QUICK DEFINITIONS OF MACDONALD POLYNOMIALS

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This note presents the minimal skeleton of definitions needed to introduce Macdonald polynomials and q,t-Kostka polynomials. We give two equivalent definitions of (modified) Macdonald polynomials, one algebraic and one combinatorial.

1. Partitions

Fix a positive integer n. A list of integers $\mu = (\mu_1, \ldots, \mu_n)$ is a *partition of* n, written $\mu \vdash n$, iff $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq 0$ and $\mu_1 + \cdots + \mu_n = n$. We let $\ell(\mu)$ be the largest index i such that $\mu_i > 0$. The *diagram* of μ, denoted $D(\mu)$, consists of $\ell(\mu)$ rows of boxes in the first quadrant of the xy-plane, left-justified, with μ_i boxes in the i'th row from the bottom. The *transpose of* μ, denoted μ', is the partition whose diagram is obtained by reflecting the diagram of μ about the line $y = x$. If λ and μ are partitions of n, we write $\lambda \geq \mu$ and say λ *dominates* μ iff $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for all $i \leq n$.

2. Abstract Symmetric Functions

Let F be the field $\mathbb{Q}(q,t)$, whose elements are formal quotients of polynomials in two indeterminates q and t with rational coefficients. We now give an "abstract" definition of the ring Λ of symmetric functions with coefficients in F. We simply define Λ to be the polynomial ring $F[p_1, p_2, \ldots, p_k, \ldots]$ in countably many indeterminates p_k. The p_k's are algebraically independent by definition. We make Λ into a graded ring by setting $\deg(p_k) = k$ (in contrast to typical polynomial rings, where each indeterminate has degree 1). Let Λ^n denote the F-subspace of Λ consisting of homogeneous elements of degree n (including zero). The set $\{p_\mu : \mu \vdash n\}$ is a basis for the vector space Λ^n, where $p_\mu = \prod_{i=1}^{\ell(\mu)} p_{\mu_i}$ is an *abstract power-sum symmetric function.*

3. Plethysm

The universal mapping property for polynomial rings states that, for any F-algebra A and any function $h : \{p_1, p_2, \ldots, p_k, \ldots\} \to A$, there exists a unique F-algebra homomorphism $h' : \Lambda \to A$ extending h. For historical reasons, homomorphisms h' obtained in this way are often encrypted using *plethystic notation.* We will only need the following two special cases of this notation. First, if $A = \Lambda$ and h is the function such that $h(p_k) = (q^k - 1)p_k$, then we write $f[X(q-1)]$ instead of $h'(f)$. Second, if $A = \Lambda$ and h is the function such that $h(p_k) = (t^k - 1)p_k$, then we write $f[X(t-1)]$ instead of $h'(f)$. To compute $f[X(q-1)]$

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for $f \in \Lambda$, one must expand f as a linear combination of p_μ's and then use multiplicativity, linearity, and the definition of h on the p_k's.

4. Concrete Symmetric Polynomials

It is often easier to visualize elements of Λ^n as being elements of "concrete" polynomial rings $R_N = F[z_1, \ldots, z_N]$, where $N \geq n$. More precisely, the evaluation homomorphism $\Lambda \to R_N$ defined by $p_k \mapsto p_k(z_1, \ldots, z_N) = z_1^k + \cdots + z_N^k$ restricts to an isomorphism of the vector space Λ^n onto a subspace of R_N. More specifically, the image of Λ^n in R_N consists of all polynomials in $z_1, \ldots, z_N$ that are homogeneous of degree n and invariant under all permutations of the z_i's. Using these concrete instantiations of Λ^n, we can define more F-bases of Λ^n. For example, define the *monomial symmetric polynomial* $m_\mu(z_1, \ldots, z_N) \in R_N$ to be the sum of all distinct monomials obtained by rearranging the exponents of the monomial $z_1^{\mu_1} \cdots z_N^{\mu_N}$. One easily sees that the preimage of $m_\mu(z_1, \ldots, z_N)$ in Λ^n (denoted by m_μ) is independent of N, and that $\{m_\mu : \mu \vdash n\}$ is an F-basis of Λ^n. Similarly, we can give a concrete definition of the *Schur function* $s_\mu(z_1, \ldots, z_N)$ as the sum of the weights of all semistandard tableaux of shape μ with entries in $\{1, 2, \ldots, N\}$. The preimage s_μ of $s_\mu(z_1, \ldots, z_N)$ in Λ^n is independent of N, and one can show that $\{s_\mu : \mu \vdash n\}$ is another F-basis of Λ^n. The *Hall inner product* $\langle \cdot, \cdot \rangle$ on Λ^n is defined by requiring that $\{s_\mu\}$ be an orthonormal basis.

5. Algebraic Definition of Modified Macdonald Polynomials

We can now state the theorem used to define the *modified Macdonald polynomials* $\tilde{H}_\mu$. The theorem asserts that there exists a unique F-basis $\{\tilde{H}_\mu : \mu \vdash n\}$ for Λ^n satisfying the following three conditions:

(A1) $\tilde{H}_\mu[X(q-1)] = \sum_{\lambda \leq \mu'} c_{\lambda,\mu} m_\lambda$ for some $c_{\lambda,\mu} \in F$
(A2) $\tilde{H}_\mu[X(t-1)] = \sum_{\lambda \leq \mu} d_{\lambda,\mu} m_\lambda$ for some $d_{\lambda,\mu} \in F$
(A3) The coefficient of z_1^n in $\tilde{H}_\mu(z_1, \ldots, z_N)$ is 1.

