NOTES ON KOSTKA POLYNOMIALS AND FUSION PRODUCTS

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This point of view on Kostka polynomials is to think of them as a refinement of the multiplicity of irreducible $\mathfrak{g}$-modules in the tensor product of several $\mathfrak{g}$-modules, or more generally, the multiplicities of irreducible integrable affine algebra modules in the fusion products of several integrable representations.

Let $\mathfrak{g}$ be a simple Lie algebra $\{V_i\}_{i=1}^N$ be a set of finite-dimensional $\mathfrak{g}$-modules. Consider the integers $K_{\lambda,\{V_i\}}$ which describe the decomposition of the tensor product of $\mathfrak{g}$-modules into irreducible modules V_λ:

$$V_1 \otimes \cdots \otimes V_N \simeq \bigoplus_\lambda V(\lambda)^{\oplus K_{\lambda,\{V_i\}}}.$$

These integers can be computed from the Littlewood-Richardson rules.

For example, if $\mathfrak{g} = \mathfrak{sl}_n$, and if $V_i = V(\mu_i\omega_1)$, ($\mu_i \in \mathbb{N}$, with $\mu_i \geq \mu_{i+1}$) are the irreducible, symmetric power representations of $\mathfrak{sl}_n$, then $K_{\lambda,\{V_i\}} = K_{\lambda,\mu}$, where $\mu = (\mu_1, ..., \mu_N)$, are the Kostka numbers.

Kostka polynomials and their generalizations can be viewed as a refinement of these multiplicities, which is obtained by defining a $\mathfrak{g}$-equivariant grading on the tensor product. The graded components are still $\mathfrak{g}$-modules, and the decomposition into irreducible components can be written as

$$V_1 \star \cdots \star V_N \simeq \bigoplus_\lambda \bigoplus_{n\geq 0} V(\lambda)^{\oplus \widetilde{K}_{\lambda,\{V_i\}}[n]},$$

where $\star$ refers to the graded tensor product.

define the Poincare polynomial $\widetilde{K}_{\lambda,\{V_i\}}(q) = \sum_n \widetilde{K}_{\lambda,\{V_i\}}[n] q^n$. This can be viewed as a generalization of the Kostka polynomial in special cases.

1. Definition of the graded tensor products

The definition of the graded tensor product is motivated by the definition of the fusion product of integrable modules in conformal field theory [BPZ]. It was defined by Feigin and Loktev [FL]. See [FJKLM] for further details of the definition.

Let $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$ be the subalgebra of positive currents in the affine algebra associated with $\mathfrak{g}$. Let $a_i \in \mathbb{C}$ be complex numbers. The $\mathfrak{g}$-module V_i can be given the structure of a $\mathfrak{g}[t]$-module, called the evaluation module. The generator $x \otimes f(t) \in \mathfrak{g}[t]$, with $x \in \mathfrak{g}$ and $f(t) \in \mathbb{C}[t]$, acts on $v \in V_i$ as

$$x \otimes f(t)\ v = f(a_i) x\ v, \qquad v \in V_i.$$

Thus, the action on V_i depends on the complex numbers a_i, and thus the $\mathfrak{g}[t]$-module structure is denoted by $V_i^{a_i}$.

On the tensor product $\otimes V_i^{a_i}$, $\mathfrak{g}[t]$ acts by the co-product,

$$x \otimes f(t)\ v_1 \otimes \cdots \otimes v_N = \sum_{i=1}^N f(a_i) v_1 \otimes \cdots \otimes x\ v_i \otimes \cdots \otimes v_n,$$

where $v_i \in V_i$.

Now assume that V_i are cyclic modules with cyclic vectors v_i, and that a_i are pairwise distinct. Then it is a simple lemma that

$$U(\mathfrak{g}[t])v_1 \otimes \cdots \otimes v_N \simeq V_1 \otimes \cdots \otimes V_N. \tag{1}$$

Note that this is not true if a_i are not pairwise distinct.

The algebra $U(\mathfrak{g}[t])$ is filtered by homogenous degree in t. Let $U^{\leq n}$ denote the filtered component, consisting of polynomials of degree less than or equal to n. The module (1) inherits this filtration, where the cyclic vector is taken to have degree 0.

$$\mathcal{F}^{\leq n} = U^{\leq n} v_1 \otimes \cdots \otimes v_N.$$

The definition of the graded tensor product is as the associated graded space of this filtered space.

1.1. **Examples.** Let $\mathfrak{g} = \mathfrak{sl}_n$ and let V_i be the symmetric power representations as above, with the cyclic vectors to be the highest weight vectors of V_i. Then the resulting Poincare polynomial of the graded tensor product is $\widetilde{K}_{\lambda,\mu}(q)$, the co-charge Kostka polynomial. This is proved in [Ke].

