Theory and Algorithms of Linear Matrix Inequalities

Questions and Discussions of the Literature

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Preface

This booklet contains some questions, expositions, references, and ideas for teaching supplied by participants at the “Theory and Algorithms of Linear Matrix Inequalities” workshop, held at the American Institute of Mathematics on August 1 – 5, 2005.

Contents

Chapter 1

Questions and Open Areas

1.1 Joseph Ball

In recent years I (and my collaborators) have been looking for multivariable generalizations (N-D linear systems, operator tuples, analytic functions of several variables) of what we are familiar with in 1D and applications thereof. Somewhat but not completely surprisingly, we found that noncommutative versions of these ideas (systems with evolution along a free semigroup rather than over an integer lattice, noncommuting operator tuples rather than commuting operator tuples, formal power series in noncommuting indeterminants rather than analytic functions of several (commuting) complex variables) turn out to be easier to deal with and lead to results having a stronger parallel with the single-variable case.

Open Areas

As applications of the above, there is the potential for a complete analogue of classical linear control theory (Kalman state-space theory, LQR and H^∞ feedback control theory, etc.) for linear systems with evolution along a free semigroup.

More relevant, however, are exciting new applications of formal power series to classical linear control (e.g., analysis of time-varying structured uncertainty for classical linear plants, systematic search for dimensionless Linear Matrix Inequalities in control theory), where the noncommutative system theory ideas is merely a tool for dealing with problems concerning formal power series (a noncommutative version of the state-space method—now applied to formal power series rather than to rational or analytic matrix-valued functions).

1.2 John Helton, Scott McCullough, and Victor Vinnikov

See the paper "Noncommutative Semialgebraic Geometry and Convexity vs LMI's" by Helton, McCullough, and Vinnikov.

1.3 Didier Henrion

Alternative descriptions of the cone of positive polynomials

It is well-known that the convex cone of globally non-negative one-variable polynomials is semidefinite representable (in the sense of Nesterov, Nemirovskii, Ben-Tal) in the space of polynomial coefficients, i.e. it is the projection of the solution set of an LMI. This LMI involves lifting variables, namely entries of a positive semidefinite matrix ensuring a decomposition as a sum of squares of polynomials. Lifting variables can be removed using e.g. quantifier elimination, yielding an explicit semialgebraic description of this cone. Such a description is likely to be intricate, with a large number of polynomial inequalities of large degree.

1. Under which conditions does there exist an explicit LMI representation of this cone, without using lifting variables ?
2. Can we derive efficient, e.g. matrix-wise, algebraic reformulation of these positivity conditions, without using lifting variables ?
3. A related problem is that of finding algebraic conditions on two real-valued symmetric matrices A and B such that there exist a real scalar x ensuring positive semidefiniteness of the matrix A+xB. Note that matrix B is sign indefinite in general.

Applications include the design of restricted complexity robust control laws in the H-infinity framework.

Conditioning estimates for LMI problems

Following Renegar's work, a theory of conditioning is available for conic optimization, in particular for LMI problems. Unfortunately, it seems that there is no satisfying cheap estimate of the conditioning of an LMI problem: the most efficient methods to date consist in solving at least two LMI problems of the same dimensions as the original LMI problem for which we want evaluate the conditioning. This is in sharp contrast with linear system of equations, where cheap conditioning estimates are available and implemented in standard numerical linear algebra package such as LAPACK.

4. Could it be possible to compute cheap but still useful conditioning estimates for LMI problems, using basic linear algebra ?

Detecting convexity of polynomial matrix inequalities

Several difficult design problems in control theory boil down to solving polynomial matrix inequalities (PMIs) with a few decision variables (typically between 5 and 15). This is the case for e.g. static output feedback or low order H2 optimal controller design, when formulated in a polynomial framework using Hermite's stability criterion.

Particular cases of PMIs are LMIs (degree one) and bilinear matrix inequalities (BMI, degree two). Several powerful semidefinite programming packages are available for control engineering willing to solve LMIs. This is in sharp contrast with BMIs, for which there is almost nothing. The main reason could be that BMI problems are generally non-convex, in contrast with LMI problems which are always convex. It may happen however that some BMI or PMI problems are convex in the set of decision variables. This is the case for example for PMI problems arising from static output feedback design.

5a. Is it possible to detect efficiently convexity of the solution set of a PMI ?

