A GLOSSARY OF SOME RELEVANT TERMS IN THE MODULI SPACE OF CURVES

This glossary contains brief definitions and explanations of many technical terms that arise is the study of the moduli space of curves. It was produced as part of the activities during the Topology and Geometry of the Moduli space of Curves workshop in March 2005; it is a component of a larger resource site at http://www.aimath.org/WWN/modspacecurves/

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1. Ample cone

Suppose X is a projective variety, with $\pi : X \hookrightarrow \mathbb{P}^n$ an inclusion (a "closed immersion"). Projective space has a natural line bundle $\mathcal{O}(1)$, and the pullback $\pi^*\mathcal{O}(1)$ is said to be a *very ample* line bundle on X. That is, a line bundle is very ample if it can be obtained by pulling back $\mathcal{O}(1)$ via a closed immersion into projective space. Equivalently, a line bundle is very ample if its global sections $s_0, \dots, s_n$ determine a closed immersion into projective space $[s_0, \dots, s_n] : X \hookrightarrow \mathbb{P}^n$. The tensor product of two very ample line bundles is again very ample.

A line bundle on a projective variety is *ample* if some tensor power of it is very ample. The ample cone is the convex cone in $H^2(X, \mathbb{Q})$ generated by $\{c_1(L) : L \text{ an ample line bundle on } X\}$.

The ampleness of a line bundle L is determined only by its first Chern class. More precisely, a line bundle L is ample if and only if, for every subvariety Z, $c_1(L)^k \cap [Z] > 0$, where $\dim Z = k$.

2. Arithmetic genus

The arithmetic genus of an irreducible, projective curve C is $\dim H^1(C, \mathcal{O})$, where $\mathcal{O}$ is the structure sheaf of holomorphic functions on C. For a smooth curve, this is the same as the geometric genus; however, unlike the geometric genus, the arithmetic genus has the nice feature that it remains constant in families of curves with possibly singular fibers. Intuitively this means that the arithmetic genus of a nodal curve is the geometric genus of the curve obtained by smoothing out the nodes.

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3. Boundary divisors

The boundary of $\overline{\mathcal{M}}_{g,n}$ is the complement of the open subset $\mathcal{M}_{g,n}$. It is of pure (complex) codimension 1. It consists of irreducible components Δ_i, $i = 0, \dots, [g/2]$ where a generic curve in Δ_0 is a (geometric) genus $g-1$ curve with a single node and a generic curve in $\Delta_i, i \neq 0$ consists of a genus i curve attached to a genus $g-i$ curve at a single node. Each boundary divisor is a finite-group quotient of a product of $\overline{\mathcal{M}}_{g',n'}$'s for $g' < g$ and $n' \leq n+1$. The subspace $\mathcal{M}_g - \Delta_0$ is called the locus of *curves of compact type.*

4. Classifying space

The classifying space BC of a category C is the geometric realization of the nerve $N(C)$. That is,

$$BC = (\amalg \triangle_k \times N(C)_k)/\sim$$

where the equivalence relation $\sim$ glues the k-simplices togetehr as specified by the face and degeneracy maps of $N(C)$. For a group G, we can consider the category with a single object and morphisms given by elements of G; in this case, this construction recovers the Borel construction BG. More generally, given a group G acting on a space X, we can construct a (topological) category whose objects are given by points in X and whose morphisms are given by elements of G. The classifying space of this category is the homotopy quotient $X//G = EG \times_G X$. If C is a strict symmetric monoidal category then BC will be an infinite loop space.

Note that in algebraic geometry, "BG" often refers to the stack-theoretic quotient [point$/G$].

5. Coarse and fine moduli spaces

Suppose we have some kind of algebraic object that we want to parametrize (say, curves or vector spaces). A *moduli functor* is a contravariant functor F : Schemes $\to$ *Sets* which sends a scheme X to the set of isomorphism classes of families of these objects parametrized by X (e.g. families of curves over X). A morphism f of schemes is sent to the set map induced by pulling back families via f.

