we have
\[
\sum_{i \neq \ell \in A_j} |\langle f_i, f_\ell \rangle|^2 \leq \frac{B^2}{r}.
\]
Define an operator $S f = \sum_{i=1}^n \langle f, f_i \rangle f_i$. Then,
\[
\langle S f, f \rangle = \sum_{i=1}^r |\langle f, f_i \rangle|^2,
\]
and since $\{f_i\}_{i=1}^n$ is a Riesz basis with bounds $A, B$ we have
\[
A^2 I \leq S \leq B^2 I.
\]
Now, by Step 1,
\[
\sum_{i \neq \ell \in A_j} |\langle f_i, f_\ell \rangle|^2 \leq \frac{1}{r} \left[ \sum_{i \neq \ell \in A_j} |\langle f_i, f_\ell \rangle|^2 + \sum_{j \neq k=1}^r \sum_{\ell \in A_k} |\langle f_i, f_\ell \rangle|^2 \right]
\leq \frac{1}{r} \sum_{i=1}^n |\langle f_i, f_\ell \rangle|^2
\leq \frac{1}{r} \|S\| \|f_i\|^2 \leq \frac{B^2}{r}.
\]

Step 3: For each $j = 1, 2, \cdots, r$ and all $i \in A_j$, if $P_{ij}$ is the orthogonal projection of span $\{f_\ell \}_{\ell \in A_j}$ onto span $\{f_\ell \}_{i \neq \ell \in A_j}$ then
\[
\|P_{ij} f_i\|^2 \leq \frac{B^4}{A^4 r}.
\]
Define the operator $S_{ij}$ on span $\{f_\ell \}_{\ell \in A_j}$ by
\[
S_{ij}(f) = \sum_{i \neq \ell \in A_j} \langle f, f_\ell \rangle f_\ell.
\]
Then $A^2 I \leq S_{ij} \leq B^2 I$ and $\{S_{ij}^{-1} f_\ell \}_{i \neq \ell \in A_j}$ are the dual functionals for the Riesz basic sequence $\{f_\ell \}_{i \neq \ell \in A_j}$. Also, as in Step 1, $A^2 I \leq S_{ij} \leq B^2 I$. So by
Step 2,
\[
\|P_{ij}f_i\|^2 = \| \sum_{i \neq \ell \in A_j} \langle f_i, f_\ell \rangle S_{ij}^{-1} f_\ell \|^2 \\
\leq \| S_{ij}^{-1} \|^2 \| \sum_{i \neq \ell \in A_j} \langle f_i, f_\ell \rangle f_\ell \|^2 \\
\leq \frac{B^2}{A^4} \sum_{i \neq \ell \in A_j} | \langle f_i, f_\ell \rangle |^2 \leq \frac{B^4}{A^4r}.
\]

Step 4: Fix 1 \leq j \leq r and let \( A_j = \{i_1, i_2, \ldots, i_k\} \). If we Gram-Schmidt \( \{f_\ell\}_{\ell=1}^k \) to produce an orthonormal basis \( \{e_\ell\}_{\ell=1}^k \) then for all 1 \leq m \leq k we have
\[
| \langle f_m, e_m \rangle |^2 \geq 1 - \frac{\epsilon}{2}.
\]

Fix 1 \leq m \leq k and let \( Q_m \) be the orthogonal projection of span \( \{e_\ell\}_{\ell=1}^m \) onto span \( \{f_\ell\}_{\ell=1}^m = \text{span} \ \{f_\ell\}_{\ell=1}^m \). By Step 3,
\[
\|Q_m f_m\|^2 \leq \|P_{mj} f_m\|^2 \leq \frac{B^4}{A^4r}.
\]
Since
\[
f_m = \sum_{\ell=1}^m \langle f_\ell, e_\ell \rangle e_\ell,
\]
we have
\[
| \langle f_m, e_m \rangle |^2 = \|f_m\|^2 - \|Q_{m-1} f_m\|^2 \\
\geq 1 - \frac{B^4}{A^4r} \geq 1 - \frac{\epsilon}{2},
\]
where the last inequality follows from our choice of \( r \).

Step 5: We complete the proof.

Let
\[
M = (\langle f_\ell, e_\ell \rangle)_{\ell \neq t=1}^k,
\]
where by this notation we mean the \( k \times k \)-matrix with zero diagonal and the given values off the diagonal. By the Gram-Schmidt Process, \( M \) is a lower triangular matrix with zero diagonal. Define an operator \( T : \ell_2^k \rightarrow \text{span} \ \{e_\ell\}_{\ell=1}^k \) by
\[
T ((a_\ell)_{\ell=1}^k) = \sum_{\ell=1}^k a_\ell f_\ell.
\]
If $K$ is the matrix of $T$ with respect to the orthonormal basis \( \{e_i\}_{i=1}^k \) and $D = D(K)$ is the diagonal of $K$ then $M = (K - D)^*$ and so
\[
\|M\| \leq \|K\| + \|D\| = \|T\| + 1 \leq B + 1.
\]
By The Paving Conjecture for lower triangular matrices, there is a natural number $L_j$ (which is a function of $0 < \epsilon$ and $B$ only) and a partition $\{B^j\}_{\ell=1}^{L_j}$ of $\{i_1, i_2, \ldots, i_k\}$ so that
\[
\|Q_{B^j}\| \leq \frac{B}{2}.
\]
for all $\ell = 1, 2, \ldots, L_j$ ($Q_{B^j}$ was defined in the introduction). Now, for all scalars $(a_i)_{i \in B^j}$, if
\[
f = \sum_{i \in B^j} a_i f_i,
\]
then
\[
\left\| \sum_{i \in B^j} a_i f_i \right\| = \|D(f) + Q_{B^j} M Q_{B^j}(f)\|
\geq \|D(f)\| - \|Q_{B^j} M Q_{B^j}(f)\|
\geq (1 - \frac{\epsilon}{2})\|f\| - \frac{\epsilon}{2}\|f\|
\geq (1 - \epsilon)\|f\|.
\]
Similarly,
\[
\left\| \sum_{i \in B^j} a_i f_i \right\| \leq (1 + \epsilon)\|f\|.
\]
It follows that $\{f_i\}_{i \in B^j}$ is an $\epsilon$-Riesz basic sequence for all $j = 1, 2, \ldots, r$ and all $\ell = 1, 2, \ldots, L_j$. Hence, the Finite $R_\epsilon$-Conjecture holds which completes the proof of the theorem.

Let us make an observation concerning the proof of the main theorem.

**Definition 3.2.** Let $\{f_i\}_{i=1}^\infty$ be a sequence of vectors in a Hilbert space $\mathbb{H}$. For each $i = 1, 2, \ldots$ let $P_i$ be the orthogonal projection of $\mathbb{H}$ onto span $\{f_\ell\}_{i \neq \ell \in \mathbb{N}}$. Our sequence is said to be $\epsilon$-minimal if $\|P_i\| \leq \epsilon$ for all $i = 1, 2, \ldots$.

The first three steps of the proof of Theorem 3.1 yields:

**Corollary 3.3.** If $\{f_i\}_{i=1}^\infty$ is a unit norm Riesz basic sequence in a Hilbert space $\mathbb{H}$ then for every $\epsilon > 0$ there is a partition $\{A_j\}_{j=1}^r$ of $\mathbb{N}$ so that for all $j = 1, 2, \ldots, r$, the family $\{f_i\}_{i \in A_j}$ is $\epsilon$-minimal.
References


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