5b. If the solution set of a PMI is convex, does it admit an LMI representation ?

1.4 Jean Lasserre

Recent developments are:

- the representation of polynomials, nonnegative on a semi algebraic-set $S \subset R^n$ or a real variety $V \subset R^n$.
- the representation of positive semidefinite polynomial matrices, and more generally *noncommutative* polynomials.

Indeed, the latter representation results should have a profound impact on various important potential applications, notably in Control. An open area is to see how the above representation results can be applied in many important applications, when modelled as a particular instance of the *general problem of moments* with polynomial data.

Finally, I would like to emphasize the importance of being able to solve efficiently the following SDP problem:

$$min_y \{c'y : \quad M_r(y) \succeq 0; \quad Ay \leq b\}$$

where A is a matrix, $M_r(y)$ is the usual *moment matrix* associated with a sequence y, indexed in the canonical basis of monomials $\{x^\alpha\}$. Indeed, one may show that *converging SDP-relaxations* of many polynomial optimization problems can be put in the above form. What is interesting is that the data of the original problem only appear in the linear inequalities $Ay \leq b$, but *not* in the (difficult) LMI constraint $M_r(y) \succeq 0$! So an adhoc and efficient SDP solver should take into account the specific structure of the moment matrix, which is independent of the problem data.

1.5 Antonis Papachristodoulou

A topic of considerable interest is LMIs used to solve sum of squares programs. Of particular interest are, PLMIs (Parameterized Linear Matrix Inequalities), which in general take the form

$$F(x,p) \triangleq F_0(p) + x_1 F_1(p) + \cdots + x_k F_k(p) \geq 0 \tag{1.1}$$

where the F_i's are $n \times n$ given real symmetric polynomial matrices in $p = [p_1, \ldots, p_n]$, a collection of parameters typically allowed to take values in a compact set P:

$$P = \{p \in \mathbf{R}^m | g_i(p) \leq 0, i = 1, \ldots, N\} \tag{1.2}$$

Such PLMIs appear frequently in systems theory, when considering robust stability for systems with parametric uncertainty [23, 28] or analysis of time-delay systems [26], performance analysis etc.

A PLMI is an infinitely constrained LMI, and PLMI problems are known to be NP-hard in general. Several techniques have been developed in the past [22], which try to turn this problem into a standard LMI problem, possibly conservative, for particular descriptions of the set P (typically polytopic) and assumptions on the dependence of the $F_i(p)$'s on p. For example, in the special case in which $F(x, p)$ is affine in p, and P is a polytope, it is enough to test feasibility at the vertices of the polytope, which offers a significant reduction on the computational cost. Other techniques include discretization methods (not exact), convex covering (can be conservative) etc.

In many cases the set P is described by a set of polynomial inequalities as shown in (1.2) and/or the PLMI given by (1.1) is not affine in p. Our aim is to obtain potentially conservative LMI conditions that guarantee the solvability of the original PLMI. One way of generating such conditions is through the use of the sum of squares technique, related to the notion of sum of squares (SOS) matrix [24]. A PLMI $F(x, p)$ can be written as:

$$F(x, p) = (I_n \otimes Z(p))^T Q (I_n \otimes Z(p)) \tag{1.3}$$

where I_n denotes the $n \times n$ Identity matrix, $Z(p)$ is a properly chosen vector of monomials in p, and Q is a symmetric matrix of appropriate dimensions. Then $Q \succeq 0$ implies that $F(x, p) \geq 0$. In special cases (e.g. for $n = 1$), $Q \succeq 0$ is necessary and sufficient for $F(x, p) \geq 0$. In fact the condition $Q \succeq 0$ is equivalent to

$$f(v, x, p) = v^T F(x, p) v \text{ is SOS} \tag{1.4}$$

for all $v = [v_1, \ldots, v_m]$ [27, 25].

The conditions $g_i(p) \leq 0$, which may describe the set P, can be adjoined to $f(v, x, p) \geq 0$ using sum of squares multipliers $\sigma_i(v, p) = v^T \Xi_i(p) v$, where $\Xi_i(p) = \Xi_0 + p\Xi_1 + \ldots + p^l \Xi_l$ are symmetric matrices with polynomial entries of order l:

$$\tilde{f}(v, x, p) = v^T \left\{ F(x, p) + \sum_{i=1}^{N} \Xi_i(p) g_i(p) \right\} v$$

If $\tilde{f}(v, x, p)$ is a Sum of Squares then $F(x, p) + \sum_{i=1}^{N} \Xi_i(p) g_i(p) \geq 0$, which in turn implies that when $p \in P$, then $F(x, p) \geq 0$. For increasing l, a nested family of conditions can be developed.

