LLT POLYNOMIALS, RIBBON TABLEAUX AND THE QUANTUM AFFINE ALGEBRA

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This document is a brief introduction about the link between the LLT polynomials and the Fock space representation $\mathcal{F}_q$ of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}_n})$.

1. Definitions of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}_n})$ and the Fock space $\mathcal{F}_q$

Let first recall that the affine cartan matrix of type $A^{(1)}_{n-1}$ is the following $n \times n$ matrix $A = (a_{i,j})_{0\leq i,j\leq n-1}$

$$\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \text{ if } n = 2 \qquad \begin{bmatrix} 2 & -1 & 0 & \dots & 0 & 0 & -1 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ -1 & 0 & 0 & \dots & 0 & -1 & 2 \end{bmatrix} \text{ if } n \geq 3.$$

The quantum affine Lie algebra $U_q(\widehat{\mathfrak{sl}_n})$ is the associative algebra generated by generators $E_i, F_i, K_i, K_i^{-1}, D$ submitted to the relations

$$K_iK_i^{-1} = K_i^{-1}K_i = 1 \;\; ; \;\; K_iK_j = K_jK_i \;\; ; \;\; K_iE_jK_i^{-1} = q^{a_{i,j}}E_j \;\; ;$$

$$K_iE_jK_i^{-1} = q^{-a_{i,j}}F_j \;\; ; \;\; DD^{-1} = D^{-1}D = 1 \;\; ; \;\; DK_i = K_iD \;\; ;$$

$$DE_0D^{-1} = q^{-1}E_0 \;\; ; \;\; DF_0D^{-1} = qF_0 \;\; ; \;\; DE_iD^{-1} = 0 \;\; ; \;\; DF_iD^{-1} = 0 \;\; ;$$

$$E_iF_j - F_jE_i = \delta_{i,j}\frac{K_i - K_i^{-1}}{q - q^{-1}} \;\; ;$$

$$\sum_{k=0}^{1-a_{i,j}} (-1)^k \begin{bmatrix} 1-a_{i,j} \\ k \end{bmatrix} E_i^{1-a_{i,j}-k}E_jE_i^k = 0 \;\; (i \neq j) \;\; ;$$

$$\sum_{k=0}^{1-a_{i,j}} (-1)^k \begin{bmatrix} 1-a_{i,j} \\ k \end{bmatrix} F_i^{1-a_{i,j}-k}F_jF_i^k = 0 \;\; (i \neq j)$$

where $[k] = \frac{q^k - q^{-k}}{q - q^{-1}}$, $[k]! = [k][k-1]\dots[1]$ and $\begin{bmatrix} m \\ k \end{bmatrix} = \frac{[m]!}{[m-k]![k]!}$.

Let now define the Fock space $\mathcal{F}_q$ as an infinite-dimentional vector space over $\mathbb{C}(q)$ with a distinguished basis $(s_\lambda)_{\lambda\in\mathcal{P}}$ where $\mathcal{P}$ denotes the set of all the partitions.

2. An action of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}_n})$ on the Fock space $\mathcal{F}_q$

For λ a given partition, we write now each content modulo n.

Definition 1. *A position (a, b) is said to be an* ***indent i-node*** *if we can add a box of content i in this position such that the new shape is a partition.*

A position (a, b) is said to be an ***removable i-node*** *if we can remove a box of content i in this position such that the new shape is a partition.*

Example 1. *For $n = 2$ and for the following partition diagram filled with the contents modulo 2*

$$\begin{array}{cccc} 0 & 1 & 0 & \\ \boxed{1} & \boxed{0} & 1 & 0 \\ \boxed{0} & \boxed{1} & \boxed{0} & 1 \end{array}$$

The only indent 0-node is in position (3,1).
The two removable 0-nodes are in position (1,3) and (2,2).
The two indent 1-nodes are in position (1,4) and (2,3).
There is no removable 1-node.

Let i be an integer, λ and μ two partitions such that the skew shape μ/λ is an indent i-node δ (i.e μ is obtained by adding an indent i-node to the partition λ). Let us define some combinatorial quantities

- $I_i^r(\lambda, \mu) = \#\{\text{indent } i\text{-node at the right of } \delta\}$
- $R_i^r(\lambda, \mu) = \#\{\text{removable } i\text{-node at the right of } \delta\}$
- $I_i^l(\lambda, \mu) = \#\{\text{indent } i\text{-node at the left of } \delta\}$
- $R_i^r(\lambda, \mu) = \#\{\text{removable } i\text{-node at the left of } \delta\}$
- $N_i^r(\lambda, \mu) = I_i^r - R_i^r$
- $N_i^l(\lambda, \mu) = I_i^l - R_i^l$

