Algebraic and Combinatorial Macdonald Polynomials

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Generalized Kostka Polynomials
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Reference: "A Combinatorial Formula for Macdonald Polynomials" by Haglund, Haiman, and Loehr, *JAMS* **18** (2005), 735—761.

Outline

1. Background on Symmetric Polynomials

2. Algebraic definition of (modified) Macdonald polynomials

3. New combinatorial definition of Macdonald polynomials

4. Proof that the two definitions agree

5. q, t-Kostka polynomials

Symmetric Polynomials

Let n be a positive integer. Suppose:

- P is a polynomial in $x_1, \ldots, x_n$.
- P is homogeneous of degree n.
- Permuting the subscripts of the x_i's always leaves P unchanged.

Then P is *a symmetric polynomial of order n.*

Example: $(n = 3)$

$$P = 5x_1^3 + 5x_2^3 + 5x_3^3 - (1/2)x_1x_2x_3.$$

Fact: $V_n = \{\text{symm. polys. of order } n\}$
is a vector space.

Partitions

A *partition of* n is a list of positive integers $\mu = (\mu_1, \ldots, \mu_n)$ with

$$\mu_1 \geq \cdots \geq \mu_n \geq 0 \text{ and } \mu_1 + \cdots + \mu_n = n.$$

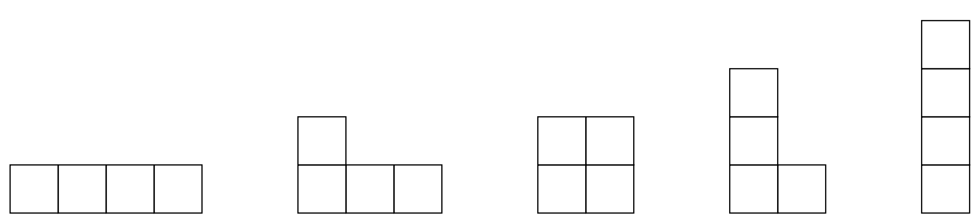
Notation: $\mu \vdash n$ means μ is a partition of n.

Example: There are 5 partitions of $n = 4$:

$$(4,0,0,0),\ (3,1,0,0),\ (2,2,0,0),$$

$$(2,1,1,0),\ (1,1,1,1).$$

Diagram of μ: Put μ_i boxes in row i.



Bases for Symmetric Polys.

Fact: $\dim(V_n) =$ number of partitions of n.
We use partitions to index bases of V_n.

The six classical bases of V_n:

- monomial basis $\{m_\mu : \mu \vdash n\}$
- elementary basis $\{e_\mu : \mu \vdash n\}$
- homogeneous basis $\{h_\mu : \mu \vdash n\}$
- power-sum basis $\{p_\mu : \mu \vdash n\}$
- Schur basis $\{s_\mu : \mu \vdash n\}$
- forgotten basis $\{f_\mu : \mu \vdash n\}$

Modern Bases for V_n

- Zonal symmetric polys. $\{Z_\mu : \mu \vdash n\}$

- Jack's symmetric polys.

- Hall-Littlewood basis

- original Macdonald basis $\{P_\mu : \mu \vdash n\}$

- dual Macdonald basis $\{Q_\mu : \mu \vdash n\}$

- integral Macdonald basis $\{J_\mu : \mu \vdash n\}$

- transformed Macdonald basis $\{H_\mu : \mu \vdash n\}$

- modified Macdonald basis $\{\tilde{H}_\mu : \mu \vdash n\}$

The Monomial Basis

For $\mu \vdash n$, the *monomial symmetric poly.* m_μ is the sum of all distinct monomials obtained by permuting the subscripts of the monomial

$$x_1^{\mu_1} x_2^{\mu_2} \cdots x_n^{\mu_n}.$$

Fact: $\{m_\mu : \mu \vdash n\}$ are linearly independent polynomials that span V_n.
They form the *monomial basis* for V_n.