The method proposed to solve PLMIs suffers from increasing computational burden when the number of parameters and/or their order is increased. Even though this increase is polynomial as one of these two factors is increased (but not both at the same time), currently only problems with 5-7 variables in 3-4 order can be dealt with efficiently. More appropriate representations of the problem, apart from reducing computational cost, can also give better numerical conditioning. For this, factors such as problem structure, algorithm efficiency etc., should be taken into account.

1.6 Mihai Putinar

Here are some problems:

1. Supports for NC Positivstellensatze:

   So far we were successful to prove NC positivstellensatze only for: polydisks, spherical isometries, or product of these supports. It would be desirable to have more general supports. The difficulty in our proof was related to the GNS construction, and the possibility of extending, from subspaces of a Hilbert space, partial isometries to full isometries.

2. A trade-off between commutative and NC SOS decompositions.

   When commutative SOS decompositions, on prescribed semi-algebraic supports fail, how far (in terms of rank) one must go with replacing the variables by matrices to obtain simple, weighted SOS decompositions?

3. Study carefully the degree bounds in the known NC SOS decompositions, such those we know for spherical isometries.

4. The optimization theory community has paid little attention to the hermitian SOS, that is of the form $|f(z)|^2$, with the function f analytic in a certain region. There are two instances where the positivity with respect to this slightly smaller cone is relevant:

   - von Neumann type inequalities, for one or several commuting operators (see the works of Agler, McCarthy, Cole, Wermer,...)
   - positive Hermitian bundles on domains of C^d and the associated isometric embeddings (see the works of Quillen, Catlin, d'Angelo)

   There are considerable advantages of working with such hermitian positivity, and the bounds one can expect from the resulting polynomial decompositions.

1.7 Bruce Reznick

My interest in Linear Matrix Inequalities is somewhat oblique. What I am really interested in is the representation of real polynomials in several polynomials as a sum of squares of polynomials. This leads directly to the *Gram matrix method.* Using multinomial notation, suppose

$$p(x) = \sum_{k=1}^{r} h_k^2(x), \qquad p(x) = \sum_{\alpha} a(\alpha)x^{\alpha}, \qquad h_k(x) = \sum_{\beta} b_k(\beta)x^{\beta},$$

and let $B(\beta) = (b_1(\beta), \dots, b_r(\beta))$. Then a direct calculation shows that

$$a(\alpha) = \sum_{\beta+\beta'=\alpha} B(\beta) \cdot B(\beta'),$$

for all α, and the matrix $[B(\beta) \cdot B(\beta')]$ is psd with rank r. Conversely, if $V = [v(\beta, \beta')]$ is a symmetric psd matrix with rank r and for all α,

$$a(\alpha) = \sum_{\beta+\beta'=\alpha} v(\beta, \beta'),$$

then p is a sum of squares of r polynomials; V is called a *Gram matrix* for p.

This algebraic subject was revolutionized by the realization that the underlying matrix problem is a semidefinite programming problem, and therefore fast computation is possible!

Suppose $p = \sum_k h_k^2$ is a sum of squares and $p(u) = 0$. Then $h_k(u) = 0$, hence

$$0 = h_k(u) = \sum_{\beta} b_k(\beta)u^{\beta} \quad (1 \le k \le r) \qquad \Longrightarrow \sum_{\beta} u^{\beta} B(\beta) = 0.$$

It should be noted that this restriction comes from the assumption that the Gram matrix is psd, and not from the linear equations its entries satisfy. Before the recent software advances, it was usually only practical to apply the Gram matrix method to psd forms with lots of zeros.

It is natural to ask whether there may be non-trivial restrictions on the Gram matrix of a polynomial, which are *not* a consequence of its zeros. Unfortunately, the answer is yes. It can be shown that the even symmetric ternary sextic

$$9(x^6 + y^6 + z^6) - 8(x^4y^2 + x^4z^2 + \dots) + 24x^2y^2z^2 =$$

$$x^2(3x^2 - 2y^2 - 2z^2)^2 + y^2(-2x^2 + 3y^2 - 2z^2)^2 + z^2(-2x^2 - 2y^2 + 3z^2)^2$$

has no non-trivial zeros, and yet has a unique Gram matrix. (For algebraic reasons, the squares of products of irreducible, indefinite polynomials also have unique Gram matrices.)