A *fine moduli space* is a scheme M representing this functor. That is to say, there is a universal family over M and, for every family of objects over a scheme X, there is a unique map $X \to M$ which induces it by pulling back the universal family. This is usually impossible (for example, when there are extra automorphisms for some objects). If we are willing to enlarge the category in which we work, we can sometimes obtain a fine moduli space by working with stacks instead of schemes. Otherwise, the *coarse moduli space* is the scheme which best approximates the fine moduli space. By this, we mean that given a family of objects over a parameter space X, there will be a unique map from X to the coarse space M, and M is "universal" with respect to this property, which is to say that if there is also M' with this property, then the map $X \to M'$ factors uniquely as $X \to M \to M'$. Note that not every map to the coarse moduli space M gives rise to a family. In particular there may not be a universal family of objects over M itself.

Another requirement for a coarse moduli space is that its points should be in bijection with the objects being parametrized. I.e. algebraically-closed-field-valued points should be in bijection with the objects defined over that algebraically closed field.

6. Deligne-Mumford compactification

The Deligne-Mumford compactification is obtained as a moduli space for stable genus g curves with marked points, (instead of considering just smooth curves, as in $\mathcal{M}_{g,n}$). A stable genus g curve is a connected, projective curve with at worst nodal singularities and finite automorphism group. This translates to: every genus 0 irreducible component has at least three marked or nodal points and every genus 1 component has at least one marked or nodal point. This space is important because it is a smooth compactification (as a stack, at least) with easy-to-understand boundary components that give an inductive structure to all the moduli spaces of curves. For instance, Deligne and Mumford used this compactification to prove that $\mathcal{M}_g$ is irreducible for any characteristic.

7. Deligne-Mumford stack

A Deligne-Mumford stack is a space that “locally has the structure of a variety modulo a finite group.” The stackiness comes from remembering this presentation, in a manner similar to the definition of an orbifold in usual topology. Many moduli problems are naturally represented by D-M stacks and not by schemes or varieties (for instance, they were defined to handle the moduli functor of stable curves).

8. Dyer-Lashof-Araki-Kudo operations

These are operations on the mod-p homology of infinite loop spaces. They measure the failure of the Pontrjagin product to be commutative at the chain-level and are analogous to the Steenrod squares. The operations are linear maps of the form

$$\beta^{\epsilon}Q^{r} : H_n(X;\mathbb{F}_p) \to H_{n+2r(p-1)-\epsilon}(X;\mathbb{F}_p)$$

for $\epsilon \in \{0,1\}$ and $r \in \mathbb{Z}_{\geq \epsilon}$.

9. Effective cone

A line bundle is effective if it has a nonzero holomorphic section. The effective cone is the convex cone generated by $\{c_1(L) | L \text{ an effective line bundle}\}$.

10. Frobenius algebra

A (commutative) Frobenius algebra is a finite-dimensional commutative, associative, unital algebra A (over a field or ring k) with a trace map $\mathrm{Tr} : A \to k$ for which the bilinear form $\mathrm{Tr}(xy)$ is nondegenerate.

11. Frobenius manifold

A Frobenius manifold is a manifold M equipped with a smoothly (or analytically) varying structure of a Frobenius algebra on each tangent space which satisfies a series of integrability conditions (e.g. the induced metric is required to be flat). The most relevant example arises in Gromov-Witten theory where the genus 0 theory of a variety X imposes a Frobenius manifold structure on $H^*(X, \mathbb{C})$ (more precisely, the structure of a formal, graded-commutative Frobenius manifold). The integrability condition in this case is equivalent to the associativity of the quantum product. Other examples arise in the deformation theory of isolated singularities and in Hodge theory.

12. General type

A variety X is of *general type* if its Kodaira dimension is equal to $\dim X$. It is a theorem of Eisenbud, Harris, and Mumford that $\mathcal{M}_g$ is of general type when $g \geq 24$.

13. Geometric genus

The geometric genus of a smooth connected curve C is what one expects and can be defined in many ways, e.g. $\frac{1}{2}\dim H^1(C)$. For an irreducible, singular curve, the geometric genus is then defined to be the genus of its normalization (the curve obtained by ungluing all the nodes).

