MOCK MODULAR FORMS

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Following are brief statements of some problems raised during the AIM Workshop *Mock modular forms in combinatorics and arithmetic geometry*, March 8-12, 2010.

1. Basic modularity questions

When faced with a particular q-series, it is often useful to the modularity properties, if any, of this function.

Problem 1.1. *Given a q-series, determine fast methods to find (heuristically) if it is modular.*

The method should require knowledge of the weight, but not require knowledge of the level or group.

Remark. Zagier has a method involving asymptotics at a point.

Remark. It is worth exploring a p-adic method. For example, more coefficients should be divisible by primes than expected by chance.

Remark. One might consider the distribution of a_p and compare to Sato-Tate.

Problem 1.2. *Find a method to prove modularity directly from the sum (q-hypergeometric) expansion.*

For motivation, consider the Rogers-Ramanujan identities

$$G(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty},$$

$$H(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty};$$

it is apparent from the right hand side that $q^{-1/60}G(q)$ and $q^{11/60}H(q)$ are modular. How can we see this from the left hand side?

Problem 1.3. *Is there any kind of relations between the modularity of a series and the expansion in terms of q-orthogonal polynomials?*

Problem 1.4. *For vector-valued modular forms, do the matrices involved have interesting digitalization?*

For example, in the Rogers-Ramanujan example, the transformation $z \mapsto z+1$ leads to a diagonal matrix, where as $z \mapsto -1/z$ does not. Is the diagonalization of this interesting?

2. Bloch Group Method Problems

Work by Werner Nahm gives us a way to attack modularity questions with Algebraic K-theory. Let $B(F)$ denote the Bloch group for a field F. For more about the Bloch group, see Nahm's article *Conformal Field Theory and Torsion Elements of the Bloch Group*, which is available on the arXiv.

Let $A \in M_{r\times r}(\mathbb{Q})$ be a positive definite symmetric matrix. Let $B \in \mathbb{Q}^r$ and $C \in \mathbb{Q}$; define

$$f_{A,B,C}(\tau) := \sum_{n_!,\ldots,n_r \geq 0} \frac{q^{\frac{1}{2}\vec{n}A\vec{n}^t + B\vec{n} + C}}{(q)_{n_1}\cdots(q)_{n_r}},$$

where $q := e^{2\pi i\tau}$ and $(q)_m := (1-q)(1-q^2)\cdots(1-q^m)$, taking $(q)_0 := 1$.

In the case $r = 1$, Zagier proved that this function is only modular for seven choices of A, B, C, such as $(2, 0, -1/60)$ and $(2, 1, 11/60)$; the proof involves looking at the behavior of the function as q tends to one.

Nahm conjectures that for general r, there is an A for which $f_{A,B,C}(\tau)$ is modular if and only if the image of A in the Bloch group is torsion. If such an A exists, there may be multiple choices of B and C for which $f_{A,B,C}(\tau)$ is indeed modular.

Motivation for this idea comes from the study of the dilogarithm function and other related functions. For more on this, see Don Zagier's chapter '*The Dilogarithm Function* in "Frontiers in number theory, physics, and geometry II."

The problems throughout this section assume the notation above.

Problem 2.1. *Given a particular A, can we bound the number of possible B? Similarly, given a particular A, can we bound the dimension of the vector space determined by the B?*

The bounds may or may not be effective.

Problem 2.2. *Consider the Laplace transform of the q-hypergeometric series above. Note that the Mellin transform will not converge. Look at the properties, in particular the "jumps;" what can we say about the dependence on A, B, and C?*

Problem 2.3. *Look at explicit examples where $r \geq 2$ and $f_{A,B,C}(\tau)$ is modular. Can we write these in terms of natural objects? Similarly, can we related them to combinatorial identities; in the case $r = 1$, the two triples given relate to the Rogers-Ramanujan identities.*

Problem 2.4. *Relate Bailey pair techniques with Bloch group techniques. In particular, develop a dictionary. Perhaps this might might help with the computation of torsion. Similarly, connect orthogonal polynomials to Bloch group techniques.*

Problem 2.5. *Determine how mock modular forms fit into this theory.*

Problem 2.6. *How do q-analogues of the dilogarithm figure into the theory?*