As a token of appreciation in the other direction, the following result can be proved using sums of squares, and, ultimately, the Fundamental Theorem of Algebra. For an $n \times n$ square matrix $M = [m_{ij}]$, say that the *diagonal sums* are the sums of the $2n-1$ southwest-northeast diagonals. Then, if M is a psd matrix, then there is another psd matrix M' with rank 2, and the same diagonal sums as M. (In general, for $n \ge 2$, there are 2^{n-2} such matrices.)

1.8 Carsten Scherer

It is indisputable that linear matrix inequalities have played a fundamental role for recent advances in robust control. Many interesting problems could be successfully subsumed, at least conceptually, to the the generic LMI framework. It has turned out, however, that the suggested generic schemes suffer in their practical application from substantial trouble with computational complexity and numerical reliability.

It is our strong believe that the main reason for these deficiencies are rooted in a rather incomplete understanding of how a particular control theoretic structure can be effectively reflected in LMI solvers for reduced complexity and improved numerical stability. Research in these directions could have a profound impact on the further dissemination of LMI techniques within the whole field of control.

The following questions could provide an initial thrust for progress:

1. If considering an interconnection of linear time-invariant dynamical systems, can one quantify the relaxation error if searching for structured storage functions in order to prove dissipativity of the interconnection?

2. How can *external* structure of a system be systematically translated into *internal* structural properties of some/all storage functions?

3. Does there exists an H_∞-synthesis algorithm for which the the increase of the McMillan degree of weighting functions only leads to a moderate increase of computational complexity?

1.9 Konrad Schmüdgen

My research interest at the workshop concerns two topics:

1. Noncommutative Positivstellensaetze

   This means possible generalizations of classical Positivstellensaetze to noncommmutative star algebras such as the Weyl algebra or enveloping algebras of Lie algebras. In two recent papers I have obtained such generalizations of Bruce Reznicks uniform denumerator theorem and of Putinar and Vasilescus results in their Annals paper (2000).

2. Moment problems for closed semialgebraic sets

   It is the question for which noncompact closed semialgebraic sets the moment problem is solvable or the assertion of the Archimedean Positivstellensatz holds.

1.10 Levent Tunçel

Even though it was difficult to pick just a few topics and open problems for the workshop, I eventually decided to mention the following two areas:

- Computational complexity theory for LMI based convex relaxation methods
- Representation theory for LMIs

LMI Based Convex Relaxation Methods

There are many methods that solve systems of polynomial inequalities by computing the convex hull of the solution set. In particular, the procedures proposed by Lasserre [11, 12], Parrilo [15], and [9, 10] (the last approach works with systems of quadratic inequalities, so the original polynomial system needs to be reformulated using additional variables) do the job. The convergence theory for the procedures of Lasserre as well as Parrilo rely on the various foundational theorems of Putinar [17], Schmüdgen [19], Curto and Fialkow, Reznick, etc.

Currently, there is a rough complexity analysis for the Successive Convex Relaxation Methods based on the quadratic inequality representation (see Kojima and Takeda [8]; for an extension to discretized version see [20]). Given any Polynomial Optimization Problem, we can first write the polynomial system as a quadratic inequality system and then apply the existing theory. However, the existing theory can certainly improved. Moreover, it would be more desirable to directly analyze the computational complexity of the procedures on the original Polynomial Optimization Problem.

There are some obvious complexity measures one can propose for such analysis. It seems that interesting measures of *distance to convexity of a set*, *diameter*, *Lipschitz constants*, *amount of nonconvexity of the initial formulation*, etc. will come into play.

Representation Theory for LMIs

Generalized Lax conjecture "all hyperbolic cones can be expressed as a linear subspace intersected with a positive semidefinite cone" is perhaps one of the most interesting open problems here. For progress on this conjecture (including a proof of the original Lax conjecture), see [7, 13, 21]. (Also mentioned by Henrion and Hilton-McCullough-Vinnikov.)