Example. For $n = 3$,

$$\begin{aligned} m_{(3,0,0)} &= x_1^3 + x_2^3 + x_3^3 \\ m_{(2,1,0)} &= x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_1 \\ &\quad + x_2^2 x_3 + x_3^2 x_1 + x_3^2 x_2 \\ m_{(1,1,1)} &= x_1 x_2 x_3 \end{aligned}$$

The Power-Sum Basis

For $k > 0$, the k'th *power-sum* is

$$p_k(x_1, \ldots, x_n) = x_1^k + x_2^k + \cdots + x_n^k.$$

Define $p_0 = 1$. For $\mu \vdash n$, define the *power-sum symmetric poly.* p_μ to be

$$p_\mu = \prod_{i=1}^{n} p_{\mu_i}(x_1, \ldots, x_n).$$

Fact: $\{p_\mu : \mu \vdash n\}$ is a basis for V_n.

Example: For $n = 3$,

$$\begin{aligned}
p_{(3,0,0)} &= x_1^3 + x_2^3 + x_3^3 = m_{(3,0,0)} \\
p_{(2,1,0)} &= (x_1^2 + x_2^2 + x_3^2)(x_1 + x_2 + x_3) \\
&= m_{(3,0,0)} + m_{(2,1,0)} \\
p_{(1,1,1)} &= (x_1 + x_2 + x_3)^3 \\
&= m_{(3,0,0)} + 3m_{(2,1,0)} + 6m_{(1,1,1)}
\end{aligned}$$

The Parameters q and t

Macdonald polynomials involve variables $x_1, \ldots, x_n$ and two extra parameters q and t.

Formally, let $F = \mathbb{Q}(q,t)$ be the field whose elements are formal quotients of polynomials in two variables q and t.

Examples: 4, $3t-1$, $\frac{qt+5}{q^2-(3/7)t}$ lie in F.

From now on, view V_n as a vector space over this field.

Example:
$(3t-1)x_1^2 + (3t-1)x_2^2 + \frac{qt+5}{q^2-(3/7)t}x_1x_2 \in V_2$.

Two Special Linear Maps

We can define linear maps on the vector space V_n by specifying their effect on any basis. Define:

$$A_t(p_\mu) = \left(\prod_{i:\mu_i>0} [t^{\mu_i} - 1] \right) p_\mu.$$

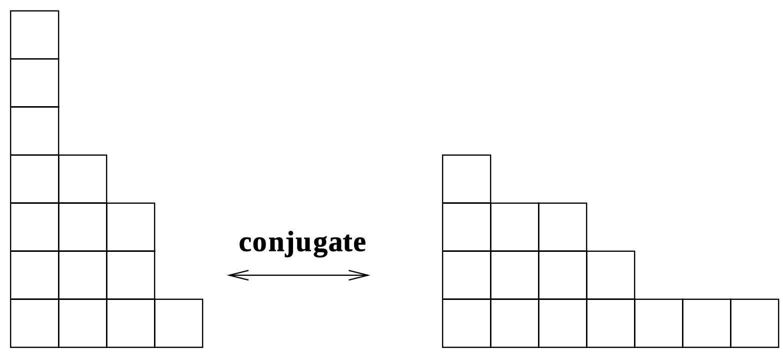
$$A_q(p_\mu) = \left(\prod_{i:\mu_i>0} [q^{\mu_i} - 1] \right) p_\mu,$$

Note: Terms in parentheses are elements of F (scalars)!

Extending by linearity, we get two linear maps A_q and A_t mapping V_n into itself.

Conjugation; Domination

Conjugate of μ: μ' is the partition whose parts are the columns in the diagram of μ.



Dominance partial ordering: For $\lambda, \mu \vdash n$,
$\lambda \preceq \mu$ iff $\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$ for all i.