Let us now define an action of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}_n})$ on the Fock space $\mathcal{F}_q$. The operator F_i acts on the basis by

$$F_i \cdot s_\lambda = \sum_\mu q^{N_i^r(\lambda,\mu)} s_\mu$$

where the sum is over all the partitions μ such that μ/λ is a removable i-node, and D acts by

$$D \cdot s_\lambda = q^{N_0(\lambda)} s_\lambda$$

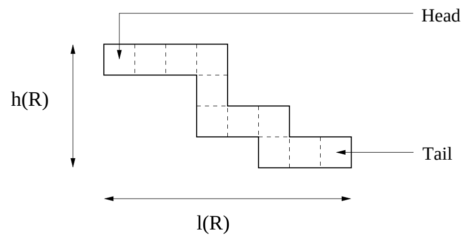
where $N_0(\lambda)$ is the number of indent 0-nodes minus the number of removable 0-nodes.

For this action, $s_\emptyset$ is a highest weight vector. So, the vector space $U_q(\widehat{\mathfrak{sl}_n}) \cdot s_\emptyset$ is an irreductible $U_q(\widehat{\mathfrak{sl}_n})$-module .

3. A family of operators on the Fock space $\mathcal{F}_q$

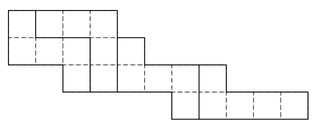
In order to obtain others irreducible $U_q(\widehat{\mathfrak{sl}_n})$-modules, we now introduce a family of operators $(V_k)_{k\in\mathbb{N}}$.

Definition 2. *Let n be an integer. A n-ribbon is a skew partition of size n which does not contains a 2×2 square. The **head** of a ribbon is the upper left box and the **tail** is the lower right box. The **spin** of a ribbon is the number of rows minus 1 divided by 2.*



Definition 3. *A horizontal n-ribbon strip of size k is a tiling of a skew partition by k n-ribbons such that the tail of each ribbon is not upon an other ribbon. The spin of a horizontal ribbon strip is the sum of the spins of all its ribbons.*

Example 2. *A horizontal 5-ribbon strip of weight 4 and spin 7.*



Let us now define the action of the operators V_k on the Fock space $\mathcal{F}_q$ by

$$V_k \cdot s_\lambda = \sum_\mu (-q)^{-\mathrm{spin}(\mu/\lambda)} s_\mu$$

where the sum is over all partitions μ such that μ/λ is an horizontal n-ribbon strip of size k. These operators commute with the action of the quantum affine algebra on the Fock space, so

$$v_\lambda := V_{\lambda_1} \dots V_{\lambda_r} \cdot s_\emptyset$$

is a highest weight vector. Consequently, we obtain the following decompositions into irreducible components

$$\mathcal{F}_q = \oplus_{\lambda \in \mathcal{P}} U_q(\widehat{\mathfrak{sl}}_n) \cdot v_\lambda \quad .$$

Definition 4. *A n-ribbon tableau of skew shape λ/μ is a tiling of the skew shape λ/μ by labelled n-ribbons such that the head of a ribbon labelled i must not be on the right of a ribbon labelled $j > i$ and its tail must not be on the top of a ribbon labelled $j \geq i$. The weight of a n-ribbon tableau is the vector ν such that ν_i is the number of n-ribbons labelled i.*

The set of all n-ribbon tableaux with shape λ and weight μ is denoted by $\mathrm{Tab}^{(n)}(\lambda, \mu)$. Let us define the spin polynomial $G^{(n)}_{\lambda,\mu}(q)$ and the cospin polynomial $\tilde{G}^{(n)}_{\lambda,\mu}(q)$, as the generating polynomials of the set $\mathrm{Tab}^{(n)}(\lambda, \mu)$ with the spin or the cospin statistics, i.e,

$$G^{(n)}_{\lambda,\mu}(q) = \sum_{T \in \mathrm{Tab}^{(n)}(\lambda,\mu)} q^{\mathrm{spin}(T)} \quad \text{and} \quad \tilde{G}^{(k)}_{\lambda,\mu}(q) = \sum_{T \in \mathrm{Tab}^{(n)}(\lambda,\mu)} q^{\mathrm{cospin}(T)} \ .$$

Property 1. *The LLT polynomials* $G_\lambda^{(n)}(X;q)$ *are symmetric functions, with the following expansion on the monomials basis*

$$G_\lambda^{(n)}(X;q) = \sum_\mu G_{\lambda,\mu}^{(n)}(q)\ m_\mu(X)\ .$$

Now the expansion of the highest weight vector v_λ on the basis of Schur functions is

$$v_\lambda = \sum_\mu G_{\lambda,\mu}^{(n)}(-q^{-1}) s_\mu$$

where μ run on all partitions of weight $|\lambda| = \lambda_1 + \ldots + \lambda_n$.

References

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