However, I would like to mention another related problem. Let's us call the above representation *LMI representation* and define a more general tool of representations. Namely, those which also allow projecting away some of the variables (for some fundamental results on these representations, pertaining to optimization, see [4, 6]):

Definition 1.10.1. $G \subset \mathbb{R}^d$ *is said to* admit a lifted-LMI representation *if there exists* $\mathcal{L} : \mathbb{R}^d \oplus \mathbb{R}^m \to \mathbb{R}^n$ *a linear map such that*

$$x \in \text{int}(G) \iff \mathcal{L}(x, u) \succ 0 \text{ for some } u \in \mathbb{R}^m.$$

Note that a convex cone G admits a lifted-LMI representation iff its dual cone G^* does. In addition to this symmetry property, lifted-LMI representations cover a larger class of convex sets than the LMI representations and in terms of optimization algorithms which are used to solve the underlying optimization problems, lifted-LMI representations provide no additional difficulty (at least from a theoretical viewpoint).

G admits a *poly-time, lifted-LMI representation* if

$$\max\{m, n\} = O(\text{poly}(d)),$$

where poly$(\cdot)$ is a polynomial.

Open Problem 1.10.1. *Characterize all d-dimensional, pointed, closed, convex cones in $\mathbb{R}^d$ which admit a lifted-LMI representation. I believe that this set of cones strictly contain the hyperbolic cones.*

Open Problem 1.10.2. *Characterize all d-dimensional, pointed, closed, convex cones in $\mathbb{R}^d$ which admit a poly-time, lifted-LMI representation. I believe that this set of cones strictly contain the hyperbolic cones.*

Most specifically:

Open Problem 1.10.3. *Are all Hyperbolic Feasibility Problems polynomial-time equivalent to LMI problems?*

This last question needs some definitions and clarifications.

Definition 1.10.2. *Let $p_1, p_2, \ldots, p_m : \mathbb{R}^d \to \mathbb{R}$ be given polynomials. Then the problem "does there exist $x \in \mathbb{R}^d$ such that $p_i(x) \geq 0, \forall i \in \{1, 2, \ldots, m\}$ is a* Hyperbolic Feasibility Problem (HFP) *if every p_i is a hyperbolic polynomial.*

Next, we define the size(HFP). The "size" should involve the basic complexity measures needed to bound the amount of computational effort required (in the Blum-Shub-Smale real computation model) to "solve" HFP to $\epsilon \in (0, 1)$ accuracy using some general class of well-established algorithms. For instance, we can define

$$\text{size}(HFP) := \max\{m, \ln(1/\epsilon), \ln(R)\},$$

where $R > 1$ denotes the volume of a given ellipsoid E_0 which determines the region in which we will decide the solvability of HFP. I.e., our problem is to find $\bar{x} \in E_0$ satisfying all the inequalities. We require that after

$$\text{poly}(\text{size}(HLP)) \text{ operations}$$

the algorithm either outputs $\bar{x} \in \mathbb{R}^d$ such that $p_i(\bar{x}) \geq 0, \forall i \in \{1, 2, \ldots, m\}$ or it outputs

"there does not exist a ball of volume at least ϵ which is contained in

$$E_0 \cap \left\{x \in \mathbb{R}^d : p_i(x) \geq 0, \forall i \in \{1, 2, \ldots, m\}\right\}."$$

In this context, when we say *HFP is polynomial-time equivalent to LMI* we mean that for every HFP (with m, R and a given $\epsilon \in (0, 1)$), we can explicitly describe an LMI such that

- the formulated LMI can be solved to ϵ accuracy in time poly (size(HFP)),
- solving the LMI within accuracy ϵ, solves the original HFP.

This notion of poly-time equivalence is quite important in optimization theory.

A problem analogous to Problem 1.10.3 was solved in [5] by showing that Second Order Cone Programming is poly.-time equivalent to Linear Programming.

1.11 Hugo Woerdeman

One of the questions I am interested in is how to approximate numerically the Schur complement of a positive definite operator supported on a finite subspace. Of course there is the formula that for a block matrix [A B ; C D] the Schur complement is A B inv(D) C, but this requires determining the inverse of the infinite operator D. The way this question arose is through attempts to develop multivariable analogs of the Gohberg-Semencul formula. One way to prove the Gohberg-Semencul formula is by determining the Schur complement of the inverse of a Toeplitz operator whose symbol is the reciprocal of a positive trigonometric polynomial. In several variables the analogous attempt runs into difficulties. In order to at least get a decent numerical approximation of the inverse of a doubly Toeplitz matrix, one may try to find reasonable numerical methods to determine finitely based Schur complements of infinite operators.