Example: $(1,1,1) \preceq (2,1,0) \preceq (3,0,0)$. But
$(3,1,1,1,0,0) \not\preceq (2,2,2,0,0,0)$,
$(2,2,2,0,0,0) \not\preceq (3,1,1,1,0,0)$.

Algebraic Definition of Modified Macdonald Polys.

Def./Thm. There exists a unique basis $\{\tilde{H}_\mu : \mu \vdash n\}$ of V_n satisfying these axioms:

(1) The coefficient of x_1^n in $\tilde{H}_\mu$ is 1.

(2) Let $A_t(\tilde{H}_\mu) = \sum_{\lambda \vdash n} c_{\lambda,\mu} m_\lambda$ ($c_{\lambda,\mu} \in F$).
Then $c_{\lambda,\mu} = 0$ *except* when $\lambda \preceq \mu$.

(3) Let $A_q(\tilde{H}_\mu) = \sum_{\lambda \vdash n} d_{\lambda,\mu} m_\lambda$ ($d_{\lambda,\mu} \in F$).
Then $d_{\lambda,\mu} = 0$ *except* when $\lambda \preceq \mu'$.

The $\tilde{H}_\mu$'s are the *modified Macdonald polynomials*.

Comments/Complaints

The algebraic definition for $\tilde{H}_\mu$ just given:

- requires a hard **proof** to justify (uniqueness easy, but existence unclear!)
- is completely **non-explicit**
- seems totally **unmotivated**
- gives us **no** intuition about $\tilde{H}_\mu$

Yet, this definition was the only one available for the last 16 years! (1988 — 2004)

The New Definition

We're about to give Haglund's conjectured combinatorial definition for $\tilde{H}_\mu$, which:

- proves the existence claim in the earlier definition by giving a construction for $\tilde{H}_\mu$
- is an explicit sum of weighted combinatorial objects
- shows that $\tilde{H}_\mu$ is in $\mathbb{N}[q,t][x_1,\ldots,x_n]$, not just in $\mathbb{Q}(q,t)[x_1,\ldots,x_n]$.
- has intuitive appeal due to its concreteness and simplicity
- exhibits the combinatorial significance of the cryptic algebraic axioms defining $\tilde{H}_\mu$
- leads to elegant proofs of results on Jack's polys., Hall-Littlewood polys., etc.

Combinatorial Definition

Haglund's combinatorial formula:

$$\tilde{H}_\mu = \sum_{\text{objects } T} q^{\text{qwt}(T)} t^{\text{twt}(T)} \vec{x}^{\,\text{xwt}(T)}$$

where the objects and weights depend on μ.

The objects: all fillings of the boxes of μ with integers from 1 to n, repeats allowed.

The $\vec{x}$-weight of T:

$$\text{xwt}(T) = x_1^{\#\text{ of 1's in } T} \cdots x_n^{\#\text{ of } n\text{'s in } T}.$$

Example: $n = 10$, $\mu = (3, 3, 3, 1, 0, \ldots, 0)$.

T =

4		
2	**5**	**2**
2	**4**	**9**
3	**1**	**2**

$$\text{xwt}(T) = x_1^1 x_2^4 x_3^1 x_4^2 x_5^1 x_9^1.$$

Major Index and Inversions

Let $w = w_1, w_2, \ldots, w_s$ be a list of integers.

The *major index* maj(w) is the sum of all $i < s$ such that $w_i > w_{i+1}$.

The *inversions* of w, inv(w), is the number of pairs $i < j$ with $w_i > w_j$.

