AIM Workshop on Spectra of Families of Matrices described by Graphs, Digraphs and Sign Patterns

Open Questions

(prepared by In-Jae Kim and Amy Wangsness Wehe)

September 3, 2007

Abstract

The workshop, *Spectra of Families of Matrices described by Graphs, Digraphs and Sign Patterns*, was held at the American Institute of Mathematics Research Conference Center on October 23 to October 27, 2006, focused on three problems: 1. Determination of the minimum rank of real symmetric matrices described by a graph; 2. The $2n$-conjecture for spectrally arbitrary sign patterns; and 3. The energy of a graph. This is a report for the workshop that is mainly concerned with the open questions raised during the workshop regarding those three problems. This report is based on the group reports by the participants and the final report by the organizers.

1 Minimum Rank of Symmetric Matrices described by a Graph

The workshop has extended research on the relationship between spectral properties of real symmetric matrices and the combinatorial arrangements of their nonzero entries described by graphs in several directions. Research projects developed at the workshop are concerned with (1) minimum rank over fields other than the field of real numbers $\mathbb{R}$, (2) effect of graph operations on minimum rank such as the complement and powers of a graph, (3) inertially balanced graphs and (4) computation of the minimum rank of a graph.

Let G be a (simple) graph and let $S_n(\mathbb{F})$ be the set of symmetric matrices over a field $\mathbb{F}$. The *minimum rank* of G over field $\mathbb{F}$ is

$$\mathrm{mr}^{\mathbb{F}}(G) = \min\{\mathrm{rank}(A) : A = [a_{ij}] \in S_n(\mathbb{F}) \text{ and for } i \neq j,\ a_{ij} \neq 0 \text{ if and only if } ij \text{ is an edge of } G\}.$$

When $\mathbb{F} = \mathbb{R}$, mr(G) denotes $\mathrm{mr}^{\mathbb{R}}(G)$. The following two questions concern a relationship among the minimum ranks of a graph over various fields. Henceforth, the questions **Q** are open problems raised at the workshop.

Q1. Does there exist a graph G for which $\mathrm{mr}^{\mathbb{R}}(G) > \mathrm{mr}^{\mathbb{C}}(G)$?

Q2. Does there exist a graph G for which $\mathrm{mr}^{\mathbb{Q}}(G) > \mathrm{mr}^{\mathbb{R}}(G)$?

It was shown at the workshop that if G is a connected graph with at most 6 vertices and $\mathbb{F}$ is an infinite field of characteristic not 2, then $\mathrm{mr}^{\mathbb{Q}}(G) = \mathrm{mr}^{\mathbb{R}}(G) = \mathrm{mr}^{\mathbb{C}}(G)$.

Let $|G|$ denote the number of vertices of a graph G and let $\delta(G)$ be the minimum degree of a vertex in graph G.

Q3. For any graph G and infinite field $\mathbb{F}$, $\mathrm{mr}^{\mathbb{F}}(\mathrm{G}) \leq |\mathrm{G}| - \delta(\mathrm{G})$.

Question **Q3** is true for any bipartite graph and some graphs with some further restrictions on $|G|$, a cut vertex and/or $\delta(G)$. Minimum rank of non-symmetric matrices described by a graph was also investigated. A non-symmetric version of **Q3** was established and used to prove the result on bipartite graphs.

The basic question regarding the minimum ranks of a graph G and its complement $\overline{G}$ is

Q4. How large can $\mathrm{mr}(\mathrm{G}) + \mathrm{mr}(\overline{\mathrm{G}})$ be? There are two possibilities:

1. Does there exist a constant $c \geq 2$ such that $\mathrm{mr}(G) + \mathrm{mr}(\overline{G}) \leq |G| + c$? if so, find the smallest such c.
2. If not, find the best constant $d \leq 2$ such that $\mathrm{mr}(G) + \mathrm{mr}(\overline{G}) \leq d|G|$.

