Questions arising in open problem sessions in AIM workshop on L^2-harmonic forms in geometry and string theory

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1 March, 17, 2004 – L. Saper as moderator

Question [Sobolev-Kondrakov embedding theorem]:
Theorem: If (M^n, g) is a compact Riemannian manifold then

$$L_k^p \to L_l^q$$

is compact if p, q, k, l satisfy $k - \frac{n}{p} > l - \frac{n}{q}$ and $k > l$. (Here L_k^p denotes the Sobolev space of functions and their k-derivatives bounded in the L^p-norm.)
Question: How does this theorem generalize from compact manifolds to complete open manifolds?
Answers:
1. (Carron) One needs geometrical assumptions on (M, g) to make this work. It turns out that bounded Ricci-curvature is not a sufficient condition. Donnelly [3] had shown that if a complete non-compact Riemannian manifold of dimension n has $Ric \geq -(n-1)g$ then the essential spectrum of the Laplace operator (on functions) has a non-empty intersection with $[0, (n-1)^2/4]$. Moreover, Donnelly and Li [4] had shown that if a Cartan-Hadamard manifold (i.e. a complete simply-connected Riemannian manifold with non-positive curvature) has its curvature going to $-\infty$ at infinity then the essential spectrum of the Laplace operator is empty.
2. It can also be generalized introducing weighted Sobolev spaces – with special conditions on the weights depending on the geometry at infinity (see Joyce's book [7] for the case of ALE manifolds).
3. (Melrose) There is no good general answer.

Question [L^2-cohomology]:
Is there a proper way to define L_2-homology as opposed to L^2-cohomology?
Motivation: There is a way to define it for triangulated manifolds $(M, \mathcal{T})$, by going up to the universal cover $(\widetilde{M}, \widetilde{\mathcal{T}})$ and taking the L_2-chains to be

$$C^i_{(2)}(\widetilde{M}) = \{\sum a_i \sigma_i | \sum |a_i|^2 < \infty\}.$$

(and possibly some boundary terms condition coming in also?)
Question to ask: How does this depend on the metric? And when do you get something dual to the L^2-cohomology.

Question [L^2-cohomology]:
Is there any relation between L^2-cohomology and group-cohomology.
Answer: see Lück in his book [8].

Questions [Non-parabolicity and exactness of the excision sequence for reduced L^2-cohomology – G. Carron's lecture]:

Let (M,g) Riemannian manifold so that $d+d^*$ is non-parabolic at infinity with respect to K. For a compact $\widetilde{K}$ so that $K \subset \widetilde{K}$ we define the norm

$$N_{\widetilde{K}}(\alpha) := ||\alpha||_{L^2(\widetilde{K})} + ||(d+d^*)\alpha||_{L^2(M\setminus\widetilde{K})}.$$

Then on $\mathcal{C}_0^\infty(\Lambda(M))$ all these norms are equivalent. Let W be the completion of $\mathcal{C}_0^\infty(\Lambda(M))$ with respect one of them. One of the main points was that

$$d+d^* : W \to L^2$$

is Fredholm. Several related questions arose:
1. Is there any other description for W?
2. Is the reduced L^2-cohomology $\dot{H}^k_{(2)}(M,g)$ the cohomology of a complex/ of many complexes?
3. Can the space W be put into a complex?

Question [reduced L^2-cohomology – G. Carron's lecture]:
When is there a Mayer-Vietoris sequence for $\dot{H}_{(2)}(M,g)$?
Answer: (Carron) The condition of non-parabolicity at infinity is not sufficient for this. One needs a better control over noncompact overlaps. Perhaps if also Range(d) is closed on $U \cap V$ then it is sufficient??

Questions [Self-dual Gravitational Instantons – S. Cherkis's lecture]:
1. Why do you expect them to arise as monopole moduli spaces?
Answer: (Cherkis) string theory argument.

2. Are all 8-dimensional hyperkähler manifolds (non-compact, with some sort of well-behaved asymptotic behaviour) moduli spaces?
Answer: (Cherkis) yes, from string theory intuition.