Examples:

$$\begin{aligned} \text{maj}(4,2,2,3) &= 1, & \text{inv}(4,2,2,3) &= 3 \\ \text{maj}(5,4,1) &= 3, & \text{inv}(5,4,1) &= 3 \\ \text{maj}(2,9,2) &= 2, & \text{inv}(2,9,2) &= 1 \end{aligned}$$

The t-weight

Given an object T, let $w^{(j)}$ be the list of integers in column j of μ, from top to bottom. Define

$$\text{twt}(T) = \text{maj}_\mu(T) = \sum_{j=1}^{\mu_1} \text{maj}(w^{(j)}).$$

Example:

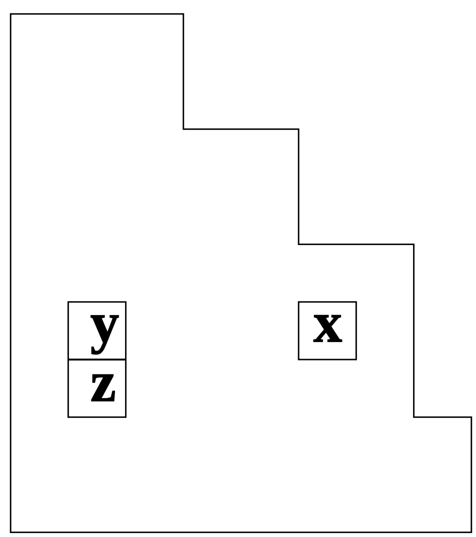
T =

4		
2	**5**	**2**
2	**4**	**9**
3	**1**	**2**

$$\text{maj}_\mu(T) = 1 + 3 + 2 = 6.$$

Inversion Triples

Consider a configuration of cells in T like this:



These three cells form an *inversion triple* of T

iff $x < y \leq z$ or $y \leq z < x$ or $z < x < y$.

The q-weight

Given an object T, let $w^{(0)}$ be the list of integers in the lowest row of μ, from left to right. Suppose T has K inversion triples. Define

$$\mathrm{qwt}(T) = \mathrm{inv}_\mu(T) = \mathrm{inv}(w^{(0)}) + K.$$

Example:

T =

4		
2	**5**	**2**
2	**4**	**9**
3	**1**	**2**

$$\mathrm{inv}_\mu(T) = 2 + 3 = 5.$$

Full weight of T: $q^5 t^6 x_1^1 x_2^4 x_3^1 x_4^2 x_5^1 x_9^1$.

Example: $\tilde{H}_{(2,1,0)}$

$\begin{array}{\|c\|c\|}\hline 1 & \\ \hline 1 & 1 \\ \hline\end{array}$ $q^0t^0x_1x_1x_1$	$\begin{array}{\|c\|c\|}\hline 2 & \\ \hline 2 & 2 \\ \hline\end{array}$ $q^0t^0x_2x_2x_2$	$\begin{array}{\|c\|c\|}\hline 3 & \\ \hline 3 & 3 \\ \hline\end{array}$ $q^0t^0x_3x_3x_3$	$\begin{array}{\|c\|c\|}\hline 1 & \\ \hline 1 & 2 \\ \hline\end{array}$ $q^0t^0x_1x_1x_2$	$\begin{array}{\|c\|c\|}\hline 1 & \\ \hline 2 & 1 \\ \hline\end{array}$ $q^1t^0x_1x_1x_2$
$\begin{array}{\|c\|c\|}\hline 2 & \\ \hline 1 & 1 \\ \hline\end{array}$ $q^0t^1x_1x_1x_2$	$\begin{array}{\|c\|c\|}\hline 1 & \\ \hline 1 & 3 \\ \hline\end{array}$ $q^0t^0x_1x_1x_3$	$\begin{array}{\|c\|c\|}\hline 1 & \\ \hline 3 & 1 \\ \hline\end{array}$ $q^1t^0x_1x_1x_3$	$\begin{array}{\|c\|c\|}\hline 3 & \\ \hline 1 & 1 \\ \hline\end{array}$ $q^0t^1x_1x_1x_3$	$\begin{array}{\|c\|c\|}\hline 2 & \\ \hline 2 & 3 \\ \hline\end{array}$ $q^0t^0x_2x_2x_3$
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$$\tilde{H}_{(2,1,0)} = 1m_{(3,0,0)} + (1+q+t)m_{(2,1,0)}$$

$$+(1+2q+2t+qt)m_{(1,1,1)}.$$

Steps in the Proof

Let C_μ denote Haglund's formula for $\tilde{H}_\mu$.