It was shown at the workshop that all the graphs for which the minimum ranks of both $\mathrm{mr}(\mathrm{G})$ and $\mathrm{mr}(\overline{\mathrm{G}})$ are known so far satisfy $\mathrm{mr}(\mathrm{G}) + \mathrm{mr}(\overline{\mathrm{G}}) \leq |\mathrm{G}| + 2$.

Consider a more general set of graphs. Given a set S of size n, together with a collection $\mathcal{S} = \{S_1, \dots S_m\}$ of subsets of S, let $G_{\mathcal{S}}$ be the graph with vertex set $\mathcal{S}$, where two subsets S_i and S_j are adjacent whenever they intersect. Thus if every S_i has size 2, then G_s is a line graph.

If we restrict $\mathcal{S}$ to all subsets S with a given size k, then $G_{\mathcal{S}}$ is the complement of the Kneser graph $K(n, k)$. It is known that if $\mathcal{S}$ consists of all subsets of S, then $\mathrm{mr}(G_{\mathcal{S}}) = n$. It is also known that the minimum rank equals $n - 2$ if $k = 2$. It is not difficult to show that the minimum rank is at most $n - 1$ and at least $n - 2k + 2$ (provided $1 < k \leq n/2$).

Q5. Is the lower bound on the minimum rank of such graphs sharp?

The conjecture if true if $k = 2$ and $k = n/2$.

Let $G = (V, E)$ be a graph with vertex set $V = \{1, 2, \dots, n\}$ and edge set E. The *j-th power of graph G* is the graph $G^j = (V, F)$ where $uv \in F$ $(u \neq v)$ if and only if there is a walk of length j between u and v. The question considered on powers of a graph G are

Q6. What is the relationship between $\mathrm{mr}(G)$ and $\mathrm{mr}(G^j)$?

Q7. Characterize the graphs G for which $\mathrm{mr}(G^j) \geq \mathrm{mr}(G^{j+1})$ for all $j \geq 1$.

For trees T, **Q7** can be rephrased as the following.

Q8. Is it the case that for each tree $T \neq K_{1,n-1}$, the sequence $\mathrm{mr}(T), \mathrm{mr}(T^2), \dots$ decreases strictly until it hits 2?

In order to determine the minimum rank for several classes of graphs, a central role is played by the relative position of the packet of zero eigenvalues in the spectrum of a matrix (so called an *optimal matrix*) that achieves the minimum rank. The *inertia* of a real symmetric matrix A is an ordered integer triple $(i_+(A), i_-(A), i_0(A))$ where $i_+(A)$ (resp. $i_-(A)$ and $i_0(A)$) is the number of positive (resp. negative and zero) eigenvalues of A. A matrix A is *inertially balanced* if $i_-(A) \leq i_+(A) \leq i_-(A) + 1$. An *inertially balanced graph* is a graph that has an inertially balanced optimal matrix. A graph G is *minimal non-inertially-balanced* if G is not inertially balanced, while all proper induced subgraphs of G are inertially balanced. The fact that at present there are no examples of graphs that fail to be inertially balanced leads us to the following question.

Q9. Are all graphs inertially balanced?

Balanced inertia is related to the notion of *rank-spread* $r_v(G)$ of a graph G at a vertex v that is defined to be $r_v(G) = \mathrm{mr}(G) - \mathrm{mr}(G \setminus v)$, where $G \setminus v$ is the induced subgraph of G obtained by deleting vertex v and all the incident edges to v. For instance, it was shown at the workshop that a minimal non-inertially-balanced graph has no (so called *rank-strong*) vertices v with $r_v = 2$ and no (pendant) vertices of degree one. From this result, it can be shown that the existence of a sequence of rank-strong vertices provides a method for the construction of optimal inertially balanced matrices. A natural question arising in this context is

Q10. Do there exist graphs with only rank-strong vertices?

2 Spectrally Arbitrary Sign Patterns and the $2n$-Conjecture

An $n \times n$ *zero-nonzero pattern* $\mathcal{A}$ *over a field* $\mathbb{F}$ is an $n \times n$ matrix whose entries are in $\{*, 0\}$ where $*$ denotes a nonzero element in $\mathbb{F}$. If $\mathbb{F} = \mathbb{R}$ and $*$ is replaced by $\pm$, then $\mathcal{A}$ is a *sign pattern*. Spectrally arbitrary sign (or zero-nonzero) patterns allow every possible spectrum of a real matrix, or equivalently allow every monic real polynomial as the characteristic polynomial. Inertially arbitrary sign (or zero-nonzero) patterns allow every possible inertia.

