Problems from the workshop

"Mahler's conjecture and duality in convex geometry"

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$\mathbf{I}K$: the intersection body of K
$\mathbf{\Pi}K$: the projection body of K
$\mathcal{P}(K)$: the volume product of K
$u^\perp$: the hyperspace perpendicular to $u \in S^{n-1}$
$K|u^\perp$: the projection of K onto $u^\perp$
St_u : Steiner symmetrization with respect to $u^\perp$
B_p^n : the unit ball of ℓ_p^n (n-dimensional ℓ_p-ball)
$\langle\cdot,\cdot\rangle$: the scalar product in $\mathbb{R}^n$

1. Is it true that if $\mathbf{I}K \subset \mathbf{I}L$ and $\mathbf{\Pi}K \subset \mathbf{\Pi}L$ then $K \subset L$?

2. Is it true that if $\mathbf{I}K \subset \mathbf{I}L$ and $\mathbf{\Pi}K \subset \mathbf{\Pi}L$ then $\mathrm{vol}(K) \leqslant \mathrm{vol}(L)$? (True if $n \leqslant 4$)

3. Is it true that for any convex (symmetric) body K there exists a direction $u \in S^{n-1}$ such that
$$\mathrm{vol}(\mathrm{St}_u(K)^*) \leqslant C\mathrm{vol}(K^*)?$$

4. What about an average version of the previous question, i.e.:
$$\int_{S^{n-1}} \mathrm{vol}(\mathrm{St}_u(K)^*)du \leqslant C\mathrm{vol}(K^*)?$$

5. Is the average of $\mathcal{P}(K)$ for symmetric polytopes K greater than or equal to $\mathcal{P}(B_\infty^n)$? More precisely, let K be a random symmetric polytope of a fixed number of vertices, where each vertex is independent Gaussian random vector is it true that $E\mathcal{P}(K) \geqslant \mathcal{P}(B_\infty^n)$?

6. Given even log-concave probability measure μ, estimate $\mu(K)\mu(K^*)$ from above. (It is known that $\mu(K)\mu(K^*) \leqslant \mu(B_2^n)$ if μ is unconditional)

7. In the above question, characterize the equality case and find lower bound for the unconditional case.

8. Consider μ^* in the sense of Artstein-Milman and answer to the same questions about $\mu(K)\mu^*(K)$.

9. Let K and L be symmetric convex bodies in $\mathbb{R}^n$ and denote by $N(K,L)$ the covering number of K by L, the minimum number of the translates of L needed to cover K. Is it true that
$$N(K,L) \leqslant [N(L^*, cK^*]^\delta,$$
where c, δ are absolute constants?

10. Given random vectors X and Y uniformly distributed in K and K^* respectively, look at $\langle X, Y\rangle$, up to a "100"-moment. Are the first "100" moments the same $(1+\varepsilon)$ as those for Gaussian random vectors?

11. Prove Mahler's conjecture for finite-dimensional normed spaces that embed in L_p, $p<1$ and, in particular, for convex intersection bodies (which correspond to the case $p=-1$).

12. What if $1<p<2$? Is it true that $\mathcal{P}(B_X) \geqslant \mathcal{P}(B_p^n)$ if X is embedded into L^p, $1<p<2$ and $dimX = n$?

13. (T. Tao) Let K be a symmetric convex body in $\mathbb{R}^n$. Consider the covering radius of $\mathbb{Z}^n$ for K^*, the smallest $r>0$ such that
$$\bigcup_{v\in\mathbb{Z}^n\setminus K} rK^* + v = \mathbb{R}^n.$$
Is it true that K^* has the largest covering radius when K is a cube, among all symmetric convex bodies in $\mathbb{R}^n$?

14. Does $\mathbf{I}K = K$ imply $K = cB_2^n$ when $n \geqslant 3$? Notice that if $\mathbf{\Pi}K$ is considered instead of $\mathbf{I}K$, it is not true because the projection body of a cube is a dilate of a cube. It is a special case of Problem 8.7 in [G]. There are a few comments about the more general problem in [G, Note 8.6]. Moreover the analogous more general question for projection bodies is [G, Problem 4.5], and in [G, Note 4.6] it is stated that Weil solved this for polytopes.