1. Show the coefficient of x_1^n in C_μ is 1.

2. Prove C_μ is *symmetric* (i.e., $C_\mu \in V_n$).

3. Interpret $A_t(C_\mu)$ and $A_q(C_\mu)$ as sums of *signed*, weighted objects.

4. $A_t(C_\mu) = \sum_{\lambda \vdash n} a_{\lambda,\mu} m_\lambda, \ A_q(C_\mu) = \sum_{\lambda \vdash n} b_{\lambda,\mu} m_\lambda.$

   Use *cancellation of objects* to show

   $$a_{\lambda,\mu} \neq 0 \Rightarrow \lambda \preceq \mu, \quad b_{\lambda,\mu} \neq 0 \Rightarrow \lambda \preceq \mu'.$$

5. C_μ satisfies all axioms, so $C_\mu = \tilde{H}_\mu$.

Interpreting $A_t(C_\mu)$

One can prove that $A_t(C_\mu)$ is a sum of *signed*, weighted objects:

$$A_t(C_\mu) = \sum_{\text{objects } T} \mathrm{sgn}(T) q^{\mathrm{qwt}(T)} t^{\mathrm{twt}(T)} \vec{x}^{\,\mathrm{xwt}(T)}$$

The objects: fillings of μ using the alphabet

$$\{1, 2, \ldots, n, \overline{1}, \overline{2}, \ldots, \overline{n}\}$$

consisting of *positive* and *negative* letters.

The $\vec{x}$-weight: $\prod_{i=1}^{n} x_i^{\#\text{ of } i\text{'s and } \bar{i}\text{'s in } T}$.

Example: $n = 10$, $\mu = (3, 3, 3, 1, 0, \ldots, 0)$.

$$\mathbf{T} = \begin{array}{|c|c|c|} \hline \mathbf{4} & & \\ \hline \mathbf{2} & \mathbf{5} & \overline{\mathbf{2}} \\ \hline \overline{\mathbf{2}} & \mathbf{4} & \overline{\mathbf{9}} \\ \hline \overline{\mathbf{3}} & \mathbf{1} & \overline{\mathbf{2}} \\ \hline \end{array}$$

$$\mathrm{xwt}(T) = x_1^1 x_2^4 x_3^1 x_4^2 x_5^1 x_9^1.$$

Signs and Weights

Consider an object T with P positive letters and N negative letters.

q-weight: $\mathrm{qwt}(T) = \mathrm{inv}_\mu(T)$.

t-weight: $\mathrm{twt}(T) = \mathrm{maj}_\mu(T) + P$.

Sign: $\mathrm{sgn}(T) = (-1)^N$.

Example:

$$\mathbf{T} = \begin{array}{|c|c|c|} \hline \mathbf{4} & \multicolumn{2}{c}{} \\ \hline \mathbf{2} & \mathbf{5} & \bar{\mathbf{2}} \\ \hline \bar{\mathbf{2}} & \mathbf{4} & \bar{\mathbf{9}} \\ \hline \bar{\mathbf{3}} & \mathbf{1} & \bar{\mathbf{2}} \\ \hline \end{array}$$

Full weight of T is $(-1)^5 q^2 t^{8+5} x_1^1 x_2^4 x_3^1 x_4^2 x_5^1 x_9^1$.

Cancelling Pairs of Objects

Idea: Cancel pairs of terms in

$$A_t(C_\mu) = \sum_{\text{objects } T} \mathsf{sgn}(T) q^{\mathsf{qwt}(T)} t^{\mathsf{twt}(T)} \vec{x}^{\,\mathsf{xwt}(T)}$$

with equal weights and opposite signs.