The $2n$-conjecture asserts that an $n \times n$ spectrally arbitrary pattern must contain at least $2n$ nonzero entries. It is known that an $n \times n$ irreducible spectrally arbitrary matrix has at least $2n - 1$ nonzero entries, and numerous examples of $n \times n$ spectrally arbitrary sign patterns with $2n$ nonzero entries are known. The crux of each known proof for the existence of at least $2n - 1$ nonzero entries is the simple fact that if a polynomial function $f : \mathbb{R}^k \to \mathbb{R}^n$ is surjective, then necessarily $k \geq n$. The particular polynomial function of interest for an $n \times n$ sign (or zero-nonzero) pattern $\mathcal{A}$ with m nonzero entries is constructed by choosing a collection of $n - 1$ nonzero entries that correspond to a spanning tree in the underlying graph of the digraph of $\mathcal{A}$; setting $A(x_1, \ldots, x_{m-n+1})$ to be the matrix with the chosen $n-1$ entries equal to 1 and the remaining $m-n+1$ nonzero entries being indeterminates $x_1, \ldots, x_{m-n+1}$; and then defining $f_{\mathcal{A}} = (p_1(x_1, \ldots, x_{m-n+1}), \ldots, p_n(x_1, \ldots, x_{m-n+1}))$, where $\det(xI - A) = x^n + p_1(x_1, \ldots, x_{m-n+1})x^{n-1} + \cdots + p_n(x_1, \ldots, x_{m-n+1})$. If the irreducible $2n$-conjecture is true, then it is suspected that the polynomial function $f_{\mathcal{A}}$ for a spectrally arbitrary pattern $\mathcal{A}$ has special properties.

Q1. What are the necessary and sufficient conditions for a polynomial function f to be equal to $f_{\mathcal{A}}$ for some spectrally arbitrary sign (or zero-nonzero) pattern $\mathcal{A}$?

Q2. Is there anything special about $f_{\mathcal{A}}$ coming from $\det(xI - A(x_1, \ldots, x_{m-n+1}))$ rather than $\chi(xI - A(x_1, \ldots, x_{m-n+1}))$ where χ is an immanent?

In order to gain additional insight into the $2n$-conjecture, the following related problem was posed and studied at the workshop. Let $\mathcal{A}$ be an $n \times n$ sign (or zero-nonzero) pattern and let $\beta = \{i_1, \ldots, i_k\}$ be a subset of $\{1, \ldots, n\}$. Then $\mathcal{A}$ is a β-spectrally arbitrary pattern provided that for each k-tuple $(r_1, \ldots, r_k)$ of real numbers, there is a realization A of $\mathcal{A}$ whose characteristic polynomial $x^n + \sum_{i=1}^{n} \alpha_i x^{n-i}$ satisfies $\alpha_{i_j} = r_j$ for $j = 1, \ldots, k$.

Q3. For an integer k with $1 \leq k \leq n$, what is the minimum number of nonzero entries in an $n \times n$ irreducible sign (or zero-nonzero) pattern that is β-arbitrary for some β with $|\beta| = k$?

The notion of a spectrally arbitrary zero-nonzero pattern can be extended to an arbitrary field as follows. An $n \times n$ zero-nonzero pattern $\mathcal{A}$ is *spectrally arbitrary over a field* $\mathbb{F}$ provided that every monic polynomial of degree n in $\mathbb{F}[x]$ is the characteristic polynomial of some matrix with zero-nonzero pattern $\mathcal{A}$ and entries in $\mathbb{F}$.

Q4. Does every $n \times n$ zero-nonzero pattern over an infinite field with nonzero characteristic have at least $2n$ nonzero entries?

