PROBLEMS ON FRAMES AND THE KADISON-SINGER PROBLEM

Abstract. Please send problems, which should be posted in this section, to Pete Casazza at pete@math.missouri.edu. We would like to refer to the introduction to this section and [2] for an introduction to frames and the Kadison-Singer Problem as well as the notation used in this section.

1. Frame Theory Problems Which Are Equivalent to the Kadison-Singer Problem

Problem 1.1: [Feichtinger Conjecture] [3]. Can every unit norm frame be written as a finite union of Riesz basic sequences?

Problem 1.2: [Weak Feichtinger Conjecture] [3]. Can every unit norm Bessel sequence be written as a finite union of Riesz basic sequences?

Problem 1.3: [6] Can every unit norm Bessel sequence be written as a finite union of frame sequences?

Problem 1.4: [4] Does there exist an \( \epsilon > 0 \) and a natural number \( r \) so that for all equal norm Parseval frames \( \{f_i\}_{i=1}^{2n} \) for \( \ell_2^N \) there is a partition \( \{A_j\}_{j=1}^r \) of \( \{1, 2, \cdots , 2n\} \) so that \( \{f_i\}_{i\in A_j} \) has Bessel bound \( \leq 1 - \epsilon \) for all \( j = 1, 2, \cdots , r \)?

Problem 1.5: [Finite Feichtinger Conjecture] [3] For every \( B, C > 0 \) is there a natural number \( M = M(B, C) \) and an \( A = A(B, C) > 0 \) so that whenever \( \{f_i\}_{i\in I} \) is a frame for \( \ell_2^N \) (\( N \in \mathbb{N} \)) with upper frame bound \( B \) and \( \|f_i\| \geq C \) for all \( i \in I \), then \( I \) can be partitioned into \( \{A_j\}_{j=1}^M \) so that for each \( 1 \leq j \leq M \), \( \{f_i\}_{i\in A_j} \) is a Riesz basic sequence with lower Riesz basis bound \( A \) (and upper Riesz basis bound \( B \))? 

Problem 1.6: [7] Are there universal constants \( B \) and \( \epsilon > 0 \) and a natural number \( r \) so that the following holds? Let \( \{f_i\}_{i=1}^M \) be elements of \( \ell_2^N \) with \( \|f_i\| \leq 1 \) for \( i = 1, 2, \cdots , M \) and suppose \( \{f_i\}_{i=1}^M \) is a B-Bessel sequence (or a B-tight frame). There is a partition \( \{A_j\}_{j=1}^r \) of \( \{1, 2, \cdots , n\} \) so that for all \( j = 1, 2, \cdots , r \) the family \( \{f_i\}_{i\in A_j} \) has Bessel bound \( B - \epsilon \).

Problem 1.7: [7] Are there universal constants \( B \geq 4 \) and \( \epsilon > \sqrt{B} \) and an \( r \in \mathbb{N} \) so that the following holds? Whenever \( \{f_i\}_{i=1}^M \) is a unit norm B-tight
frame for $\ell_p^2$, there exists a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \ldots, M\}$ so that for all $j = 1, 2, \ldots, r$ the family $\{f_i\}_{i \in A_j}$ has Bessel bound $B - \epsilon$.

**Problem 1.8:** [5] For every unit norm $B$-Bessel sequence $\{f_i\}_{i=1}^M$ in $\mathbb{H}_N$ and every $\epsilon > 0$, does there exist $r = r(B, \epsilon)$ and a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \ldots, M\}$ so that for every $j = 1, 2, \ldots, r$ and all scalars $\{a_i\}_{i \in A_j}$ we have

$$\sum_{n \in A_j} \sum_{n \neq m \in A_j} |\langle f_n, a_m f_m \rangle|^2 \leq \epsilon \sum_{m \in A_j} \|a_m f_m\|^2.$$ 

**R$_\epsilon$-Conjecture** [Casazza, Vershynin] For every $\epsilon > 0$, every unit norm Riesz basic sequence is a finite union of $\epsilon$-Riesz basic sequences.

2. **Frame Theory Problems Which May Be Easier Than the Kadison-Singer Problem**

**Problem 2.1:** Can every unit norm Bessel sequence be written as a finite union of $\omega$-independent sets?
[Casazza, Kutyniok, Nikolskii, Speegle, Tremain]

**Problem 2.2:** Can every unit norm Bessel sequence be written as a finite union of minimal systems with constant $\delta$?
[Casazza, Tremain]

**Problem 2.3:** Can every unit norm Bessel sequence which is a minimal system with constant $\delta$ be written as a finite union of Riesz basic sequences?
[A positive solution to KS is equivalent to having positive solutions to both Problems 2.2 and 2.3.]
[Casazza, Tremain]

3. **Frame Theory Problems Related to the Kadison-Singer Problem**

**Problem 3.1:** Can every frame of translates be written as a finite union of Riesz basic sequences?

**Problem 3.2:** Can every regular (or irregular) Bessel Gabor system be written as a finite union of Riesz basic sequences?
[Feichtinger]

**Problem 3.3:** Assume $\{x_i\}_{i \in I}$ is a sequence in $\mathbb{H}$ such that the operator

$$\sum_{i \in I} x_i \otimes x_i$$
is bounded below on the closed linear span of \( \{x_i\}_{i \in I} \). Find a "nice" criterion for which \( \{x_i\}_{i \in I} \) is a Riesz basis for its closed linear span.

[Larson]

**Problem 3.4:** [1] For every unit norm frame \( \{f_i : i \in \mathbb{N}\} \) indexed by the natural numbers, does there exist a set \( K \subset \mathbb{N} \) such that \( \{f_i : i \in K\} \) is a Riesz basic sequence and

\[
\lim_{n \to \infty} \frac{\#(K \cap [1, n])}{n} > 0?
\]

[It is not known if a positive solution to KS would yield a positive solution to this problem. It is easy to see that there is a partition of the natural numbers into two sets, neither of which satisfy (3.1).]

**References**