Example:

T =

$\mathbf{4}$		
$\mathbf{2}$	$\mathbf{5}$	$\overline{\mathbf{2}}$
$\overline{\mathbf{2}}$	$\mathbf{4}$	$\overline{\mathbf{9}}$
$\overline{\mathbf{3}}$	$\mathbf{1}$	$\overline{\mathbf{2}}$

T contributes the term $-q^2 t^{8+5} x_1^1 x_2^4 x_3^1 x_4^2 x_5^1 x_9^1$.

U =

$\mathbf{4}$		
$\overline{\mathbf{2}}$	$\mathbf{5}$	$\overline{\mathbf{2}}$
$\overline{\mathbf{2}}$	$\mathbf{4}$	$\overline{\mathbf{9}}$
$\overline{\mathbf{3}}$	$\mathbf{1}$	$\overline{\mathbf{2}}$

U contributes the term $+q^2 t^{9+4} x_1^1 x_2^4 x_3^1 x_4^2 x_5^1 x_9^1$.

Terms for T and U **cancel** in $A_t(C_\mu)$!!!

Finding Matched Pairs

To cancel an object T:

- Choose i minimal† such that i or $\bar{i}$ appears above the lowest i rows in T.

- Find the topmost and then leftmost occurrence of i or $\bar{i}$ in T.

- Flip the sign of this symbol to get U.

- Check: sign reverses, but weights are preserved!

†If no such i exists, then T contributes an *uncancelled term* to $A_t(C_\mu)$.

Proving Axiom 2 for C_μ

$A_t(C_\mu) = \sum_{\lambda \vdash n} a_{\lambda,\mu} m_\lambda$. **Show:** $a_{\lambda,\mu} \neq 0 \Rightarrow \lambda \preceq \mu$.

1. $a_{\lambda,\mu} \neq 0 \Rightarrow$ the coefficient of $x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ in

   $$A_t(C_\mu) = \sum_{\text{objects } T} \mathrm{sgn}(T) q^{\mathrm{qwt}(T)} t^{\mathrm{twt}(T)} \vec{x}^{\,\mathrm{xwt}(T)}$$

   is nonzero.

2. So, there must be an *uncancelled* object T with $\mathrm{xwt}(T) = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$.

3. For each i, the number of i's and $\bar{i}$'s in T is exactly λ_i.

4. For each i, letters in $\{1, \ldots, i, \bar{1}, \ldots, \bar{i}\}$ occur in the lowest i rows of μ.

5. So $\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i \quad \forall i$, which says that $\lambda \preceq \mu$!

Interpreting $A_q(C_\mu)$

As with $A_t(C_\mu)$, we can prove that

$$A_q(C_\mu) = \sum_{\text{objects } T} \mathrm{sgn}(T) q^{\mathrm{qwt}(T)} t^{\mathrm{twt}(T)} \vec{x}^{\,\mathrm{xwt}(T)}$$

Objects: fillings of μ with entries from $\{1, \ldots, n, \overline{1}, \overline{2}, \ldots, \overline{n}\}$.

$\vec{x}$**-weight:** $\prod_{i=1}^{n} x_i^{\#\text{ of } i\text{'s and } \bar{i}\text{'s in } T}$.

Sign: $(-1)^{\#\text{ of negative letters in } T}$.

q**-weight:** $\mathrm{inv}_\mu(T)+$
(# of positive letters in T).

t**-weight:** $\mathrm{maj}_\mu(T)$.

Cancelling Pairs of Objects

Idea: Cancel pairs of terms in $A_q(C_\mu)$ with equal weights and opposite signs.