14. Group completion

Given a topological monoid M, the group completion is the space ΩBM, where BM is the classifying space of M (thinking of M as a topological category). If M is already a topological group, then this operation does not change M up to homotopy equivalence. Under some assumptions, we have the following description of the homology of the group completion. If we treat the monoid $\pi_0(M)$ as a directed system (with maps given by the monoid operation), then

$$\lim_{\alpha \in \pi_0(M)} H_*(M_\alpha) = H_*((\Omega BM)_0)$$

in situations where the direct limit on the left-hand side is well-defined. In particular, if we consider the monoid $\amalg_g B\Gamma_{g,2}$, the homology of the group completion is precisely the stable homology. The plus construction can often be used to give an alternative construction of the group completion.

15. Gromov-Witten invariants

Gromov-Witten invariants count (in a loose sense only) holomorphic maps from genus g Riemann surfaces to a variety X which pass through a given collection of cycles on X. In order to define these, we compactify the space of maps from a variable pointed curve C to X by allowing the domain curve to degenerate to a nodal curve so that the corresponding map always has finite automorphism group. For a fixed genus g, image homology class β, and number of marked points n, this gives the moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(X,\beta)$ which is typically a highly singular Deligne-Mumford stack. The Gromov-Witten invariants of X are given by integrals

$$\int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} ev_1^*(\alpha_1)\cdots ev_n^*(\alpha_n)$$

where $ev_i : \overline{\mathcal{M}}_{g,n}(X,\beta) \to X$ is evaluation at the i^{th} marked point and the α_i are elements of $H^*(X;\mathbb{Q})$. An important point of the theory is that this integral is defined via cap product with a distinguished homology class known as the virtual fundamental class of $\overline{\mathcal{M}}_{g,n}(X,\beta)$. The *descendent Gromov-Witten Invariants* are obtained by inserting monomials in the Witten classes ψ_i into the integral.

16. Harer stability

Harer stability states that the degree d homology of the mapping class group $\Gamma_{g,n}$ is independent of g and n if d is small compared to g. More precisely, consider the following maps on classifying spaces. First, we construct a map $B\Gamma_{g,b} \to B\Gamma_{g,b-1}$ by adjoining a disk to a given boundary component. Second, we can construct a map $B\Gamma_{g,b} \to B\Gamma_{g+1,b}$ by gluing a torus with two boundary components along a given boundary component of our original Riemann surface. Harer's stability theorem asserts that both of these maps induce an isomorphism on $H_d(-,\mathbb{Z})$ for $2d < g-1$. In particular, it allows us to talk about the stable homology/cohomology of the moduli space of curves, as in Mumford's conjecture.

17. Hodge bundle

This is another natural bundle on $\mathcal{M}_{g,n}$; the fibre over a curve C of genus g is the rank g vector space of differentials (holomorphic 1-forms) on C. This bundle can be extended naturally to a vector bundle over $\overline{\mathcal{M}}_{g,n}$ by considering certain meromorphic differentials on curves in the boundary.

18. Hodge classes, λ_k

These are the Chern classes of the Hodge bundle $\mathbb{E}$; that is, $\lambda_k = c_k(\mathbb{E})$. They live in the cohomology of either the compactified or non-compactified moduli space.

19. Homology fibration

A map of topological spaces $f : X \to Y$ is a homology fibration if, for every $y \in Y$, the natural map $f^{-1}(y) \to Pf_y$ from the fiber over y to the homotopy fiber over y induces an isomorphism on homology groups. By a theorem of McDuff and Segal, this is implied, for instance, by the condition that for sufficiently small neighborhoods U of y, the inclusion $f^{-1}(y) \to f^{-1}(U)$ induces an isomorphism on homology.

20. Hurwitz numbers

Hurwitz numbers give a count of genus g, degree d covers of $\mathbf{P}^1$ with ramification profile $\mu_1, \ldots \mu_b$ over fixed branch points $p_1, \ldots p_b$. Covers with automorphism group G are counted with weight $|G|$. The Hurwitz numbers can be calculated explicitly in terms of the character theory of the symmetric group. Remarkably, they can also be expressed in terms of tautological integrals on $\overline{\mathcal{M}}_{g,n}$ by the ELSV formula (see references below).