It was shown at the workshop that the $2n$-conjecture is true for zero-nonzero patterns over finite field, and that the classes of spectrally arbitrary zero-nonzero patterns over $\mathbb{R}$ and over $\mathbb{C}$ are different.

The recent discoveries of very sparse, reducible inertially arbitrary zero-nonzero patterns and a spectrally arbitrary sign pattern (SAP for short) that is a direct sum of a non-SAP and an SAP suggest that one might be able to use direct sums of non-SAPs to construct an $n \times n$ SAP with fewer than $2n$ nonzero entries. Research in this direction is concerned with the following two problems.

Q5. Does there exist a reducible $n \times n$ spectrally arbitrary sign pattern with fewer than $2n$ nonzero entries?

Q6. Does there exist a spectrally arbitrary sign pattern that is a direct sum of two sign patterns that are not spectrally arbitrary?

The necessary conditions for a sign pattern $\mathcal{A} = [\alpha_{ij}]$ to be inertially or spectrally arbitrary are that (N1) there exist indices i, j such that $\alpha_{ii} = +$ and $\alpha_{jj} = -$ and (N2) there exist indices k, ℓ such that $\alpha_{k,\ell}\alpha_{\ell,k} = -$. Since a full sign pattern (with no zeros) has so much freedom in choosing values for its nonzero entries, it is natural to ask if a full sign pattern $\mathcal{A}$ satisfying the necessary conditions (N1) and (N2) is spectrally arbitrary. An investigation suggests that we need at least one addtional condition that $\mathcal{A}$ has a nilpotent realization.

Q7. Is a potentially nilpotent full sign pattern $\mathcal{A}$ with the conditions (N1) and (N2) necessarily spectrally arbitrary (or inertially arbitrary)?

Another question regarding full sign patterns is:

Q8. Is there a spectrally (inertially) arbitrary sign pattern having a (full) superpattern that is not spectrally (inertially) arbitrary?

3 Energy of Graphs

The *energy* $E(G)$ *of a graph* G is the sum of the absolute values of the eigenvalues of the adjacency matrix. It has applications to chemistry. Certain quantities of importance to chemists, such as the heat of formation of a hydrocarbon, are related to π-electron energy that can be calculated as the energy of an appropriate "molecular" graph. The following questions arose as a result of the study on the effect on energy of adding, removing or subdividing an edge of a graph. The subgraph of a graph G obtained by deleting an edge e is denoted by $G \setminus e$.

Q1. If e is an edge of a connected graph G such that $E(G) = E(G \setminus e) + 2$, then is it true that $G = K_2$ where K_2 is the complete graph of order 2?

Q2. Are there any graphs G such that

$$E(G \setminus e) = E(G) + 2$$

for some edge e of G?

Q3. Which connected graphs have an edge e such that $E(G \setminus e) = E(G)$?

Q4. Let H be a graph obtained by subdividing an edge of a connected graph G. If $E(H) < E(G)$, then what are necessary conditions for G?

A quantity investigated at the workshop is the maximal energy per vertex over k-regular graphs G on n vertices,

$$f(k) = \max \frac{E(G)}{n}.$$

Q5. If q is a prime power, then is it true that

$$f(q+1) = \sqrt{q} + \frac{1}{q + \sqrt{q} + 1}?$$

Let G be a graph on n vertices with m edges and let L be the Laplacian matrix of G with eigenvalues $\lambda_1, \dots, \lambda_n$. The *Laplacian energy* $E_L(G)$ of graph G is the sum of the absolute values of eigenvalues $\lambda_1, \dots, \lambda_n$ minus the average degree $\frac{2m}{n}$. Let $d = (d_1, \dots, d_n)$ be the degree sequence of G, $\lambda = (\lambda_1, \dots, \lambda_n)$ and let $d^* = (d_1^*, \dots, d_n^*)$ be the conjugate sequence of d.

Q6. (Grone-Merris conjecture) Is λ majorized by d^*?

It was shown at the workshop that if the Grone-Merris conjecture is true, then

$$E_L(G) \leq \sum_{i=1}^{n} \left| d_i^* - \frac{2m}{n} \right|.$$