SCHUR POSITIVITY PROBLEMS

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1. Introduction

An important and ubiquitous problem in algebraic combinatorics is to show that certain symmetric functions are *Schur-positive.* If f is any symmetric function with coefficients in the field $\mathbb{Q}(q,t)$ (say), we can uniquely write f as a linear combination of Schur functions:

$$f = \sum_{\mu} a_\mu s_\mu \qquad (a_\mu \in \mathbb{Q}(q,t)).$$

We say that f is Schur-positive iff every coefficient a_μ is actually a polynomial in q and t with nonnegative integer coefficients; i.e., $a_\mu \in \mathbb{N}[q,t]$. This definition can be modified in the obvious way when the coefficients come from $\mathbb{Q}(q)$ or $\mathbb{Q}$.

In combinatorial settings, we often define an element $f \in \mathbb{Q}(q,t)[x_1,\ldots,x_N]$ by taking the weighted sum of a suitable collection of objects. For example, the Schur functions themselves can be defined by the formula $s_\mu = \sum_T x^T$ where T ranges over semistandard tableaux of weight μ and the monomial x^T records the content of T. For another example, the modified Macdonald polynomials can be defined by setting $\tilde{H}_\mu = \sum_T q^{\mathrm{inv}(T)} t^{\mathrm{maj}(T)} x^T$, where T ranges over all fillings of the diagram of μ and inv and maj and suitable statistics. There exist similar combinatorial definitions (sometimes conjectural) for LLT polynomials, generalized Kostka polynomials, $\nabla(e_n)$, etc.

Whenever we give such a combinatorial definition, it is obvious that the resulting polynomial f is an element of $\mathbb{N}[q,t][x_1,\ldots,x_N]$ (or $\mathbb{Z}[q,t][x_1,\ldots,x_N]$ if we allow signed objects). But it is usually not clear *a priori* that the polynomial in question is actually *symmetric* in the x_i's. If we can prove that f is symmetric, we automatically obtain a combinatorial interpretation for the expansion of f in terms of the *monomial* symmetric functions m_λ. However, we are then faced with the difficult problem of characterizing the coefficients in the Schur expansion of f.

2. Specific Open Problems

We would like to prove that the following entities are Schur-positive symmetric functions. Ideally, the proofs would also yield explicit combinatorial and/or representation-theoretical interpretations for the coefficients of the Schur expansion.

(1) **arbitrary LLT polynomials**, which can be defined either as the generating functions for ribbon tableaux weighted by (co)spin, or as the generating function for

k-tuples of semistandard skew tableaux weighted by d-inversions (see Lam's lecture notes).

(2) **modified Macdonald polynomials** $\tilde{H}_\mu$, which can be defined either combinatorially using Haglund's statistics on fillings of the diagram of μ, or by various indirect algebraic characterizations (see Loehr's lecture notes). Mark Haiman has proved Schur-positivity by identifying $\tilde{H}_\mu$ as the Frobenius character of a suitable doubly-graded S_n-module, but we would still like combinatorial interpretations for the Schur coefficients and/or a more elementary proof of Schur-positivity.

(3) **generalized Kostka polynomials**, which can be defined (in the case of rectangles that concatenate to a partition) as the Schur coefficients in the generating function for k-tuples of tableaux weighted by coenergy (see Shimozono's lecture notes). In the general case where the indexing set is a tuple of partitions that concatenate to give another partition, generalized Kostka polynomials can also be defined using suitable creation operators (see Zabrocki's lecture notes).

(4) $\nabla(e_n)$, which can be defined (conjecturally) as the generating function for certain labelled Dyck paths relative to area and d-inversions (see the paper by Haglund, Haiman, Loehr, Remmel, and Ulyanov). This polynomial is known to be Schur-positive by an appeal to Kazhdan-Lusztig theory, but a combinatorial interpretation of the Schur coefficients is still lacking.

All of the objects mentioned are known to be *symmetric* polynomials, but the proofs of symmetry are usually non-trivial. It would also be nice to have more direct combinatorial proofs of this symmetry.

3. Relations between Problems

Virtually all of the problems just mentioned can be reduced to the problem of describing the Schur coefficients in various LLT polynomials.

(1) Each modified Macdonald polynomial $\tilde{H}_\mu$ can be written as a weighted sum of LLT polynomials indexed by k-tuples of ribbons (where $k = \mu_1$). Therefore, Schur-positivity of Macdonald polynomials is a special case of Schur-positivity of LLT polynomials.

(2) The conjectured combinatorial formula for $\nabla(e_n)$ can also be written as a weighted sum of LLT polynomials indexed by shifted column shapes. Therefore, Schur-positivity of $\nabla(e_n)$ is a special case of Schur-positivity of LLT polynomials.

(3) It is conjectured that the generating function for generalized Kostka polynomials indexed by rectangles are equal to the corresponding LLT polynomials indexed by the same rectangles (see the $K = LLT$ conjecture in the list of open problems.) The truth of this conjecture would reduce Schur-positivity of generalized Kostka polynomials to Schur-positivity of LLT polynomials.

4. Known Special Cases

Here are some special cases in which Schur expansions of LLT polynomials are known.

(1) An LLT polynomial indexed by a single partition shape is obviously the Schur function indexed by that shape.
(2) An LLT polynomial indexed by a single skew shape is a skew Schur function, hence the desired expansion is provided by the Littlewood-Richardson rule.
(3) An LLT polynomial indexed by two skew shapes corresponds to a domino tableaux; this case was solved in the papers of Carré, Leclerc, and van Leeuwen (see also Section 9 of "A combinatorial formula for Macdonald polynomials").
(4) An LLT polynomial indexed by n copies of the partition (1) is simply $\sum_{w\in\mathbb{Z}_+^n} q^{\mathrm{inv}(w)}x^w$. Using a bijection of Foata, this equals $\sum_{w\in\mathbb{Z}_+^n} q^{\mathrm{maj}(W)}x^w$, which can be shown to be Schur-positive with the aid of the RSK algorithm.
(5) Consider an LLT polynomial enumerating n-ribbon tableaux on a shape μ with empty n-core. Leclerc and Thibon showed that the coefficient of s_λ in this LLT polynomial is essentially given by a parabolic Kazhdan-Lustzig polynomial, hence is positive.

5. Possible Approaches to Schur Positivity

(1) **Crystals.** Let f be some polynomial defined as a weighted sum of combinatorial objects in a set B. Suppose we can define a crystal structure on B such that the q, t-weights are constant on connected components of B. Suppose further that we can define a crystal homomorphism from B into $\mathbb{Z}_+^n$ with its standard crystal structure. We can then obtain automatically a formula for the Schur expansion of f in terms of suitably defined "Yamanouchi objects". See Corollary 9.8 in HHLRU; note that the proof also yields the fact that f is a symmetric function, which is usually not obvious at the outset.
(2) **Modules.** Another way to prove Schur-positivity of a symmetric polynomial f is to identify f as the Frobenius character of a suitable (doubly-graded) S_n-module. In several papers, Mark Haiman used this approach to prove Schur-positivity of $\tilde{H}_\mu$ (which are the Frobenius characters of the Garsia-Haiman modules M_μ) and of $\nabla(e_n)$ (which are the Frobenius characters of the modules of diagonal coinvariants). This method often requires substantial machinery from algebraic geometry and, unfortunately, typically does not provide an immediate combinatorial interpretation for the coefficients in question.