Ekedahl, Lando, Shapiro, Vainshtein, *On Hurwitz numbers and Hodge integrals.* C. R. Acad. Sci. Paris Sr. I Math. 328 (1999), no. 12, 1175–1180.

Hurwitz numbers and intersections on moduli spaces of curves. Invent. Math. 146 (2001), no. 2, 297–327.

21. Hurwitz schemes

The Hurwitz scheme $\mathcal{H}_{d,g}$ parametrizes simply branched covers of $\mathbf{P}^1$, i.e. pairs (C, π) of a genus g curve C and a degree d map $\pi : C \to \mathbf{P}^1$ with simple ramification. Variations on this definition include allowing more complicated ramification behavior and higher genus targets.

22. Infinite loop space

A space X is an infinite loop space if there is a sequence $X_0 = X, X_1, \ldots$ with homotopy equivalences $X_n \to \Omega X_{n+1}$, i.e. X can be de-looped arbitrarily many times. By adjunction, we have maps $\Sigma X_n \to X_{n+1}$ so $\{X_n\}$ form a spectrum. Conversely there is a functor from spectra to infinite loop spaces given by sending a spectrum $\mathcal{E}$ to $\lim \Omega^n E_n$.

23. Kappa classes

Given a genus g surface F, consider the universal F-bundle X over $B\mathrm{Diff}(F)$ and let $T^v X$ be its vertical tangent bundle with Euler class $e \in H^2(X)$. We define cohomology classes κ_i on $B\mathrm{Diff}(F)$ by

$$\kappa_i = \int_F e^{i+1} \in H^{2i}(B\mathrm{Diff}(F))$$

where $\int_F$ denotes the Gysin push-forward map (i.e. integration over the fiber). Mumford's conjecture states that these classes freely generate the stable cohomology ring of the mapping class group. A natural extension of these classes to the Deligne-Mumford compactified moduli space was given by Arbarello and Cornalba.

24. Kodaira dimension

Given an n-dimensional smooth projective variety X, we can study the canonical line bundle K_X of holomorphic n-forms. The dimensions of the spaces of global sections of $K_X^{\otimes r}$ are useful birational invariants of X which aid in the classification of varieties (birational means they only depend on a Zariski-open subset of X). As $r \to \infty$, these numbers either behave asymptotically like $C \cdot r^k$ for a unique integer k or are eventually zero. We define the Kodaira dimension to be this integer k in the first case and $-\infty$ in the second case.

Another intepretation is as follows. For each r, we have a rational map of X into projective space given by

$$[s_0, \ldots, s_m] : X \dashrightarrow \mathbb{P}^n$$

where $s_0, \ldots, s_n$ are the global sections of $K_X^{\otimes r}$. The Kodaira dimension is the supremum, as $r \to \infty$, of the dimension of the image of X under these maps. Hence the Kodaira dimension of X takes values in $\{-\infty, 0, \ldots, n\}$.

25. Mapping class group

Given a Riemann surface $F_{g,n}$ of genus g with n boundary components, the mapping class group $\Gamma_{g,n}$ is the group of diffeomorphisms of $F_{g,n}$ which fix the boundary pointwise, modulo the subgroup of diffeomorphisms isotopic to the identity. Equivalently, it is the group of connected components $\pi_0 \mathrm{Diff}(F_{g,n})$.

26. Moduli space of admissible covers

Admissible covers were developed by Harris and Morrison. The space of admissible covers is a modular compactification of the Hurwitz scheme obtained by allowing the target $\mathbf{P}^1$ to degenerate when branch points approach each other. The map $\mathcal{H}_{d,g} \to \mathcal{M}_g$ given by forgetting the map to $\mathbf{P}^1$ extends to a map $\overline{\mathcal{H}}_{d,g} \to \overline{\mathcal{M}}_g$.