3. What about dimension > 8?

4. One way to get gravitational instantons is by solving the vacuum Einstein equation in Lorentz space, and then performing a wick rotation (meaning $t \to it$). This worked in the case of Lorentzian Taub-NUT to get the Riemannian Taub-NUT. Which other instantons arrise this way?
Answer: In order to get a Riemannian metric, one needs $t \to t+c$ (time translation) be an isometry. Therefore it does work for A_k-cases but not for the others (D_k, E_6, E_7, E_8).

5. Can they arrive as "non-trivial" hyperkähler reductions of finite dimensional linear spaces?
Answer: Yes, for ALE spaces (Kronheimer's construction). Yes, for ALF if certain (Dancer) spaces are included with the linear spaces.

6. Is it true that every gravitational instanton has an asymptotic local triholomorphic isometry?
Reason: If you have a local $\mathbb{R}$-action at infinity, then the metric is given in terms of a harmonic function, which arises from the hyperkähler moment map associated to this action (see the work of Gibbons).

7. When is it true that {Solutions of the Einstein equation} > {Self-Dual metrics}?
Reason: In general $\geq$.
Answers:
(1) In the case of ALF the answer is >, since the Euclidean Schwarzschild metric is not self-dual.
(2) Nakajima conjectured that in dimension 4, ALE and Ricci-flat implies self-dual.
(3) In dimension > 4 and even, the conjecture is that ALE and Ricci-flat implies Kähler.
(4) (Carron) For odd dimensional manifolds an ALE Ricci-flat metric has to be flat since
(a) by the Cheeger-Gromoll theorem this manifold has only one end (if not it splits isometrically as $\mathbb{R} \times N$ with N compact);
(b) The topology at infinity must be of the type $(R,\infty) \times S^{2n}/G$ where G is a finite subgroup of $O(2n+1)$ acting freely on S^{2n}. If $g \in SO(2n+1)$ it must have 1 as an eigenvalue hence the only such G is $\{I, -I\}$; but the quotient is the real projective space, and there is no odd dimensional compact manifold whose boundary is the real projective space. Therefore G is trivial and the Bishop-Gromov inequality

implies that M is the Euclidean space. Recall that the Bishop-Gromov inequality states that in a manifold M^n with non-negative Ricci curvature $r \mapsto \text{vol}B(x,r)/w_n r^n$ is decreasing with equality everywhere if and only if M^n is the Euclidean space. This ratio goes to 1 when $r \to 0$, and in the ALE case we have that this ratio goes to 1/card G when $r \to \infty$.

Questions [of S. Cherkis]:
1. There exists an explicit construction of Atiyah [1] and Page [10] for the Green's function on certain ALE and ALF spaces. Page's construction comes from physics and constructs by hand the Green's functions for both $A_k ALE$ and $A_k ALF$. Atiyah has a formalism which in theory allows one to construct the Green's function on any self-dual gravitational instanton – this formalism involves Serre classes on the corresponding twistor spaces. In the paper he constructs only the Green's function in the $A_k ALE$ case. (According to Etesi the use of Atiyah's Serre-class formalism is significantly more difficult for the ALF case and has not yet been carried out. The main problem he says is that one cannot find a nice compacification for the twistor space of an ALF self-dual space. Carrying out this construction would allow us to construct L^2 harmonic forms over the new $D_k ALF$ gravitational instantons constructed by Cherkis and Hitchin.)
Question: Is there a way to generalize this formalism to 8 dimensional hyperkähler manifolds?
Reason: In the non-compact 4-dimensional case or $K3$ case, the coefficient of the leading order term in the Green's function gave information about the values of the parameters (e.g. sizes of the cycles, metric).

2. The same story for quaternionic manifolds. How to modify Atiyah's formalism?
Reason: There is a reasonable twistor theory for these manifolds.

3. Is it true that any 4-dimensional hyperkähler manifold (non-compact, with reasonable decay at infinity) can be deformed to a quaternionic one?
For example: HK has $Ric = 0$, while quaternionic manifolds satisfy $Ric = \lambda g$ with λ a constant (non-zero cosmological constant). One would like to obtain any HK as a limit of quaternionic spaces with $\lambda \to 0$.
Example: $\mathbb{R}^4$ is the limit of $\mathbb{H}P^1$ (positive λ's) and $\mathbb{H}\mathbb{H}^1$ (negative λ's).

2 March 18, 2004 – S. Cherkis as moderator.

Question (R. Mazzeo):
What are the asymptotics at infinity of the multi-monopole moduli spaces?
Answer: (Cherkis) They should behave like a N-body problem.

