Remarks on hyperenergetic circulant graphs

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Abstract

We first settle an open problem of Balakrishnan from Linear Algebra Appl. 387 (2004) 287–295. Further, if $C(n, k_1, k_2, \ldots, k_m), n \in \mathbb{N}, k_1 < k_2 < \cdots < k_m < n/2, k_i \in \mathbb{N}$ for $i = 1, 2, \ldots, m,$ denotes a circulant graph with the vertex set $V = \{0, 1, \ldots, n - 1\}$ such that a vertex $u$ is adjacent to all vertices of $V \setminus \{u\}$ except $u \pm k_i \pmod{n}, i = 1, 2, \ldots, m,$ we show that for any given $k_1 < k_2 < \cdots < k_m$ almost all circulant graphs $C(n, k_1, k_2, \ldots, k_m)$ are hyperenergetic.

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Here we consider only simple graphs. The eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of a graph $G$ with $n$ vertices are the eigenvalues of its adjacency matrix $A(G)$. For other undefined notions, see [2]. The energy $E(G)$ of a graph $G$ is defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$
Energy of a complete graph $K_n$ is equal to $2(n-1)$. Earlier [4] it was conjectured that $K_n$ has the largest energy among all $n$ vertex graphs. After this conjecture has been disproved in [5], graphs for which $E(G) > 2(n-1)$ are called hyperenergetic graphs. (There is a typo in line 6 of [1, p. 288] in the definition of non-hyperenergetic graphs where it stands $E$ been disproved in [5], graphs for which $E(G) ≤ (2n-1)$ instead of $E(G) ≤ 2(n-1)$.)

In [1] Balakrishnan considered graphs $K_n - H$, where $H$ is a Hamilton cycle of $K_n$ and, based on computations, posed an open problem that $K_n - H$ is not hyperenergetic for $n ≥ 4$. We first solve this problem by showing that $K_n - H$ is indeed hyperenergetic for almost all $n ∈ N$.

Graph $\overline{C}(n,k_1,k_2,...,k_m)$, $n ∈ N$, $k_1 < k_2 < ... < k_m < n/2$, $k_i ∈ N$ for $i = 1, 2, ..., m$, is a circulant graph with the vertex set $V = \{0, 1, ..., n - 1\}$ such that a vertex $u$ is adjacent to all vertices of $V \setminus \{u\}$ except $u ± k_i (mod n)$, $i = 1, 2, ..., m$. Note that $K_n - H$ is actually $\overline{C}(n,1)$.

Adjacency matrix of $\overline{C}(n,k_1,k_2,...,k_m)$ is a circulant matrix with first row having 0s on positions 0, $k_1, ..., k_m$, $n - k_1, ..., n - k_m$ and 1s elsewhere. Thus, if $\omega = e^{i\frac{2\pi}{n}}$ is a primitive $n$th root of unity, eigenvalues of $\overline{C}(n,k_1,k_2,...,k_m)$ are of the form

$$\sum \{ \omega^k: 1 ≤ k ≤ n - 1, k ≠ k_i, n - k_i \text{ for } i = 1, 2, ..., m \},$$

for $j = 0, 1, ..., n - 1$. For $j = 0$ we get an eigenvalue $n - 1 - 2m$, and for $j = 1, ..., n - 1$ from $\sum_{k=1}^{n-1} \omega^k = -1$ we get an eigenvalue $-1 - \sum_{i=1}^{m} 2 \cos \frac{2\pi j}{n}$.

Thus,

$$E(\overline{C}(n,1)) = n - 3 + \sum_{j=1}^{n-1} -1 - 2 \cos \frac{2\pi j}{n}.$$

Note that

$$\frac{2\pi}{n - 1} \sum_{j=1}^{n-1} -1 - 2 \cos \frac{2\pi j}{n}$$

is an integral sum which tends to

$$\int_{0}^{2\pi} |-1 - 2 \cos x| \, dx$$

for $n → ∞$. So,

$$\lim_{n→∞} \frac{E(\overline{C}(n,1))}{n - 1} = 1 + \frac{1}{2\pi} \int_{0}^{2\pi} |-1 - 2 \cos x| \, dx = \frac{4\sqrt{3}}{2\pi} + \frac{4}{3} ≈ 2.43599,$$

which implies that for some $n_0 ∈ N$ it holds that $E(\overline{C}(n,1)) > 2(n - 1)$ for each $n ≥ n_0$. Our computations show that $n_0 = 10$.