A different compactification of the Hurwitz scheme was given by Abramovich and Vistoli. Their compactification has two advantages: (i) it is smooth, and (ii) it is a moduli space. The disadvantage is that is is a Deligne-Mumford stack ratehr than a scheme.

27. Nerve of a category

Given a category C, its nerve $N(C)$ is the simplicial set constructed as follows. The set $N(C)_k$ of k-simplices is the set of diagrams

$$A_0 \to A_1 \to \cdots \to A_k$$

of objects and morphisms from C. The face maps $\partial_i : N(C)_k \to N(C)_{k-1}$ are given by composition of morphisms atthe i^{th} node in the diagram (or dropping the first or last arrow if $i = 0$ or k respectively), and the degeneracy maps are given by inserting identity morphisms. The intuition here is that a k-simplex in $N(C)$is precisely a commutative diagram in C with the shape of a k-simplex. If C is a topological category, we can enrich $N(C)$ to be a simplicial space.

28. Partial compactifications of $\mathcal{M}_{g,n}$

These are geometrically defined intermediate subvarieties between $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ whose topology and tautological rings may also have a nice structure. They include the moduli space $\mathcal{M}^c_{g,n}$ of curves of compact type (i.e. $\overline{\mathcal{M}}_{g,n} - \Delta_0$, or those curves whose dual graph is a tree), $\mathcal{M}^{\leq k}_{g,n}$ (curves with $\leq k$ rational components), and $\mathcal{M}^{\mathrm{rat}}_{g,n}$ (curves consisting of a single genus g component attached to trees of rational tails).

29. Plus construction

Let X be a space and $H \lhd \pi_1(X)$ a normal subgroup that is perfect (i.e. $[H,H] = H$). The plus construction is the essentially unique space X^+ with fundamental group $\pi_1(X)/H$ and equipped with a map $X \to X^+$ which induces an isomorphism on homology for all coefficients. It is useful in situations where one is given a map $A \to B$ which is an isomorphism on homology groups but acts wildly on homotopy groups; in some situations, applying the plus construction can replace this homology equivalence with a homotopy equivalence—this is how the plus construction plays a role in group completion.

30. Pontrjagin-Thom construction

This is the construction which identifies cobordism groups of manifolds with the homotopy groups of certain spectra (written MO, MU, etc). The result is the following, along with many variations obtained by considering some additional structure on the stable normal bundle.

Theorem 1 (Pontrjagin-Thom)**.** *$\pi_n MSO$ is canonically isomorphic to the oriented cobordism group of n-manifolds.*

Given a manifold M^d smoothly embedded in Euclidean space $\mathbb{R}^{n+d}$, there exists a tubular neighborhood $U \supset M$ homeomorphic to the total space of the normal bundle ν^n of M. If we collapse everything outside of U to a point, we obtain a map from S^{n+d} to the one-point compactification of U, which is just $\mathrm{Th}(\nu^n)$, the Thom space of ν^n. If we compose this map with the map from $\mathrm{Th}(\nu)$ to the universal Thom space $\mathrm{Th}(U^d)$ over the Grassmannian (these fit together to form a spectrum MSO, MU, etc), this rephrases the data of an embedded manifold in terms of a map $f : S^{n+d} \to \mathrm{Th}(U^d)$. A cobordism of M can be embedded into $\mathbb{R}^{n+d} \times I$ and similarly gives rise to a homotopy between the maps constructed at either end of the cobordism.

Going in the other direction, a map $f : S^{n+d} \to \mathrm{Th}(U^d)$ can be modified by a homotopy to be transversal to the zero-section. The inverse image of the zero-section is an embedded manifold M^d, whose cobordism class depends only on the homotopy class of f.

Thom was able to use this homotopy theoretic reformulation of the cobordism groups, together with algebraic properties of the Steenrod algebra, to completely compute the structure of the oriented cobordism ring.

In the proof of Mumford's conjecture, the Pontrjagin-Thom construction is used to construct a map $B\Gamma_g \to \lim_{n\to\infty} \Omega^{n+2}\mathrm{Th}(U_{2,n}^{\perp})$.