Questions (K. Lee)
For these questions, $\mathbb{R}^4_B$ denotes the non-commutative $\mathbb{R}^4$. Here $B = B_{\mu\nu} dx^\mu dx^\nu$ is a 2-form which gives the non-commutative structure. Then, $[x_\mu, x_\nu] = B_{\mu\nu}$. We define non-commutative instantons using the Nahm transform of instanton data

$$\mu_1 = [T_1, T_2] - [T_3, T_4] - j^* j = B_{12},$$

plus two other similar equations (see the work Nekrasov-Schwarz on non-commutative instantons [9]). Let $\mathcal{M}_B^{k,n}$ be the moduli space of centered k-instantons on $\mathbb{R}^4_B$ with gauge group $G = U(n)$. (The fact that the instantons are centered means we consider only instantons with center of mass at the origin)

1. What is the instanton number in the non-commutative set-up?
Motivation: The space $\mathcal{M}_B^{k,n}$ is singular for $B = 0$ but non-singular for $B \neq 0$. One would like to understand the limit as $B \to 0$.

2. Does there exists a unique harmonic form in the middle dimension of $\mathcal{M}_B^{k,n}$?
Note: The answer to this question is Yes for the following cases:
(i) 1-instantons and any G;
(ii) 2-instantons and $G = U(1)$ (see the work of Lee et al).

3. *Conjecture:* The moduli space of centered 1-instantons on $\mathbb{R}^3 \times S^1_B$ with "given holonomy around S^1_∞" (holonomy is not in the center of $G = U(n)$) has exactly n L^2-harmonic forms in middle dimension.
Remarks:
(i) There exists a physics proof.
(ii) For $n = 2$, this is a special case and the conjectured has been verified.

4. The "massless monopoles" are defined as solutions to the Bogomolny equation:

$$F = *\nabla\phi,$$

plus an auxiliary condition. We think of Φ as a section of the bundle of endomorphisms of the Lie algebra of G. Let $(\lambda_1, \ldots, \lambda_n)$ be the limit of the eigenvalues of ϕ as $r \to \infty$. For the massless monopole condition we need $\lambda_i = \lambda_j$ for some $i \neq j$.
Question: When does the moduli space of massless monopoles have harmonic forms in middle dimension?
Answer: Sergey Cherkis seems to have an answer using a physics argument.
Conjecture: When all the λ_i's are equal the moduli space is empty.

Question (M. Singer):
Is the space of all L^2-harmonic forms on a manifold M with non-negative sectional curvature finite dimensional?
Hints:
(i) By Cheeger-Gromoll, these spaces have finite topology.
(ii) Moreover, by the solution of the Soul Conjecture such manifolds are diffeomorphic to normal bundles to totally geodesic compact submanifolds in M.
(iii) If $Ric \geq 0$ then there exist no L^2-harmonic 1-forms (Bochner type argument).

Question:
Introduction: When (Σ, g) is a Riemannian surface (possibly with boundary) then a result of Duistermaat-Guillemin [5] gives a relation between the length spectrum $\{l_i\}$ of Σ (that is, the lengths of all closed geodesics of Σ) and the spectrum of the Laplacian $\{\lambda_i\}$ on $C^\infty(\Sigma)$. Define the (trace of the) wave kernel by

$$Z(t) = \sum_{i=0}^{\infty} e^{\pm t\sqrt{\lambda_i}}.$$

Generically,

$$Z(t) \sim \sum \frac{d_i}{t - l_i}.$$

The wave kernel associated to the spectrum of the eigenvalues of Δ is a distribution whose singular support must be contained within some set related to the length spectrum??
Question: Give a more precise statement of the relationship?

Questions: (S. Cherkis)
1. When M is a 3-manifold are there some 2-dimensional objects that are related to Spec(Δ) in a similar way?
2. Does there exist some analogue for the "area spectrum" of minimal surfaces in M^3 (with some suitable geometric assumptions imposed on M^3)?
3. Compute the number of minimal surfaces $N(n_1, n_2, g)$ in $T^4 = T^2 \times T^2$ endowed with the flat metric which represent the homology class $n_1[T_1] + n_2[T_2]$, where T_i are the two 2-tori factors of T^4.
Remarks:
(i) (Cherkis) There is some physical intuition for these kind of relationships.
(ii)(Mazzeo) Closed totally geodesic surfaces are not the correct 2-dimensional objects to consider for #1, since there are too few of them.