15. For $n \geqslant 3$, is it true that $\mathbf{I}^m K \to B_2^n$ in the Banach-Mazur distance as $m\to\infty$?

16. Let $n \geqslant 5$. Construct an example of a polytope K which is an intersection body, not a polar body of a zonotope.

17. (G. Kuperberg) Let K be a symmetric convex body in $\mathbb{R}^n$. For random vectors X and Y distributed in K and K^* respectively, consider the random variable $\langle X, Y\rangle$. Then the conjecture is that $\langle X, Y\rangle$ has the largest variance when K is an ellipsoid. One may consider a weaker conjecture: the integral of $|\langle X, Y\rangle|^2$ over $K \times K^*$ is maximized when K is an ellipsoid. More generally, G. Paouris asked the same questions for the quantity $|\langle X, Y\rangle|^p, (p > 0)$, instead of $|\langle X, Y\rangle|^2$.

    Furthermore two other variations can be considered: Given $0 < c < 1$, consider either the probability that $\langle X, Y\rangle = c$, or the volume of the region in which it is so. One may again conjecture that either the probability or at least the volume is maximized when K is an ellipsoid.

18. (G. Kuperberg) In $\mathbb{R}^3$, consider a class $\mathcal{A}$ of centrally symmetric polytopes of a given combinatorial type, excluding the type of the cube and the octahedron. Then the conjecture is that the volume product does not have a local minimum in $\mathcal{A}$. Furthermore, one may ask if each such $\mathcal{A}$ has a unique critical point up to affine transformations, and it is a local maximum.

19. (R.J. Gardner) Does the following theorem hold for star bodies, or perhaps more generally still? It comes from [G, Problem 7.1] and there are some relevant comments in [G, Note 7.1].

    Theorem (Rogers, 1965.) *Suppose that $2 \leqslant k \leqslant n-1$ and that K and L are compact convex sets in $\mathbb{R}^n$ containing the origin in their relative interiors. If all k-sections $K \cap S$ and $L \cap S$ of K and L are homothetic (or translates), then K and L are homothetic (or translates, respectively).*

20. (R.J. Gardner)[G, Problem 7.6] If K and L are origin-symmetric star bodies in $\mathbb{R}^3$ whose sections by every plane through the origin have equal perimeters, is $K = L$? In particular, This was proved by R. Howard, F. Nazarov, D. Ryabogin and A. Zvavitch for bodies of revolution and by V. Yaskin for polytopes.

21. (S. Reisner) Improve the following theorem by finding the number of vertices $N(n) > n+3$.

    Theorem (M. Meyer, S. Reisner)[MR] *Let K be a convex polytope in $\mathbb{R}^n$, with at most $n+3$ vertices (or facets) and non-empty interior. Then*

$$\mathcal{P}(K) \geqslant \frac{(n+1)^{n+1}}{(n!)^2}$$

with equality if and only if K is an n-dimensional simplex.

22. (S. Reisner) Quantify the shadow movement theorem (see [MR]).

23. (H. Koenig) For $Q \subset S^{n-1}$, define its dual Q° on the sphere S^{n-1} by

$$Q^\circ = \{x \in S^{n-1} : \langle x, y \rangle \geqslant 0 \ \forall y \in Q\}$$

and define its volume product by $\mathcal{P}(Q) = \mathrm{vol}(Q)\mathrm{vol}(Q^\circ)$. For any $\Delta_n \subset S^{n-1}$ which is determined by distinct n hyperplanes passing through the origin, is it true $\mathcal{P}(\Delta_n) \leqslant \mathcal{P}(\mathbb{R}^n_+ \cap S^{n-1})$? In particular, if $n = 3$, it is true.

References

[G] R.J. Gardner, *Geometric tomography.* Second edition. Encyclopedia of Mathematics and its Applications, **58**. Cambridge University Press, New York, 2006.

[MR] M. Meyer and S. Reisner, *Shadow systems and volumes of polar convex bodies.*, Mathematika **53** (2006), 129-148.