31. Quantum Cohomology

The quantum cohomology ring of a symplectic manifold X is a deformation of the usual cohomology ring over $\mathbb{C}[[H_2(X)]]$ with structure constants given by the genus 0 Gromov-Witten invariants.

32. Spectrum

A spectrum $\mathcal{E}$ is (roughly) a sequence of based spaces $E_n, n \in \mathbb{N}$, provided with maps $f_n : \Sigma E_n \to E_{n+1}$ (where Σ denotes suspension). There are many different definitions of the category of spectra, but they all yield the same homotopy category, known as the *stable homotopy category.* The homotopy category of spectra forms a triangulated category (with shifts given by suspension and looping); if we associate to a space X the *suspesion spectrum* $\Sigma^\infty X$ with n^{th}-space $(\Sigma^\infty X)_n = \Sigma^n X$, the homotopy classes of maps between the suspension spectra of X and Y are the stable homotopy classes of maps between X and Y. There is a correspondence between generalized (co)homology theories and spectra as follows. Given a generalized cohomology theory h^n, the Brown representability theorem gives a (universal) space E_n such that $h^n(X) = [X, E_n]$; the suspension axiom provides the required structure maps for E_n to form a spectrum. Conversely, for any spectrum $\mathcal{E}$, the functor $h^n(X) = [X, \Omega^n\mathcal{E}]$ is a generalized cohomology theory, and $h_n(X) = \pi_n(X \wedge \mathcal{E})$ is a generalized homology theory.

33. String equation

Given a Gromov-Witten invariant involving $n+1$ marked points but with no constraints on the last point, the string equation expresses this in terms of invariants with fewer marked points. It is equivalent to annihilation by the differential operator L_{-1} in the Virasoro conjecture. (The string equation is a known fact; this part of the Virasoro conjecture is a proved theorem.)

34. String topology operations

These are operations on the homology and S^1-equivariant homology of the free loop space of a manifold M defined via intersection theory. For example, using these operations,$H_*(LM)$ has the structure of a graded commutative algebra and a graded Lie algebra.

35. Slope conjecture

The slope conjecture is about the possible homology classes of hypersurfaces in the moduli space of curves. Given an effective line bundle L on $\overline{\mathcal{M}}_g$, we can find non-negative a, b_i for which

$$c_1(L) = a\lambda - b_0\delta_0 \cdots - b_{[g/2]}\delta_{[g/2]} \in H^2(\overline{\mathcal{M}}_g, \mathbb{Q})$$

where λ is the Hodge class $c_1(\mathbb{E})$ and δ_i is the class Poincare-dual to the boundary divisor Δ_i. The slope of the divisor L is $s(L) = \frac{a}{\min b_i}$. The slope conjecture states that

$$s_g = \inf_{L,a\neq 0} s(L) \geq 6 + \frac{12}{g+1}.$$

For $g \leq 22$ this would imply that the Kodaira dimension of $\overline{\mathcal{M}}_g$ is $-\infty$. As it happens, Farkas and Popa have recently constructed several counterexamples to the slope conjecture. However, one can still ask for other weaker lower bounds on s_g. It is known that $s_g \geq 4$.

36. Tautological rings

The tautological ring $R^*(\overline{\mathcal{M}}_{g,n})$ is the subring of the the Chow ring $A^*(\overline{\mathcal{M}}_{g,n})$ which is meant to contain all of the natural geometric information. Faber and Pandharipande gave the following elegant formulation (which is equivalent to previous definitions). There are forgetful morphisms $\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n-1}$ and gluing morphisms $\overline{\mathcal{M}}_{g,n+1} \times \overline{\mathcal{M}}_{h,m+1} \to \overline{\mathcal{M}}_{g+h,n+m}$ and $\overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g+1,n}$. The system of tautological rings is then the smallest system of $\mathbb{Q}$-subalgebras of the Chow rings which is closed under the gluing and pushforward maps and which contains all of the Witten classes ψ_i. Tautological rings for the uncompactified moduli space and its partial compactifications are defined by restriction.