(iii) (Mazzeo) There is some version of the Selberg trace formula which applies here.
(iv) (Mazzeo) Would also be interesting to look at the eta-invariant.

Questions (M. Jardim):
Let $\mathcal{M}$ denote the moduli space of k SU(n) (commutative) instantons on $\mathbb{R}^4$. Take the Dirac operator D_A coupled with the instanton. Then we have $\ker D_A = (0)$. Let $\lambda = \min \operatorname{Spec} D_A$. λ is a function on $\mathcal{M}$. The questions arrising are:
1. What can we say about the function λ? Is is bounded?
Motivation: this is the lightest quark we can have.
2. Same question as above but for instantons on T^4 instead of $\mathbb{R}^4$.

3 Wrap-up Session, March 21 2004. R. Mazzeo as moderator.

Question (T. Hausel)
Motto: "toric" hyperKähler quotients are perhaps the simplest examples of complete hyperKähler metrics.
Problem: Understand the Hodge cohomology of these spaces from a systematic point of view.
Possible first steps:
1. Understand the asymptotics of these spaces (for a start see work of Bielawksi-Dancer [2] and Gibbons-Rychnenkova [6]).
2. (Mazzeo) Combine asymptotics with N-body techniques.
3. (Hausel) We know that all the L^2 Hodge cohomology of these spaces must be in the middle dimension and that any L^2-harmonic form in middle dimension is $\pm$ self-dual. So it is enough to understand the L^2-signature. This will have consequences for the sign of $\sigma_{L^2}(X)$ and is related to combinatorics and matroids.
More general problem: Prove a suitable index theorem on such manifolds. Will there be contributions from subsytems??

Questions (G. Etesi):
1. It is known that there exists a good compactification of ALF spaces for L^2-cohomology. What about its twistor space Q?
(a) Does it compactify?
(b) Does the complex structure extend over the compactification?

2. Study the Yang-Mills functional on open manifolds. For example, for manifolds which are conformally equivalent to a manifold with one cylindrical end, the energy of Yang-Mills instantons (if finite) is congruent mod $\mathbb{Z}$ to one of the Chern-Simons invariants of the boundary. What about different asymptotic geometries (ALF spaces, etc)?
Remarks:
(i) Without smoothness and irreducibility assumptions for the YM instantons, the energy cannot be quantized by the Chern-Simons invariants.
(ii) It is conjectured that the Chern-Simons invariants of a compact 3-manifold are rational, and therefore the same is conjectured for finite energy YM instantons on manifolds with one cylindrical end. If one considers finite energy Yang-Mills instantons on manifolds with different asymptotic geometeries is the YM action still quantized?

Questions (E. Hunsicker):
1. Is there a topological interpretation for the L^2-cohomology of higher dimensional monopole modulii spaces?
(i) What are the right tools to use here?
(ii) Can we use Saper's L-module techniques? (see [11, Section 12])

2. These questions involve complete manifolds with special holonomy.
(i) Understand asymptotic geometry of these manifolds.

(ii) Understand their L^2-cohomology.
(iii) Find general vanishing theorems (possibly even if we do not understand part 1 completely).
Remarks:
(a) Some G_2 metrics fiber over S^3 with hyperKähler fibres (although some fibres can degenerate). This may be a good place to test whether L-module techniques are appropriate.
(b) In the case of SU(2)-monopole moduli space over $\mathbb{R}^3$, the Segal-Selby paper [12] proves (by pure topology) that there is a harmonic form (coming from the topology of the moduli space) as predicted by Sen's conjecture. What remains is to prove that these are the only L^2-harmonic forms (surjectivity of the map in the Selby-Segal construction).

Question: Study extended L^2-harmonic forms on noncompact spaces.
Definition: a form ω is *extended L^2-harmonic* if

$$(d+d^*)\omega = 0, \qquad \omega = \lim \omega_j \text{ locally}, \quad \text{where } \ \omega_j \in C_0^\infty,$$

and

$$(d+d^*)\omega_j \to 0 \text{ in } \ L^2.$$

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