Let $\mathfrak{g}$ be any simple Lie algebra and let V_i be a graded Kirillov-Reshetikhin module, parametrized by the highest weight $m_i\omega_{j_i}$ with ω_{j_i} fundamental weights. (The definition of graded KR modules can be deduced from [Ch]). Then the Poincare polynomial is the generalized Kostka polynomial of [SW, KS, SS]. This is proved in [AK].

2. Some open problems and generalizations

In their paper [FL] defining the graded tensor and fusion multiplicities, the authors made several conjectures about the properties of the graded multiplicities. The most basic one is that the mutliplicities are independent of the parameters a_i. This can be proved by direct computation, but is does not follow from any geometric arguments.

The authors also conjectured that there should exist a relation between the graded tensor multiplicities and the generalized (restricted) Kostka polynomials. This has now been established, by direct computation, for the cases where the comparison is valid (see [Ke, AKS, AK]) for the non-level-restricted case. (It also seems to follow from the form of the energy operator on the tensor product of crystals (Okado, private communication).) In the level-restricted case, the comparison is currently a work in progress, but is known to hold in special cases ($\mathfrak{sl}_n$, λ a rectangular representation).

In general, the graded tensor product multiplicities are defined for the tensor product of any irreducible representations, not just the Kirillov-Reshetikhin ones. By definition, they manifestly have positive coefficients. It is now established that there is no fermionic formula in these more general cases. (Fermionic formulas have the major advantage that they are manifestly postive.) Nevertheless, the (restricted) graded tensor product defined above does provide a new combinatorial method of computing these coefficients. It is worthwhile to investigate whether this is an effective new combinatorial algorithm for computing (graded) Littlewood-Richardson and Verlinde coefficients.

It is possible to define the graded tensor product for double-loop algebras, making a connection with Haiman's diagonal coinvariants in the case where all the representations are the vector representation of $\mathfrak{gl}_n$ for sufficiently large n [FL2].

However, The appropriate definition of the grading for the tensor product in general is unknown. Question: What is the graded tensor product in the case of the double loop algebra of symmetric power representations of $\mathfrak{sl}_n$? Is there a graded tensor product related to Haiman's ring R_μ?

References

[AK] Ardonne, Kedem, Fusion products of graded Kirillov-Reshetikhin modules (in progress).

[AKS] E. Ardonne, R. Kedem, M. Stone, Fermionic characters and arbitrary highest-weight integrable affine sl_{r+1} modules (to appear in *Comm. Math. Phys.*). math.RT/0504364.

[BPZ] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov. Infinite conformal symmetry in two-dimensional quantum field theory. *Nuclear Phys. B*, 241(2):333–380, 1984.

[Ch] V. Chari, On the fermionic formula and the Kirillov-Reshetikhin conjecture. Internat. Math. Res. Notices 2001 12, 629–654.

[FJKLM] B. Feigin, M. Jimbo, R. Kedem, S. Loktev, and T. Miwa. Spaces of coinvariants and fusion product. affine sl_2 character formulas in terms of kostka polynomials. *Duke Math. J.*, 125(3):549–588, 2004.

[FL] B. Feigin and S. Loktev. On generalized Kostka polynomials and the quantum Verlinde rule. In *Differential topology, infinite-dimensional Lie algebras, and applications*, volume 194 of *Amer. Math. Soc. Transl. Ser. 2*, pages 61–79. Amer. Math. Soc., Providence, RI, 1999.

[FL2] B. Feigin and S. Loktev. Deformation of Weyl modules and generalized parking functions. Int. Math. Res. Not. 2004 2719–2750.

[HKOTY] Hatayama, G., Kuniba, A., Okado, M., Takagi, T. and Yamada, Y.: Remarks on fermionic formula. Contemp. Math. 248, 243–291 (1999)

[KL] D. Kazhdan and G. Lusztig, Tensor structures arising from affine Lie algebras. I, II. J. Amer. Math. Soc. 6 (1993), 905–947, 949–1011.

[Ke] R. Kedem, Fusion products, cohomology of $GL(N)$ flag manifolds, and Kostka polynomials, IMRN 2004:25 (2004) 1273-1298.

[KS] Anatol N. Kirillov and Mark Shimozono. A generalization of the Kostka-Foulkes polynomials. *J. Algebraic Combin.*, 15(1):27–69, 2002.

[SS] A.Schilling and M. Shimozono, Fermionic formulas for level-restricted generalized Kostka polynomials and coset branching functions. Comm. Math. Phys. 220 (2001), 105–164.

[SW] Anne Schilling and S. Ole Warnaar. Inhomogeneous lattice paths, generalized Kostka polynomials and A_{n-1} supernomials. *Comm. Math. Phys.*, 202(2):359–401, 1999.

[FS] A. V. Stoyanovskiĭ and B. L. Feĭgin. Functional models of the representations of current algebras, and semi-infinite Schubert cells. *Funktsional. Anal. i Prilozhen.*, 28(1):68–90, 96, 1994.