Chapter 2

Ideas for Teaching

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Mihai:  OPEN FOR ADDITIONS, CORRECTIONS, REARRANGEMENTS
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ToDo

A sketch of a plan:

1. General convexity (Hahn Banach, Minkowski separation theorem, Caratheodory's theorem on generators of convex hulls)
2. Weighted sums of squares in free *-algebras
3. The spectral theorem for commuting self-adjoint operators. Note the spectral measure in physical terms is just the power spectral density.
4. Multivariate moment problems and their dual: weighted SOS decompositions of polynomials
5. Applications (optimization, Lyapunov functions, ...)
6. Real algebra, logic and the full Positivestellensatz
7. More optimization (see Tuncel, Henrion, Lasserre)

Chapter 3

Other Remarks

3.1 Leonid Gurvits

The van der Waerden conjecture states that the permanent of $n \times n$ doubly stochastic matrix A satisfies the inequality $Per(A) \geq \frac{n!}{n^n}$ (VDW bound) and was finally proven (independently) by D.I. Falikman and G.P. Egorychev in 1981 . They both shared Delbert Ray Fulkerson prize in 1982 .

It was for more than XX years the most important conjecture about permanents. The VDW bound is the simplest and most powerful bound on permanents and therefore among the simplest and most powerful general purpose bounds in combinatorics. We introduce and prove a vast generalization of the VDW conjecture :

Consider a homogeneous polynomial $p(z_1, ..., z_n)$ of degree n in n complex variables. Assume that this polynomial satisfies the property:

$$|p(z_1, ..., z_n)| \geq \prod_{1 \leq i \leq n} Re(z_i) \text{ on the domain } \{(z_1, ..., z_n) : Re(z_i) \geq 0, 1 \leq i \leq n\}.$$

We prove that $|\frac{\partial^n}{\partial z_1 ... \partial z_n} p| \geq \frac{n!}{n^n}$.

Our generalization not only affects the world of permanents, but also has important implications concerning PDEs, stability and control theory , complexity theory. Besides, our proof is much shorter and conceptually simpler than original proofs as of the van der Waerden conjecture for permanents as well of the Bapat's conjecture on mixed discriminants, proved by the author . The paper with the proof is available at

`http://lanl.arxiv.org/abs/math.CO/0504397`,

see also the paper at

`http://xxx.lanl.gov/abs/math.CO/0404474`.

3.2 Alexandre Megretski

See separate paper "Optimal Model Order Reduction for Finite Length Segments of LTI System Unit Sample Response".

3.3 Jiawang Nie

Some topics in polynomial optimization:

1. Using gradients in SOS for approximating minimizing polynomials
2. Convergence rate of Lasserre's method for solving constrained polynomial optimzation.
3. Practical sos/moment methods for Maximum-Likelyhood estimations

3.4 Frank Vallentin

I am a beginner (or more precisely a user) in the theory of linear matrix inequalities and my main interest in the workshop is to learn about applications and possibilities of LMIs as well as about open problems in this theory.

In the past years I was a user of LMIs for problems in classical lattice geometry. In my Ph.D. thesis I developed and implemented algorithms for solving lattice packing and covering problems which are based on semidefinite programming. Currently I am interested in LMIs and convex optimization problems which have many symmetries and usually come from discrete geometry or combinatorics.

3.5 Jan Willems

I am especially interested in obtaining new insights concerning the relations between LMI's and dissipative systems. In particular, I would like to learn how the recent methods involving multivariable polynomials lead to the construction of storage functions and Lyapunov functions for systems described in terms of differential equations using polynomial matrices or matrices of rational functions.

Bibliography

[1] N. I. Akhiezer, *The Classical Moment Problem and Some Related Questions in Analysis*, Oliver and Boyd, Edinburgh, 1965.

[2] B. Fuglede, The multidimensional moment problem, *Expo. Math.* 1(1983), 47-65.

[3] C. Berg and C. Thill, Rotation invariant moment problems, *Acta Math.* 167(1991), 207-227.

From Levent Tunçel:

[4] A. Ben-Tal and A. Nemirovski, *Lectures on Modern Convex Optimization Analysis, Algorithms, and Engineering Applications*, MPS-SIAM Series in Optimization, SIAM, Philadelphia, PA, USA, 2001.