37. Teichmuller space

Teichmuller space $\mathcal{T}_g$ for genus g parametrizes pairs (C, ϕ) of a genus g Riemann surface C and a homeomorphism $\phi : C \to C_0$ to a fixed surface of genus g, up to isotopy of ϕ. This is equivalent to parametrizing Riemann surfaces C with a choice of normalized generators of $\pi_1(C)$. It is naturally an open subset of $\mathbb{C}^{3g-3}$, homeomorphic to a ball. It admits an action of the mapping class group Γ_g with finite stabilizers and with quotient $\mathcal{M}_g$.

38. Torelli group

The Torelli group I_g is the subgroup of the mapping class group Γ_g that acts trivially on the first homology of the surface F_g. It is the kernel of the map $\Gamma_g \to Sp(2g, \mathbb{Z})$ and captures (in some sense) the difference between the topology of $\mathcal{M}_g$ and that of $\mathcal{A}_g$, the moduli space of principally polarized abelian varieties

39. Unirational

A variety X is *unirational* if there is a map $U \to X$ from an open subset $U \subset \mathbb{A}^N$ of some affine space whose image contains a dense, open subset of X. For instance, $\mathcal{M}_g$ is unirational means that there is a family of curves on an open subset of affine space which contains a general curve of genus g. This is known to be the case for $g \leq 14$. Moreover, since unirational implies Kodaira dimension $-\infty$, the result of Eisenbud, Harris, and Mumford shows that $\mathcal{M}_g$ is not unirational for $g \geq 24$.

40. Virasoro conjecture

The Virasoro conjecture is a series of relations among the full Gromov-Witten invariants of a smooth projective variety X for all genera. If one encodes all Gromov-Witten invariants (including descendents) in a generating function Z_X, these relations are expressed as a sequence of differential operators L_k (for $k \geq -1$) for which

$$L_k Z_X = 0$$

and

$$[L_k, L_l] = (k-l)L_{k+l}.$$

That is, these operators give a representation of half of the Virasoro algebra of differential operators $\{x^{k+1}\frac{d}{dx}, k \in \mathbb{Z}\}$ on S^1. The conjecture is currently known for toric Fano varieties and for curves (as well as some other trivial cases). One strange feature of the Virasoro relations is that their definition relies upon the Hodge decomposition in a weak but nontrivial way. In particular, they are not defined for arbitrary symplectic manifolds, even though Gromov-Witten invariants are defined in this context.

41. Virtual fundamental class

The moduli space of maps $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is highly singular, and yet it has a homology class which behaves much like a fundamental class, and is hence called the *virtual fundamental class* $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}$ defined in terms of the deformation theory of stable maps to X. The Gromov-Witten invariants of X are defined by integrating various geometric cohomology classes over this class.

42. Witten classes

There are n natural line bundles L_i over $\mathcal{M}_{g,n}$ (or $\overline{\mathcal{M}}_{g,n}$); the fibre at C of L_i is the cotangent space $T^*_{x_i}C$, where x_i is the i^{th} marked point in C. The Witten class ψ_i is then the first Chern class of L_i. These live naturally in the cohomology of either the compactified or non-compactified moduli space.

43. Witten's Conjecture

Witten's conjecture is a recursive constraint for top intersections of ψ-classes on $\overline{\mathcal{M}}_{g,n}$. More precisely, if one considers the generating function

$$F_g = \sum_{n \geq 0} \frac{1}{n!} \left(\sum_{k_1,\ldots,k_n} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \right) t_{k_1} \cdots t_{k_n},$$

Setting $F = \sum_g \lambda^{2g-2} F_g$, Witten's conjecture is that $Z = \exp F$ is annihilated by a certain partial differential operator that also arises in studying the KdV equation in soliton theory.

It can be restated as a specialization of the Virasoro conjecture for the case where X is a point.

Multiple proofs of this conjecture now exist, due to Kontsevich, Okounkov-Pandharipande, and most recently Mirzakhani.

44. Weil-Petersson metric/symplectic form

The Weil-Petersson metric is a metric on Teichmuller space (an alternative to the Teichmuller metric). It is a Kähler metric, but it is not complete.