We remark that these conditions are equivalent to the following conditions often found in the literature:

(B1) $\tilde{H}_\mu[X(1-q)] = \sum_{\lambda \geq \mu} a_{\lambda,\mu} m_\lambda$ for some $a_{\lambda,\mu} \in F$
(B2) $\tilde{H}_\mu[X(1-t)] = \sum_{\lambda \geq \mu'} b_{\lambda,\mu} m_\lambda$ for some $b_{\lambda,\mu} \in F$
(B3) $\langle \tilde{H}_\mu, s_{(n)} \rangle = 1$.

We also remark that the uniqueness assertion in the theorem is a (relatively) routine linear algebra exercise using triangularity of suitable transition matrices, but the existence assertion of the theorem is highly non-obvious. The new combinatorial construction sketched below provides the easiest proof of the existence part of the theorem.

6. Combinatorial Definition of Macdonald Polynomials

Haglund's conjectured combinatorial interpretation for $\tilde{H}_\mu(z_1, \ldots, z_N)$ is a sum of monomials, each arising from an object weighted by powers of q, t, and the z_i's. (As usual, as $N \geq n$ varies, these interpretations all give the same preimage in the abstract space Λ^n.) A typical object is a filling of the cells in the diagram of μ using integers from $\{1, 2, \ldots, N\}$, with repeats allowed. Let $\mathcal{S}(\mu)$ be the set of all such fillings.

Suppose T is a filling in $\mathcal{S}(\mu)$. Define the *content function* c_T by letting $c_T(i)$ be the number of occurrences of the integer i in T. Let w_i be the sequence of integers in the i'th column of T, read from top to bottom. If $w_i = x_1 x_2 \cdots x_k$, let $\text{maj}(w_i)$ be the sum of all $j < k$ such that $x_j > x_{j+1}$. Define the *μ-major index* of T by

$$\text{maj}_\mu(T) = \sum_i \text{maj}(w_i).$$

Next we define the *μ-inversions* of T. First, suppose we have two cells in the lowest row of T, not necessarily adjacent, whose fillings (from left to right) are y and x. This pair of cells is a *μ-inversion pair* of T iff $y > x$. Second, suppose we have a triple of cells in T positioned as follows:

$$\begin{matrix} y & \cdots & x \\ z & & \end{matrix} \tag{1}$$

Thus, y appears somewhere to the left of x in the same row of T (not the lowest row), and z appears in the cell immediately below y. This triple of cells is a *μ-inversion triple* of T iff $x < y \leq z$ or $y \leq z < x$ or $z < x < y$. The *μ-inversion count* of T, denoted $\text{inv}_\mu(T)$, is the number of μ-inversion pairs and μ-inversion triples of T.

Haglund defined the "combinatorial Macdonald polynomial"

$$C_\mu(z_1, \ldots, z_N) = \sum_{T \in \mathcal{S}(\mu)} q^{\text{inv}_\mu(T)} t^{\text{maj}_\mu(T)} z_1^{c_T(1)} \cdots z_N^{c_T(N)}. \tag{2}$$

and conjectured that $C_\mu(z_1, \ldots, z_N) = \tilde{H}_\mu(z_1, \ldots, z_N)$.

7. Proof of Equivalence of Definitions

Haglund, Haiman, and I found a combinatorial proof of Haglund's conjecture. The main steps in the proof are as follows. First, we show that $C_\mu(z_1, \ldots, z_N)$ is a symmetric function of $z_1, \ldots, z_N$; i.e., $C_\mu \in \Lambda^n$ for all $\mu \vdash n$. Second, we derive combinatorial expressions for the coefficients of m_λ in $C_\mu[X(q-1)]$ and $C_\mu[X(t-1)]$. These expressions are similar to the description of C_μ just given, but now we sum over objects containing both positive and negative integers, with slightly different weights. Third, we define sign-reversing involutions on these new collections of objects, which cancel out everything except when $\lambda \leq \mu'$ (for (A1)) or when $\lambda \leq \mu$ (for (A2)). Fourth, it is trivial to check condition (A3) for $C_\mu(z_1, \ldots, z_N)$. Finally, since the elements $\tilde{H}_\mu$ are the *unique* elements of Λ^n_F satisfying (A1), (A2), and (A3), the desired result $C_\mu = \tilde{H}_\mu$ follows.

8. q,t-Kostka Polynomials

Suppose we expand $\tilde{H}_\mu$ in terms of the monomial basis of Λ^n:

$$\tilde{H}_\mu = \sum_{\lambda \vdash n} a_{\lambda,\mu} m_\lambda.$$

Fix any sequence $(s_1, s_2, \ldots)$ that rearranges to λ. The combinatorial formula for Macdonald polynomials shows that $a_{\lambda,\mu} = \sum q^{\mathrm{inv}(T)} t^{\mathrm{maj}(T)}$ where we sum over those fillings T of $D(\mu)$ that contain exactly s_i copies of i for all i.

To define the q,t-Kostka polynomials, we expand $\tilde{H}_\mu$ in terms of the Schur basis of Λ^n:

$$\tilde{H}_\mu = \sum_{\lambda \vdash n} \tilde{K}_{\lambda,\mu} s_\lambda.$$

The coefficients $\tilde{K}_{\lambda,\mu}$ appearing in this expansion are, by definition, the *modified q,t-Kostka numbers*. We have $\tilde{K}_{\lambda,\mu} \in F = \mathbb{Q}(q,t)$ by definition. By using the combinatorial expansion and the transition matrix from the monomial to the Schur basis, it is clear that $\tilde{K}_{\lambda,\mu} \in \mathbb{Z}[q,t]$. In fact, it can be shown that each $\tilde{K}_{\lambda,\mu}$ actually lies in $\mathbb{N}[q,t]$. This fact is called the *positivity and polynomiality* of the q,t-Kostka numbers. It is an open problem to give a nice combinatorial interpretation for the q,t-Kostka numbers (i.e., a description not involving any negative objects).