Galleries

Let $R = (V, \phi)$ be an irreducible reduced root system with positive roots ϕ^+ and Weyl group W. Denote by $Q^\vee$ the group generated by the coroots and by $Q^\vee_+ \subset Q^\vee$ the dominant cone. The affine Weyl group is the semidirect product $W^{\mathfrak{a}} = W \ltimes Q^\vee$. It acts on V^* by affine transformations. For $\lambda \in Q^\vee$ denote by $t_\lambda \in W^{\mathfrak{a}}$ the associated translation. Let $H^{\mathfrak{a}}$ be the union of all reflection hyperplanes of reflections in $W^{\mathfrak{a}}$. Then $H^{\mathfrak{a}} = \bigcup_{\alpha \in \phi^+, m \in \mathbb{Z}} H_{\alpha,m}$, where $H_{\alpha,m} = \{x \in V^* \mid \langle \alpha, x \rangle = m\}$. Let $H^{\pm}_{\alpha,m}$ be the associated affine half spaces.
The closures of the connected components of $V^* \setminus H^{\mathfrak{a}}$ are the alcoves in V^*. They are fundamental domains for the $W^{\mathfrak{a}}$-action on V^*. Denote by $\mathcal{A}$ the set of all alcoves. The fundamental alcove is $A_f = \{x \in V^* | 0 \leq \langle \alpha, x \rangle \leq 1 \, \forall \alpha \in \phi^+\} \in \mathcal{A}$. We have a bijection $W^{\mathfrak{a}} \to \mathcal{A}, w \mapsto A_w := wA_f$. Every alcove A is of the form $\mu + A_w$ for unique $\mu \in Q^\vee$ and $w \in W$. Denote this by $\mu(A) = \mu$ and $w(A) = w$.
A face F of an alcove A is an intersection $F = A \cap H$ such that $H \subset H^{\mathfrak{a}}$ is a reflection hyperplane and $\langle F \rangle_{\text{aff}} = H$. Here $\langle F \rangle_{\text{aff}}$ is the affine subspace spanned by F. A wall of A is some hyperplane $H \subset H^{\mathfrak{a}}$ such that $H \cap A$ is a face of A. The group $W^{\mathfrak{a}}$ is generated by the reflections $S^{\mathfrak{a}}$ at the walls of A_f. One has $S^{\mathfrak{a}} = S \cup \{s_0\}$, where $S = \{s_1, \ldots, s_l\}$ is the set of simple reflections of W and s_0 is the affine reflection at $H_{\theta,1}$. Here $\theta \in \phi$ is the highest root.
Let F be a face of A_f. The type of F is the reflection at $\langle F \rangle_{\text{aff}}$. Extend this definition to all faces by demanding that the $W^{\mathfrak{a}}$-action preserves types.
Right multiplication of $W^{\mathfrak{a}}$ induces an action of $W^{\mathfrak{a}}$ on $\mathcal{A}$ from the right. For $A \in \mathcal{A}$ and $s \in S^{\mathfrak{a}}$ the alcove As is the unique alcove not equal to A having a common face of type s with A. Let $F_s \subset A$ be the face of type s and $\langle F_s \rangle_{\text{aff}} = H_{\alpha,m}$ for some $\alpha \in \phi^+$ and $m \in \mathbb{Z}$. Call A negative with respect to s if A is contained in $H^-_{\alpha,m}$. Notation: $A \prec As$.

Definition. *Let $t = (t_1, \ldots, t_k)$ with $t_i \in S^{\mathfrak{a}}$.*

- *A gallery σ of type t connecting the alcoves A and B is a sequence $(A = A_0, \ldots, B = A_k)$ of alcoves such that $A_{i+1} \in \{A_i, A_i t_{i+1}\}$. The initial direction $\iota(\sigma)$ is defined to be $w(A_0)$ and the weight of σ is $\mu(A_k)$.*
- *The gallery σ is said to be positive at i if $A_{i+1} = A_i t_{i+1}$ and A_i is negative with respect to t_{i+1}, i.e. $A_i \prec A_{i+1}$.*
- *The gallery σ is folded at i if $A_{i+1} = A_i$. The folding hyperplane is the wall of A_i corresponding to the face of type t_{i+1}. The folding is positive if $A_i \succ A_i t_{i+1}$. We call σ positively folded, if all foldings occurring are positive.*
- *A gallery is said to be minimal if it is of minimal length among all galleries connecting the same alcoves.*
- $m'(\sigma) = m(\sigma) = \#\{j | \sigma \text{ is positive at } j\}$
- $n(\sigma) = \#\{j | \sigma \text{ is positively folded at } j\}$
- *$n'(\sigma)$ is the number of all foldings in σ such that the folding hyperplane is no wall of the dominant Weyl chamber $\mathcal{C}$, i.e. is not of the form $H_{\alpha_i,0}$.*
- *$p(\sigma)$ is the number of foldings in σ where the folding hyperplane is a wall of $\mathcal{C}$.*
- *If σ is positively folded, define $L_\sigma = q^{m(\sigma)}(q-1)^{n(\sigma)}$ and $C_\sigma = q^{m'(\sigma)+p(\sigma)}(q-1)^{n'(\sigma)}$.*