Example:

$$\mathbf{T} = \begin{array}{|c|c|c|} \hline \mathbf{4} & \multicolumn{2}{c}{} \\ \hline \mathbf{2} & \mathbf{5} & \overline{\mathbf{2}} \\ \hline \overline{\mathbf{2}} & \mathbf{4} & \overline{\mathbf{9}} \\ \hline \overline{\mathbf{3}} & \mathbf{1} & \overline{\mathbf{2}} \\ \hline \end{array}$$

Term for T is $(-1)^5 q^{6+5} t^6 x_1^1 x_2^4 x_3^1 x_4^2 x_5^1 x_9^1$.

$$\mathbf{V} = \begin{array}{|c|c|c|} \hline \mathbf{4} & \multicolumn{2}{c}{} \\ \hline \mathbf{2} & \mathbf{5} & \mathbf{2} \\ \hline \overline{\mathbf{2}} & \mathbf{4} & \overline{\mathbf{9}} \\ \hline \overline{\mathbf{3}} & \mathbf{1} & \overline{\mathbf{2}} \\ \hline \end{array}$$

Term for V is $(-1)^4 q^{5+6} t^6 x_1^1 x_2^4 x_3^1 x_4^2 x_5^1 x_9^1$.

Terms for T and V **cancel** in $A_q(C_\mu)$!!!
As before, not every object T can cancel.

Proving Axiom 3 for C_μ

$A_q(C_\mu) = \sum_{\lambda \vdash n} b_{\lambda,\mu} m_\lambda$. **Show:** $b_{\lambda,\mu} \neq 0 \Rightarrow \lambda \preceq \mu'$.

1. $b_{\lambda,\mu} \neq 0 \Rightarrow$ there is an uncancelled object T with $\text{xwt}(T) = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$.

2. For each i, the number of i's and $\bar{i}$'s in T is exactly λ_i.

3. The new cancellation mechanism implies that, for all i, T never has two letters in $\{i, \bar{i}\}$ in the same row.

4. The last condition easily implies

$$\lambda_1 + \cdots + \lambda_i \leq \mu'_1 + \cdots + \mu'_i \quad \forall i,$$

so that $\lambda \preceq \mu'$!

Applications of New Formula

- **best** proof of existence, integrality of $\tilde{H}_\mu$
- explicit combinatorial formulas for all five Macdonald bases (P_μ, Q_μ, J_μ, H_μ, $\tilde{H}_\mu$)
- explanation and proof of the Lascoux-Schützenberger *cocharge* statistic for Hall-Littlewood polynomials
- simple proof of Sahi and Knop's formula for Jack polynomials
- expansion of $\tilde{H}_\mu$ using LLT polynomials
- insight into Kostka-Macdonald coefficients $\tilde{K}_{\lambda,\mu}$

Monomial Expansion

Expand Macdonald polynomials in terms of monomial symmetric functions:

$$\tilde{H}_\mu = \sum_\lambda a_{\lambda,\mu} m_\lambda \qquad (a_{\lambda,\mu} \in \mathbb{Q}(q,t))$$

Lemma: The combinatorial formula for $\tilde{H}_\mu$ is symmetric in the x_i's.

Corollary: If $(c_1, \ldots, c_N)$ is any sequence that rearranges to the partition λ, then

$$a_{\lambda,\mu} = \sum_{\substack{T:\mu\to[N] \\ |T^{-1}(\{i\})|=c_i}} q^{\mathsf{inv}(T)} t^{\mathsf{maj}(T)}.$$

Schur Expansion

Now expand Macdonald polynomials in terms of Schur functions:

$$\tilde{H}_\mu = \sum_\lambda \tilde{K}_{\lambda,\mu} s_\lambda$$

The scalars $\tilde{K}_{\lambda,\mu} \in \mathbb{Q}(q,t)$ are the (modified) *q,t-Kostka polynomials*.

Theorem: [positivity and polynomiality]

$$\tilde{K}_{\lambda,\mu} \in \mathbb{N}[q,t].$$

Open Problem: Find a combinatorial formula for the q,t-Kostka polynomials.