[5] A. Ben-Tal and A. Nemirovski, On polyhedral approximations of the second-order cone, *Math. Oper. Res.* 26 (2001) 93–205.

[6] C. B. Chua and L. Tunçel, Invariance and efficiency of convex representations, *Research Report* 2004–18, Dept. of Combinatorics and Optimization, Faculty of Mathematics, University of Waterloo, Ontario, Canada, June 2004.

[7] J. W. Helton and V. Vinnikov, Linear matrix inequality representation of sets, *Technical Report*, 2002.

[8] M. Kojima and A. Takeda, Complexity analysis of successive convex relaxation of nonconvex sets, *Math. Oper. Res.* 26 (2001) 519–542.

[9] M. Kojima and L. Tunçel, Cones of matrices and successive convex relaxations of nonconvex sets, *SIAM J. Optimization* 10 (2000) 750–778.

[10] M. Kojima and L. Tunçel, Discretization and localization in successive convex relaxation methods for nonconvex quadratic optimization problems, *Math. Prog. A* 89 (2000) 79–111.

[11] J. B. Lasserre, Polynomials nonnegative on a grid and discrete optimization, *Trans. Amer. Math. Soc.* 354 (2002) 631–649.

[12] J. B. Lasserre, An explicit equivalent positive semidefinite program for nonlinear 0-1 programs, *SIAM J. Optim.* 12 (2002) 756–769.

[13] A. S. Lewis, P. A. Parrilo and M. V. Ramana, The Lax conjecture is true, *Technical Report*, April 2003.

[14] Yu. E. Nesterov and A. S. Nemirovskii, *Interior-Point Polynomial Algorithms in Convex Programming*, SIAM, Philadelphia, PA, USA, 1994.

[15] P. A. Parrilo, Semidefinite programming relaxations for semialgebraic problems. Algebraic and geometric methods in discrete optimization, *Math. Program. Ser. B* 96 (2003) 293–320.

[16] P. A. Parrilo and B. Sturmfels, Minimizing polynomial functions. Algorithmic and quantitative real algebraic geometry (Piscataway, NJ, 2001), 83–99, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., 60, Amer. Math. Soc., Providence, RI, 2003.

[17] M. Putinar, Positive polynomials on compact semi-algebraic sets, *Indiana Univ. Math. J.* 42 (1993) 969–984.

[18] J. Renegar, Hyperbolic Programs and Their Derivative Relaxations, *Technical Report*, March 2004.

[19] K. Schmüdgen, The K-moment problem for compact semi-algebraic sets, *Math. Ann.* 289 (1991) 203-206.

[20] L. Tunçel and S. Xu, Complexity analyses of discretized successive convex relaxation methods. *Research Report* 99–37, Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario, Canada, September 1999.

[21] V. Vinnikov, Self-adjoint determinantal representations of real plane curves, *Mathematische Annalen* 296 (1993) 453–479.

From Antonis Papachristodoulou:

[22] P. Apkarian and H. D. Tuan. Parameterized LMIs in control theory. *SIAM J. Control Optim.*, 38(4):1241–1264, 2000.

[23] G. Chesi, A. Garulli, A. Tesi, and A. Vicino. Polynomially parameter-dependent Lyapunov functions for robust stability in polytopic systems: An LMI approach. *IEEE Transactions on Automatic Control*, 50(3):365–370, 2005.

[24] K. Gatermann and P. A. Parrilo. Symmetry groups, semidefinite programs, and sums of squares. *Journal of Pure and Appl. Algebra*, 192(1–3):95–128, 2004.

[25] C. W. J. Hol and C. W. Scherer. Sum of squares relaxations for polynomial semidefinite programming. In *Proc. of the 16th International Symposium on MTNS*, 2004.

[26] A. Papachristodoulou, M. Peet, and S. Lall. Constructing Lyapunov-Krasovksii functionals for linear time delay systems. In *Proceedings of the American Control Conference*, 2005.

[27] A. Papachristodoulou and S. Prajna. Analysis of non-polynomial systems using the sum of squares decomposition. To appear in *Positive Polynomials in Control*, Springer-Verlag, 2004.

[28] F. Wu and S. Prajna. A new solution approach to polynomial LPV system analysis and synthesis. In *Proceedings of the American Control Conference*, 2004.