Reason for Balakrishnan’s false computations and open problem that $K_n - H$ is not hyperenergetic lies in the fact that in line 8 of [1, p. 289] (s)he overlooked that for $j = 0$ the corresponding eigenvalue of $K_n - H$ is equal to $n - 3$, and not to $-3$. 
Motivated by the above approach using integral sums, we show the following

**Theorem 1.** Given $k_1 < k_2 < \cdots < k_m$ there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$ the graph $\overline{C}(n, k_1, k_2, \ldots, k_m)$ is hyperenergetic.

**Proof.** In order to prove the theorem, it is enough to show that

$$c_{k_1, \ldots, k_m} = \lim_{n \to \infty} \frac{E(\overline{C}(n, k_1, \ldots, k_m))}{n - 1} > 2.$$ 

Since

$$E(\overline{C}(n, k_1, \ldots, k_m)) = (n - 1 - 2m) + \sum_{j=1}^{n-1} \left| -1 - \sum_{i=1}^{m} 2 \cos k_i \frac{2\pi j}{n} \right|,$$

we have that

$$c_{k_1, \ldots, k_m} = 1 + \lim_{n \to \infty} \frac{1}{n - 1} \sum_{j=1}^{n-1} \left| -1 - \sum_{i=1}^{m} 2 \cos k_i \frac{2\pi j}{n} \right|.$$ 

As before, we note that

$$\frac{2\pi}{n - 1} \sum_{j=1}^{n-1} \left| -1 - \sum_{i=1}^{m} 2 \cos k_i \frac{2\pi j}{n} \right|$$

is an integral sum which for $n \to \infty$ tends to $\int_0^{2\pi} \left| -1 - \sum_{i=1}^{m} 2 \cos k_i x \right| \, dx$. Thus,

$$c_{k_1, \ldots, k_m} = 1 + \frac{1}{2\pi} \int_0^{2\pi} \left| -1 - \sum_{i=1}^{m} 2 \cos k_i x \right| \, dx.$$ 

It remains to show that

$$\int_0^{2\pi} \left| -1 - \sum_{i=1}^{m} 2 \cos k_i x \right| \, dx > 2\pi. \quad (1)$$

Since $|x| \geq -x$, one immediately has

$$\int_0^{2\pi} \left| -1 - \sum_{i=1}^{m} 2 \cos k_i x \right| \, dx \geq \int_0^{2\pi} 1 + \sum_{i=1}^{m} 2 \cos k_i x \, dx = 2\pi.$$ 

Since the function $-1 - \sum_{i=1}^{m} 2 \cos k_i x$ is continuous, in order to prove strict inequality in (1) it is enough to show that

$$\max_{x \in [0, 2\pi]} \left| -1 - \sum_{i=1}^{m} 2 \cos k_i x \right| > 0. \quad (2)$$

We know of no elementary proof of this simple fact: in order to prove it, we shall go back to eigenvalues of circulant graphs.

Consider graph $G = \overline{C}(n, k_1, \ldots, k_m)$ for $n > 4k_m$. Its eigenvalues are $n - 1 - 2m$ and $-1 - \sum_{i=1}^{m} 2 \cos k_i \frac{2\pi j}{n}$ for $j = 1, 2, \ldots, n - 1$, and the second largest eigenvalue of $G$ is equal to $-1 - \sum_{i=1}^{m} 2 \cos k_i \frac{2\pi j_0}{n}$ for some $j_0 \in \{1, 2, \ldots, n - 1\}$. For
each \( u \in \{0, 1, \ldots, n - 1\} \), among vertices \( u, u + k_m, u + 2k_m \) and \( u + 3k_m \) of \( G \) we have that \( \{u, u + 2k_m\}, \{u, u + 3k_m\} \) and \( \{u + k_m, u + 3k_m\} \) are pairs of adjacent vertices, while \( \{u, u + k_m\}, \{u + k_m, u + 2k_m\} \) and \( \{u + 2k_m, u + 3k_m\} \) are pairs of nonadjacent vertices. Thus, the subgraph of \( G \) induced by vertices \( u, u + k_m, u + 2k_m \) and \( u + 3k_m \) is isomorphic to \( P_4 \), which second largest eigenvalue is equal to \( -1 + \sqrt{5} \approx 0.618 \). By the Interlacing theorem [3, p.19], we have that the second largest eigenvalue of \( G \) is at least the second largest eigenvalue of any induced subgraph of \( G \), and thus

\[
-1 - \frac{1}{2} \sum_{i=1}^{m} 2 \cos \frac{k_i 2 \pi j_0}{n} \geq -1 + \frac{\sqrt{5}}{2},
\]

which implies (2). \( \square \)

References