Relations to Hall-Littlewood polynomials (see [S])

Let $\mathcal{L}[Q^\vee]^W$ be the algebra of symmetric polynomials with coefficients in $\mathcal{L} = \mathbb{Z}[q, q^{-1}]$. We regard two bases:

- Monomial basis $\{m_\lambda\}_{\lambda \in Q_+^\vee}$, where $m_\lambda = \sum_{\mu \in W\lambda} x^\mu$.
- Hall-Littlewood basis $\{P_\lambda(q^{-1})\}_{\lambda \in Q_+^\vee}$, where

$$P_\lambda(q^{-1}) = \frac{1}{W_\lambda(q^{-1})} \sum_{w \in W} w\Big(x^\lambda \prod_{\alpha \in \phi^+} \frac{1 - q^{-1}x^{-\alpha^\vee}}{1 - x^{-\alpha^\vee}}\Big).$$

Here $W_\lambda \subset W$ is the stabilizer of λ and $W_\lambda(t) = \sum_{w \in W_\lambda} t^{l(w)}$.

Define Laurent polynomials $L_{\lambda\mu}$ for $\lambda, \mu \in Q_+^\vee$ as modified entries of the transition matrix from the m_μ to the $P_\lambda(q^{-1})$. More precisely, we have (with $\rho := \frac{1}{2}\sum_{\alpha \in \phi^+} \alpha$)

$$P_\lambda(q^{-1}) = \sum_{\mu \in Q_+^\vee} q^{-\langle \rho, \lambda + \mu \rangle} L_{\lambda\mu} m_\mu.$$

Let $\lambda \in Q_+^\vee$ and w^λ be the element of minimal length in the right coset $t_\lambda W$. Let σ^λ be a minimal gallery of type t^λ connecting A_f and A_{w^λ}. Let W^λ be the set of minmal representatives of W/W_λ.

Theorem. *For $\mu \in Q_+^\vee$ we have $L_{\lambda\mu} = \sum_\sigma q^{l(w_0\iota(\sigma))} L_\sigma$, where the sum is over all positively folded galleries σ of type t^λ and weight μ, starting in the origin with $w_0\iota(\sigma) \in W^\lambda$.*

Define $C^\nu_{\lambda\mu}$ for $\lambda, \mu, \nu \in Q_+^\vee$ by $P_\lambda(q^{-1})P_\mu(q^{-1}) = \sum_{\nu \in Q_+^\vee} q^{-\langle \rho, \mu - \lambda + \nu \rangle} C^\nu_{\lambda\mu} P_\nu(q^{-1})$.

Theorem. *Let $\lambda, \mu \in Q_+^\vee$. Then $C^\nu_{\lambda\mu} = \frac{W_\nu(q^{-1})}{W_\mu(q)} \sum_\sigma q^{l(w_0\iota(\sigma))} C_\sigma$, where the sum is over all positively folded galleries of type t^μ and weight ν, starting in λ and contained in the dominant chamber.*

Relations to geometry (see [GL])

Let K be an algebraically closed field. Let G be a simple, simply connected algebraic group over K with root system R corresponding to some choice of a Borel $B \subset G$ and a maximal torus $T \subset B$. Let U^- denote the unipotent radical of the opposite Borel of B. Let $\mathcal{K} = K((t))$ be the field of Laurent series and denote by $\mathcal{O} = K[[t]] \subset \mathcal{K}$ the ring of formal power series. The affine Grassmanian is the quotient $G(\mathcal{K})/G(\mathcal{O})$. Let $\lambda \in Q_+^\vee$ and assume for simplicity that λ is regular.

Theorem (Gaussent, Littelmann). *There is a surjective map from the orbit $G(\mathcal{O})\lambda \cdot G(\mathcal{O})$ to the set of all positively folded galleries of type t^λ. The preimage of a gallery σ of weight μ is contained in $U^-(\mathcal{K})\mu \cdot G(\mathcal{O})$ and it is isomorphic to $K^{l(w_0\iota(\sigma))+m(\sigma)} \times (K^*)^{n(\sigma)}$.*

References

[GL] S. Gaussent and P. Littelmann. *LS galleries, the path model, and MV cycles.* Duke Math. J., 127(1):35–88, 2005

[S] C. Schwer. *Galleries, Hall-Littlewood polynomials and structure constants of the spherical Hecke algebra*, arXiv:math.CO/0506287